Learning and the Effectiveness of Central Bank Forward Guidance

Stephen J. Cole *

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Abstract

The unconventional monetary policy of forward guidance operates through the management of expectations about future paths of interest rates. This paper examines the link between expectations formation and the effectiveness of forward guidance. A standard New Keynesian model is extended to include forward guidance shocks in the monetary policy rule. Agents form expectations about future macroeconomic variables via either the standard rational expectations hypothesis or a more plausible theory of expectations formation called adaptive learning. The results show the efficacy of forward guidance depends on the manner in which agents form their expectations. In response to forward guidance, the paths of the output gap and inflation under adaptive learning overshoot and undershoot those implied by rational expectations. The adaptive learning impulse responses of the endogenous variables to a forward guidance shock exhibit more persistence before and after the forward guidance shock has been realized upon the economy. During an economic crisis (e.g. a recession), the assumption of rational expectations overstates the effects of forward guidance relative to adaptive learning. Specifically, the output gap is higher under rational expectations than adaptive learning. Thus, if monetary policy is based on a model with rational expectations, which is the standard assumption in the macroeconomic literature, the results of forward guidance could be potentially misleading.

Keywords: Forward Guidance, Monetary Policy, Adaptive Learning, Expectations.

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*Department of Economics, University of California Irvine, 3151 Social Science Plaza, Irvine, CA 92697-5100. Email: sjcole@uci.edu

1 Introduction

Once U.S. short-term interest rates effectively reached the zero lower bound (ZLB) during the 2007-2009 global financial crisis, monetary policymakers exhausted the conventional policy tool as overnight interest rates could not be lowered. In response, central banks pursued "unconventional" policies. One of these alternatives pursued by the Federal Reserve was large-scale asset purchases (LSAPs) where the central bank purchases longer-term securities in hopes of lowering long-term yields. Another unconventional policy was forward guidance, where the central bank communicates to the public information about the future course of the policy rate. Forward guidance has been pursued by central banks such as the Federal Reserve, Bank of Canada, Bank of England, and the European Central Bank. An example of forward guidance was given in the September 2012 Federal Open Market Committee (FOMC) statement: "the Committee also ... anticipates that exceptionally low levels for the federal funds rate are likely to be warranted at least through mid-2015." In addition, Eggertsson and Woodford (2003) and Woodford (2012) argue that committing to an interest rate path that is lower than what one would commit to under normal circumstances (i.e. when overnight interest rates are away from the ZLB) can have additional stimulative economic effects. Standard New Keynesian models (e.g. Woodford [2003]) predict consumption, investment, and pricing decisions are sensitive to the expected path of short term interest rates. If agents expect low interest rates in the future, current consumption and prices all increase. This stimulative effect can be limited by a conventional monetary policy rule that adjusts interest rates in response to target variables, such as the output gap and inflation. Households and firms may rationally expect higher interest rates in response to future expansions. If a forward guidance statement, instead, keeps a low policy rate through part of the expansion, consumption today will not be as limited.

The effectiveness of forward guidance hinges on how private sector expectations about economic state variables (e.g. output and inflation) and interest rates respond to forward guidance. Therefore, it is important to study whether the economic effects of forward guidance are sensitive to the rational expectations assumption that is the standard benchmark in macroeconomic models.[1] While a reasonable benchmark that is popular among macroeconomic models, rational expectations makes strong assumptions about the amount of knowledge agents possess when forming beliefs. It is natural then to examine how effective forward guidance policies can be under a more plausible theory of expectations formation.

[1] A related issue is the credibility of policymakers to commit to a future path of interest rates (see, for instance, see Woodford (2012) and Swanson and Williams [2012]). In part, because of credibility concerns, Woodford (2012) prefers forward guidance policies that explicitly state the criteria that will underlie future policy rules. This current paper abstracts from this subject.

This paper studies the effectiveness of forward guidance in an environment where rational expectations has been replaced by an adaptive learning rule similar to one proposed by Marcet and Sargent (1989) and Evans and Honkapohja (2001). In particular, the economic environment is based on Preston (2005) who derives a New Keynesian model with (potentially) non-rational expectations. Households and firms formulate spending and pricing decisions, respectively, that depend on their subjective expectations about future economic conditions and interest rates. The novelty of this paper is to incorporate policy communication about future interest rates into agents' subjective expectations. The central bank sets interest rates according to a monetary policy rule that responds positively to the output gap and inflation. The rule is augmented with anticipated shocks as in Del Negro, Giannoni, and Patterson (2012) and Laseen and Svensson (2011).[2] The anticipated shocks define central bank communication about future deviations from a normal interest rate rule that agents know today. The shocks also represent what the Bank of England (2013) describes as "time-contingent guidance" (p. 40), where the central bank communicates a definitive forward guidance end date. In this case, communication about the future path of interest rates is for a fixed amount of periods into the future and is independent of economic conditions.[3]

Agents are assumed to form expectations via either the rational expectations hypothesis or an adaptive learning rule. The former is a strong assumption and assumes agents construct expectations with respect to the true probability distribution of the model. Rational expectations agents must know the model's deep parameters, structure of the model, beliefs of other agents, and distribution of the error terms. A popular alternative to rational expectations is adaptive learning. This approach builds from the cognitive consistency principle that agents behave as real-life economists (see, for instance, Evans and Honkapohja [2013]). An econometrician, for example, would produce forecasts of future economic variables by forming an econometric model. He or she would estimate the parameters using standard econometric techniques. As new data arrives, these forecasts would be revised. Thus, a real-life economist is engaging in a process of learning about the economy. Analogously, adaptive learning agents are assumed to behave as econometricians and formulate forecasts of future endogenous variables using standard econometric techniques. The variables in their econometric model are based on the solution found under rational expectations, but adaptive learning agents estimate the parameters using ordinary least squares. Their beliefs about future endogenous variables are appropriately revised as new data arrive.[4]

[2]The anticipated shocks are similar to the news shocks of Schmitt-Grohé and Uribe (2012).

[3]This type of forward guidance is in contrast to state-contingent forward guidance where the duration of a constant interest rate path is linked to economic conditions.

[4]Adaptive learning agents do not take into account they will update their beliefs in future periods. They believe that the beliefs they form every period are optimal. This methodology follows from the anticipated utility discussion

The results of this paper show that the desired effect of forward guidance depends on the manner in which agents form their expectations. This outcome is first shown during normal economic times.[5] The impulse responses of the endogenous variables under adaptive learning fail to capture the precise effects a forward guidance shock has on the economy. There exists more persistence in the paths of the output gap and inflation under adaptive learning than rational expectations. Differences also occur when the central bank communicates to both rational expectations and adaptive learning agents the same forward guidance information such that the interest rate will equal zero for an extended period of time. The output gap and inflation return to long-run equilibrium quicker under rational expectations than adaptive learning. Under adaptive learning, the paths of the output gap and inflation overshoot and undershoot the rational expectations paths. Consequently, there exists larger variation of the paths of the output gap and inflation under adaptive learning than rational expectations. These effects occur because rational expectations agents fully understand the precise and positive effects of forward guidance on the economy. However, adaptive learning agents fail to understand the positive effect and must continually make adjustments to their beliefs causing them to overshoot and undershoot the rational expectations paths of the output gap and inflation.

The effectiveness of forward guidance is also examined under a period of economic crisis (e.g. a recession). The policy experiment includes a scenario where forward guidance is implemented to combat the effects of a downturn in the economy. The results show the effects of forward guidance under rational expectations are overstated relative to adaptive learning. Specifically, the value of the output gap is higher under the assumption of rational expectations than adaptive learning. The reason is that rational expectations agents base their expectations of future values of the endogenous variables on the true model of the economy. They understand the economic downturn and how forward guidance will precisely alleviate the economy. However, adaptive learning agents observe the economic downturn, but fail to fully understand how forward guidance will improve the economy. They are *estimating* the effects of forward guidance on the economy as their forecasts are based on an econometric model.

Overall, the results of the paper suggest a main finding: policymakers should exercise caution when recommending forward guidance policy. If monetary policy is based on a model with the standard rational expectations hypothesis, which assumes agents know the true structure of the model, the results may be misleading relative to a more plausible theory of expectations formation

from Kreps (1998).

[5] As will be discussed in Section 4, forward guidance is assumed to start after a large number of periods have passed, that is, after a period of economic stability.

(e.g. adaptive learning). Specifically, during an economic crisis, the predicted effects of forward guidance under the rational expectations assumption are overstated in comparison to adaptive learning.

1.1 Previous Literature

This paper contributes to the growing literature on unconventional monetary policy. Eggertsson and Woodford (2003) explain that the expectations channel plays a key role on the economy when interest rates are at the ZLB and at any level. Specifically, they describe that the future path of short-term interest rates affects long-term interest rates and asset prices, and thus, the management of expectations about future interests rates affects agents' optimal decisions. De Graeve, Ilbas, and Wouters (2014) find that the effectiveness of forward guidance does not necessarily work through decreasing the long-run interest rate, contrary to previous studies. The type of forward guidance and lack of information about the underlying reasons for implementing forward guidance (e.g. monetary stimulus or sign of future economic crisis) can dampen the effects of this monetary policy tool. In addition, recent literature has found large effects from forward guidance. Carlstrom, Fuerst, and Paustian (2012) show that standard New Keynesian models with the interest rate fixed for a finite period of time result in extreme responses of output and inflation. Del Negro et al. (2012) construct a Dynamic Stochastic General Equilibrium (DSGE) model with forward guidance, which produces large responses of macroeconomic variables to forward guidance. Del Negro et al. (2012) state that the long-term bond yield drives these unusually high responses. As will be discussed in Section 4.3, this current paper suggests that the exceedingly large responses to forward guidance found in the previously mentioned articles could be due to the manner in which expectations are modeled.

The model in this paper utilizes time-contingent forward guidance since there has been recent evidence of its effectiveness. Gürkaynak, Sak, and Swanson (2005) find empirical evidence that FOMC statements about the future path of the policy rate greatly contribute to the changes in the long-term interest rates. Swanson and Williams (2012) show that recent Federal Reserve forward guidance announcements have affected medium and longer-term interest rates. Woodford (2012) shows that forward guidance has had an impact on market participants. Using overnight interest rate swaps (OIS) to measure market expectations about the policy rate in Canada, Woodford (2012) displays that OIS rates immediately changed upon release of the Bank of Canada's forward guidance statement. The work of Chang and Feunou (2013) show that the Bank of Canada's forward guidance statement in 2009 had positive effects on the economy by reducing uncertainty

about future monetary policy rates. A reduction in interest rate uncertainty can affect levels of investment, output, and unemployment in the economy as described by Baker, Bloom, and Davis (2013). Femia, Friedman, and Sack (2013) describe evidence that financial variables, such as Treasury yields and equity prices, reacted favorably to the Federal Reserve's time-contingent forward guidance announcements.

By analyzing the role of expectations formation on forward guidance, this paper builds on the adaptive learning and policy literature. Evans, Honkapohja, and Mitra (2012) examine the effects of the fiscal authority giving guidance on the future course of government purchases and taxes. The results show that a temporary change in fiscal policy leads to different effects on adaptive learning and rational expectations agents. The adaptive learning output multipliers seem to match empirical data more than its rational expectations counterparts. Eusepi and Preston (2010) investigate the link between adaptive learning and central bank communication strategies. Increased central bank communication, such as communicating the monetary policy rule and the variables within the rule, can lead to increased macroeconomic stability. Preston (2006) studies forecast-based monetary policy rules and adaptive learning. He finds that a central bank that understands the basis of private sector forecasts can aid in increasing macroeconomic stability.

The remaining sections of the paper are organized as follows. Section two presents the New Keynesian model with forward guidance. Section three discusses expectations formation under both rational expectations and adaptive learning. Section four presents the outcomes of forward guidance under both rational expectations and adaptive learning. Section five examines the results under different parameter schemes. Section six concludes.

2 Model

The aggregate dynamics of the economy are described by a New Keynesian model derived under (potentially) non-rational expectations (see Preston [2005]).[6] There is a continuum of households each of whom maximizes expected future discounted utility by choosing sequences of consumption, labor, money holdings, and bonds.[7] Consequently, each household must be forward looking in order to make optimal decisions. A household's (potentially) non-rational beliefs satisfy standard probability laws that are the same across households. However, each agent does not know the beliefs of other agents. When agents optimize every period, each household believes that its beliefs about

[6] A detailed description of the model can be found in Appendix A

[7] As is standard in many macroeconomic models, money holdings do not alleviate any transaction frictions. However, households may choose to hold money as a substitute to riskless bonds because of the equivalent return of money and riskless bonds as described in Preston (2005).

future values of the endogenous variables are optimal, but does not take into account that it will be updating beliefs every period. This assumption follows the anticipated utility discussion of Kreps (1998). In addition, each household provides labor for the production of good j, and receives a wage as compensation. A household shares in the profits from the sale of good j since all households own an equal share of each firm. Markets are assumed incomplete and households can invest in riskless bonds in order to transfer wealth between periods. Households also pay lump-sum taxes to the government.[8] The optimal decisions of households satisfy a sequence of Euler equations and the intertemporal budget constraint. A household's intertemporal budget constraint includes wealth in the form of bonds purchased from the previous period. By combining the household's Euler equation, intertemporal budget constraint, and no-Ponzi constraint, the resulting log-linearized equation for the output gap is given by

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}[(1-\beta)x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n] \tag{1}$$

where

$$r_t^n = \rho_n r_{t-1}^n + \varepsilon_t^n \tag{2}$$

and $\varepsilon_t^n \overset{iid}{\sim} N(0, \sigma_n^2)$. All variables are in terms of log deviations from steady state. Equation (1) relates the current output gap x_t to current and future expected values of the output gap, interest rate i_t, inflation rate π_t, and natural real interest rate shock r_t^n. β describes the household's discount rate and is bounded between zero and one. $\sigma > 0$ defines the intertemporal elasticity of substitution of consumption between periods. $\hat{E}_t$ denotes (potentially) non-rational expectations. Households take into account the future values of the endogenous variables infinitely far into the future when choosing optimal consumption today. Intuitively, the expected course of a household's consumption pattern matters to its optimal consumption today. A household also knows future consumption patterns are affected by future values of income, interest rates, and inflation. Thus, expectations of these variables are important for decisions today.

The production side of the economy is populated by firms that operate in a monopolistically competitive environment where each good is produced using labor from households. Each firm is subject to a Calvo (1983) pricing scheme, and thus, has an α probability of not being able to change its price every period. The representative firm that is able to adjust its price chooses its price each period to maximize expected present discounted value of profits. The resulting log-linearized

[8] The fiscal authority operates a zero net supply of bonds. It is assumed to be Ricardian so that fiscal policy does not affect the output of goods.

equation for inflation is

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} [\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + \mu_T] \tag{3}$$

where

$$\mu_t = \rho_\mu \mu_{t-1} + \varepsilon_t^\mu \tag{4}$$

and $\varepsilon_t^\mu \overset{iid}{\sim} N(0, \sigma_\mu^2)$. All variables are in terms of log deviations from steady state. Equation (3) defines the inflation rate as a function of current and future values of the output gap, inflation rate, and cost-push shock μ_t. ω describes the elasticity of a firm's real marginal cost function with respect to its own output, and $\kappa \equiv \frac{(1-\alpha)}{\alpha}\frac{(1-\alpha\beta)}{(1+\omega\theta)}(\omega + \sigma^{-1}) > 0$. θ measures the elasticity of substitution between differentiated goods. The optimal decisions by firms are shown to depend on the long-run expected path of macroeconomic variables because of the assumption of sticky prices. A firm must be concerned that it will not be able to adjust its price in future periods regardless of future economic conditions. Thus, optimal pricing decisions today requires firms to forecast future states and values of economic variables.[9]

The model is closed by describing the central bank of the economy. The central bank follows a monetary policy rule that takes the following form

$$i_t = \chi_\pi \pi_t + \chi_x x_t + \varepsilon_t^{MP} + \sum_{l=1}^{L} \varepsilon_{l,t-l}^R \tag{5}$$

The short-term nominal interest rate changes based on the output gap, inflation rate, monetary policy shock, and forward guidance shocks. ε_t^{MP} defines an unanticipated monetary policy shock and is *i.i.d.* In order to incorporate forward guidance into the model, the monetary policy rule is augmented with anticipated shocks following Del Negro et al. (2012) and Laseen and Svensson (2011). Each anticipated or forward guidance shock ($\varepsilon_{l,t-l}$) is contained in the last term in equation (5) and is *i.i.d.* Intuitively, the forward guidance shock can be thought of as an announcement by the central bank in period $t-l$ that the interest rate will change l periods later, i.e. in period t. If the central bank has been communicating guidance on the interest rate for L periods ahead, there would be $1, 2, 3, \ldots, L$ forward guidance shocks that affect the monetary policy rule in period t. Thus, L corresponds to the length of the forward guidance horizon announced by the central bank. The last term in equation (5) can also be thought of as the sum of all forward guidance

[9] Another approach to modeling learning and (potentially) non-rational expectations in macroeconomic models regards the "Euler-equation" method presented in Evans and Honkapohja (2001), where only one period ahead forecasts of the endogenous variables show up in the model's equations under both rational expectations and adaptive learning. For a comparison between the "infinite-horizon" and Euler-equation approach to learning, see Evans, Honkapohja, and Mitra (2013)

commitments stated by the central bank $1, 2, ...,$ and L periods ago that affect the nominal interest rate in period t. Following Del Negro et al. (2012) and Laseen and Svensson (2011), the system is also augmented with L state variables $v_{1,t}, v_{2,t}, ..., v_{L,t}$. The law of motion for each of these state variables is given by

$$v_{1,t} = v_{2,t-1} + \varepsilon^R_{1,t} \tag{6}$$

$$v_{2,t} = v_{3,t-1} + \varepsilon^R_{2,t} \tag{7}$$

$$v_{3,t} = v_{4,t-1} + \varepsilon^R_{3,t} \tag{8}$$

$$\vdots$$

$$v_{L,t} = \varepsilon^R_{L,t} \tag{9}$$

In other words, each component of $v_t = [v_{1,t}, v_{2,t}, ..., v_{L,t}]'$ is the sum of all central bank forward guidance commitments known in period t that affect the interest rate $1, 2, ...,$ and L periods into the future, respectively.[10] It should be noted that equations (6) − (9) can be simplified to find that $v_{1,t-1} = \sum_{l=1}^{L} \varepsilon^R_{l,t-l}$. In addition, equations (5) − (9) provide a computationally tractable method to model forward guidance. Since the forward guidance shocks equal $v_{1,t-1}$, the forward guidance shocks can be put into a vector of predetermined variables in standard state-space form. As described by Laseen and Svensson (2011), standard solution techniques then can be used to solve the final system of equations. Another reason to model forward guidance in this way is that it relieves the concern of the existence of multiple solutions. As described in Honkapohja and Mitra (2005), Woodford (2005), and Woodford (2003), indeterminacy can arise if forward guidance is instead modeled as pegging the interest rate to a certain value.[11] For instance, without a monetary policy that responds to economic fluctuations, real disturbances to the economy can produce a multitude of equilibrium responses of the endogenous variables.

The following example presents the case where the central bank's forward guidance horizon is 2 periods ahead, i.e. $L = 2$, so that the reader may gain further intuition about v_t. The model's system of equations consists of $v_{1,t}$ and $v_{2,t}$ whose laws of motion are defined as

$$v_{1,t} = v_{2,t-1} + \varepsilon^R_{1,t} = \varepsilon^R_{2,t-1} + \varepsilon^R_{1,t} \tag{10}$$

$$v_{2,t} = \varepsilon^R_{2,t} \tag{11}$$

Thus, $v^R_{1,t}$ defines the sum of all forward guidance commitments by the central bank known in

[10] In the terminology of Laseen and Svensson (2011), $v_{1,t}, v_{2,t}, ..., v_{L,t}$ are described as central bank "projections" (p. 10) of what $\sum_{l=1}^{L} \varepsilon^R_{l,t-l}$ will be $1, 2, ...,$ and L periods into the future, respectively.

[11] Carlstrom, Fuerst, and Paustian (2012) show that determinacy can arise from an interest rate peg if terminal conditions are known and a standard monetary policy rule is followed after the interest rate peg. However, unusually large responses of the output and inflation are found through this process.

period t that affect the interest rate one period later. $v_{1,t}^R$ consists of current period forward guidance affecting the interest rate one period later, $\varepsilon_{1,t}^R$, and previous period's forward guidance affecting the interest rate two periods later, $v_{2,t-1} = \varepsilon_{2,t-1}^R$. $v_{2,t}$ is the sum of all forward guidance commitments by the central bank known in period t that affect the interest rate two periods later. Since the forward guidance horizon is two periods, $v_{2,t}$ consists of current period forward guidance affecting the interest rate two periods later, $\varepsilon_{2,t}^R$.[12]

The ZLB on interest rates is also enforced. Forward guidance has gained attention due to interest rates effectively reaching the ZLB because of the 2007-2009 global financial recession. Thus, it seems natural to model the ZLB on nominal interest rates when simulating forward guidance. Specifically, equations (1) and (5) become

$$\hat{x}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}[(1-\beta)\hat{x}_{T+1} - \sigma(i_T - i^* - \hat{\pi}_{T+1}) + \hat{r}_T^n] \tag{12}$$

$$i_t = max\{i^* + \chi_\pi \hat{\pi}_t + \chi_x \hat{x}_t + \varepsilon_t^{MP} + \sum_{l=1}^{L} \varepsilon_{l,t-l}^R, 0\} \tag{13}$$

where $i^* = r^* + \pi^*$ is the steady-state nominal interest rate. The " ˆ " symbol over the variables denotes log deviations from steady state.[13]

To summarize, the aggregate dynamics of the economy with forward guidance are defined by the output gap, inflation rate, AR(1) shock processes, monetary policy rule with forward guidance, and the laws of motion of the sum of central bank commitments, that is, equations $(1) - (9)$. With enforcement of the ZLB, equations (12) and (13) are used instead of (1) and (5).

3 Expectation Formation

This paper assumes agents form expectations following either the rational expectations hypothesis or adaptive learning. The difference between the two types of expectations formation regards the amount of knowledge agents hold about the economy (See, for example, Marcet and Sargent (1989), Evans and Honkapohja (2001), and Evans, Honkapohja, and Mitra (2009).). Under rational expectations, agents know the structure of the model, parameters of the model (e.g. σ, κ, etc.), distribution of the error terms, and beliefs of other agents. They compute expectations based off the true model of the economy. Under adaptive learning, agents do not know the true model of the economy, and thus, cannot compute precise expectations as under rational expectations.

[12] A constant interest rate path can still be achieved by modeling forward guidance with equations (5)-(9). As will be described in Section 4.2.2, the forward guidance shocks can be chosen such that the interest rate equals a certain value for a fixed amount of periods into the future.

[13] In a zero steady-state inflation rate, $\pi^* = 0$. The model implied steady-state real interest rate $r^* = \beta^{-1} - 1$.

Instead, they operate as econometricians by forming an econometric model to forecast values of the endogenous variables. Their model includes the variables in the rational expectations solution. Adaptive learning agents estimate the values of the model's parameters using standard econometric methods. As new information becomes available every period, they appropriately adjust their forecasts.

Rational Expectations–The model defined by equations (1) − (9) can be simplified under the assumption of rational expectations. Agents with rational expectations understand the beliefs of other agents and are able to compute the aggregate probabilities of the model. As shown in Preston (2005), this additional information simplifies the infinite horizon model to the "benchmark" one step ahead New Keynesian model. Specifically, equations (1) and (3) become

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + r_t^n \tag{14}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \mu_t \tag{15}$$

The model with rational expectations can be solved using standard techniques, such as one suggested by Sims (2002). The model can be written in general state-space form as suggested by Sims (2002). This form is defined as

$$\widetilde{\Gamma}_0 \widetilde{Y}_t = C + \widetilde{\Gamma}_1 \widetilde{Y}_{t-1} + \widetilde{\Gamma}_2 \widetilde{\epsilon}_t + \widetilde{\Gamma}_3 \zeta_t \tag{16}$$

where

$$\widetilde{Y}_t = [x_t, \pi_t, i_t, r_t^n, \mu_t, v_{1,t}, v_{2,t}, \ldots, v_{L,t}, E_t x_{t+1}, E_t \pi_{t+1}]' \tag{17}$$

$$\widetilde{\epsilon}_t = [\varepsilon_t^n, \varepsilon_t^\mu, \varepsilon_t^{MP}]' \tag{18}$$

C defines a vector of constants of required dimensions. ζ_t defines the vector of expectational errors (e.g. $\zeta_t^\pi = \pi_t - E_{t-1}\pi_t$) of required dimensions. Using standard techniques to solve the model with rational expectations (e.g. Sims [2002]) and the parameter values in Table 1, the solution to the system under rational expectations is

$$\widetilde{Y}_t = \widetilde{C} + \xi_1 \widetilde{Y}_{t-1} + \xi_2 \epsilon_t \tag{19}$$

where the matrices $\widetilde{C}$, ξ_1, and ξ_2 are defined in Appendix B.[14]

Adaptive Learning–In order to evaluate the expectations in equations (1) and (3) under adaptive learning, agents act as econometricians by forming a model based on variables that appear in the rational expectations solution and estimate the coefficients. This model is labeled the

[14] Discussion of the parameter values can be found in Table 1 in Section 4.1.

"Perceived Law of Motion" (PLM) and is constructed from the minimum state variable (MSV) solution that exists under rational expectations.[15] The PLM is defined as

$$Y_t = a + bv_t + cw_t + dv_{1,t-1} + \varepsilon_t \tag{20}$$

where

$$Y_t = [x_t, \pi_t, i_t]' \tag{21}$$

$$v_t = [v_{1,t}, v_{2,t}, ..., v_{L,t}]' \tag{22}$$

The vector $w_t = [r_t^n, \mu_t]'$ is defined by

$$w_t = \widetilde{\phi} w_{t-1} + \bar{\varepsilon}_t \tag{23}$$

where

$$\widetilde{\phi} = \begin{bmatrix} \rho_n & 0 \\ 0 & \rho_\mu \end{bmatrix} \tag{24}$$

$$\bar{\varepsilon}_t = [\varepsilon_t^n, \varepsilon_t^\mu]' \tag{25}$$

By rewriting equations (6) − (9), the vector v_t becomes

$$v_t = \Phi v_{t-1} + \eta_t \tag{26}$$

where

$$\eta_t = [\varepsilon_{1,t}^R, \ldots, \varepsilon_{L,t}^R]' \tag{27}$$

and Φ is an L x L matrix given by

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{28}$$

$$\tag{29}$$

a, b, c, and d are unknown coefficient matrices of appropriate dimensions that agents estimate and learn about over time.[16] Furthermore, the addition of $v_{1,t-1}$ is a necessary component of the PLM

[15] This paper focuses on a version of the model that is determinate so that the PLM is based on the unique non-explosive rational expectations equilibrium. The parameter values in Table 1 verify that the rational expectations solution is determinate.

[16] In the PLM, the time subscript is left off the coefficients to emphasize that adaptive learning agents believe current period forecasts are optimal and do not take into account they will be updating their beliefs every period. However, as will be described later, the PLM coefficients will evolve over time.

since it is present in the rational expectations solution shown in Appendix B and not contained in the vector v_t.[17]

An important component of adaptive learning models regards the information available to agents when they form expectations. In this paper, adaptive learning agents are assumed to know the values of the regressors in the PLM and previous period's coefficient estimates when forming beliefs about the future. They update their parameter estimates at the end of the period. This assumption avoids the simultaneous determination of current period coefficient estimates and endogenous variables when forming expectations and making optimal decisions.[18] The *i.i.d.* monetary policy shock is also assumed to be unobserved.[19] Furthermore, the following is the timeline of events:

1. At the beginning of period t, v_t, and w_t are observed by the agents and added to their information set.

2. Agents use v_t, w_t, and $v_{1,t-1}$ as well as previous period's estimates, ϕ_{t-1}, to form expectations about the future.

3. Y_t is realized.

4. In order to update their parameter estimates, agents compute a least squares regression of Y_t on 1, v_t, w, and $v_{1,t-1}$.

Agents update their parameter estimates of the PLM by following the recursive least squares (RLS) formula

$$\phi_t = \phi_{t-1} + \tau_t R_t^{-1} z_t (Y_t - \phi_{t-1}^{'} z_t)' \tag{30}$$

$$R_t = R_{t-1} + \tau_t (z_t z_t^{'} - R_{t-1}) \tag{31}$$

where $\phi = (a, b, c, d)^{'}$ contains the PLM coefficients to be estimated. R_t defines the precision matrix of the regressors in the PLM $z_t \equiv [1, v_t, w_t, v_{1,t-1}]^{'}$. τ_t is known as the "gain" parameter and controls the response of ϕ_t to new information. The last expression in equation (30) defines the recent prediction error of the endogenous variables.

The gain parameter in equations (30) and (31) can either decrease over time or be fixed at certain values. In the decreasing gain or RLS case, $\tau_t = t^{-1}$ and past observations are equally

[17] Since this paper restricts attention to fundamentals solutions and Y_{t-1} does not appear in equations (1), (3), and (5), the PLM does not contain Y_{t-1}.

[18] An alternative is to assume that agents use the coefficient estimates from the current period when forming expectations. This results in expectations and current period parameter estimates determined simultaneously when making optimal decisions.

[19] This is similar to Milani (2007a).

weighted. Evans and Honkapohja (2001) explain that as $t \to \infty$ the coefficients in the PLM converge to the rational expectations coefficients with probability one. As is assumed in this current paper, the gain parameter can also be fixed at a certain value. Under this method called discounted or constant gain learning (CGL), $\tau_t = \bar{\tau}$ and the most recent observations play a larger role when updating agents' coefficients and expectations. Evans and Honkapohja (2001) describe that the coefficients in the PLM converge in distribution to their rational expectations values with a variance that is proportional to the constant gain coefficients. CGL may be a more realistic way to model learning since it allows agents to update their beliefs every period to new information as a real-life econometrician revising his or her forecasts every period.

Agents solve for $\hat{E}_t Y_{T+1}$ by using equation (20). For any $T \geq t$, their expectations infinite periods ahead are given by

$$\begin{aligned}\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} Y_{T+1} = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} a_{t-1} + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} b_{t-1} v_{T+1} \\ + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} c_{t-1} w_{T+1} + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} d_{t-1} v_{1,T}\end{aligned} \tag{32}$$

$$\begin{aligned}\hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} Y_{T+1} = \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} a_{t-1} + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} b_{t-1} v_{T+1} \\ + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} c_{t-1} w_{T+1} + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} d_{t-1} v_{1,T}\end{aligned} \tag{33}$$

By noting the geometric sums and expectations of v_t twelve periods ahead or greater equal the zero vector, equations (34) and (35) simplify to equal

$$\begin{aligned}\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} Y_{T+1} = (1-\beta)^{-1} a_{t-1} + b_{t-1} \Phi (I_L - \beta\Phi)^{-1} (I_L - (\beta\Phi)^{11}) v_t \\ + c_{t-1} (I_2 - \beta\widetilde{\phi})^{-1} \widetilde{\phi} w_t + d_{t-1} [1, \beta, \beta^2, \ldots, \beta^{11}] v_t\end{aligned} \tag{34}$$

$$\begin{aligned}\hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} Y_{T+1} = (1-\alpha\beta)^{-1} a_{t-1} + b_{t-1} \Phi (I_L - \alpha\beta\Phi)^{-1} (I_L - (\alpha\beta\Phi)^{11}) v_t \\ + c_{t-1} (I_2 - \alpha\beta\widetilde{\phi})^{-1} \widetilde{\phi} w_t + d_{t-1} [1, \alpha\beta, (\alpha\beta)^2, \ldots, (\alpha\beta)^{11}] v_t\end{aligned} \tag{35}$$

Equations (34) and (35) are substituted into equations (1) and (3) to give

$$Y_t = \Gamma_0(\phi_{t-1}) + \Gamma_1(\phi_{t-1}) Y_{t-1} + \Gamma_2(\phi_{t-1}) v_t + \Gamma_3(\phi_{t-1}) \widetilde{w}_t \tag{36}$$

where

$$\widetilde{w}_t = [w_t, \varepsilon_t^{MP}]' \tag{37}$$

Equation (36) is called the "Actual Law of Motion" (ALM) and describes the actual evolution of the endogenous variables implied by the PLM (20).

4 Results

4.1 Parameterization

This section details the calibration values for the model's parameters, which are shown in Table 1. The discount rate, β, is set to equal 0.99 which is a common value found in the literature. The parameter representing the intertemporal elasticity of substitution is fixed at one. This value has been assumed *a priori* in Smets and Wouters (2003). κ is set to equal 0.1. This number roughly corresponds to a high degree of price stickiness, α, found in empirical work by Klenow and Malin (2011), a value of ω found in Giannoni and Woodford (2004), and a value of θ found in the literature (e.g. Gertler and Karadi [2011]). Monetary policy positively responds to the output gap, and positively adjusts at more than a one-to-one rate to the inflation rate. $\chi_x = 0.125$ follows from Branch and Evans (2013). The value of χ_π closely follows empirical adaptive learning work by Milani (2007a). The structural disturbances are not assumed to exhibit high persistence. There is assumed to be a higher degree of persistence in the natural real interest rate shock than the cost-push shock. The distribution of the white noise shocks is not assumed to be highly dispersed. There also is no covariance between the structural shocks.

The current paper examines results for the CGL case. In regards to choosing the CGL parameter $\bar{\tau}$, this paper uses 0.02. This choice is close to the results used in the literature, such as Orphanides and Williams (2005), Milani (2007a), and Branch and Evans (2006). For robustness, the current methodology also examines the results under different values of $\bar{\tau}$.

The value for the length of the forward guidance horizon L is chosen to match time-contingent forward guidance by the Federal Reserve. This is based off the FOMC September 2012 statement:"the Committee also decided today to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that exceptionally low levels for the federal funds rate are likely to be warranted at least through mid-2015." This announcement was one of the last FOMC statements to exclusively use time-contingent forward guidance language. By taking "mid-2015" to be at most the end of the third quarter of 2015, the number of quarters from September 2012 to "mid-2015" is twelve. Thus, $L = 12$.

Table 1: Parameter Values

	Description	**Value**
σ	IES	1
β	Discount Factor	0.99
κ	Function of Price Stickiness	0.1
α	Price Stickiness	0.75
χ_π	Feedback Inflation	1.4
χ_x	Feedback Output Gap	0.125
ρ_n	Autoregressive Demand	0.5
ρ_μ	Autoregressive Cost-Push	0.2
σ_n	Demand Shock	0.01
σ_μ	Cost-Push Shock	0.01
σ_i	M.P Shock	0.001
$\sigma_{1,i}$	1 Period Ahead FG Shock	0.001
$\sigma_{2,i}$	2 Period Ahead FG Shock	0.001
$\sigma_{3,i}$	3 Period Ahead FG Shock	0.001
$\sigma_{4,i}$	4 Period Ahead FG Shock	0.001
$\sigma_{5,i}$	5 Period Ahead FG Shock	0.001
$\sigma_{6,i}$	6 Period Ahead FG Shock	0.001
$\sigma_{7,i}$	7 Period Ahead FG Shock	0.001
$\sigma_{8,i}$	8 Period Ahead FG Shock	0.001
$\sigma_{9,i}$	9 Period Ahead FG Shock	0.001
$\sigma_{10,i}$	10 Period Ahead FG Shock	0.001
$\sigma_{11,i}$	11 Period Ahead FG Shock	0.001
$\sigma_{12,i}$	12 Period Ahead FG Shock	0.001
L	FG Horizon	12
$\bar{\tau}$	CGL	0.02

Note: FG stands for forward guidance.

4.2 Normal Economic Times

4.2.1 Impulse Responses

In this section, impulse responses of the output gap and inflation rate to negative one unit monetary policy and forward guidance shocks under different expectation assumptions are examined in Figures 1 and 2.[20] The forward guidance shocks are the anticipated shocks found in equations (6) - (9). Since equation (36) exhibits a nonlinear structure, standard linear techniques to compute impulse responses under adaptive learning do not apply. To remedy this situation, this paper follows Eusepi and Preston (2011) by proceeding in the following manner. The model is simulated twice for $T+K$ periods, where K is the impulse response function horizon. The impulse responses are calculated starting in period $T+1$.[21] In the first simulation, time period $T+1$ includes a negative one unit shock. The K-period impulse response function is given by the difference between the first and

[20] A projection facility is utilized to ensure beliefs are not explosive.

[21] T is chosen to be a large number so that the adaptive learning coefficients converge to its stationary distribution.

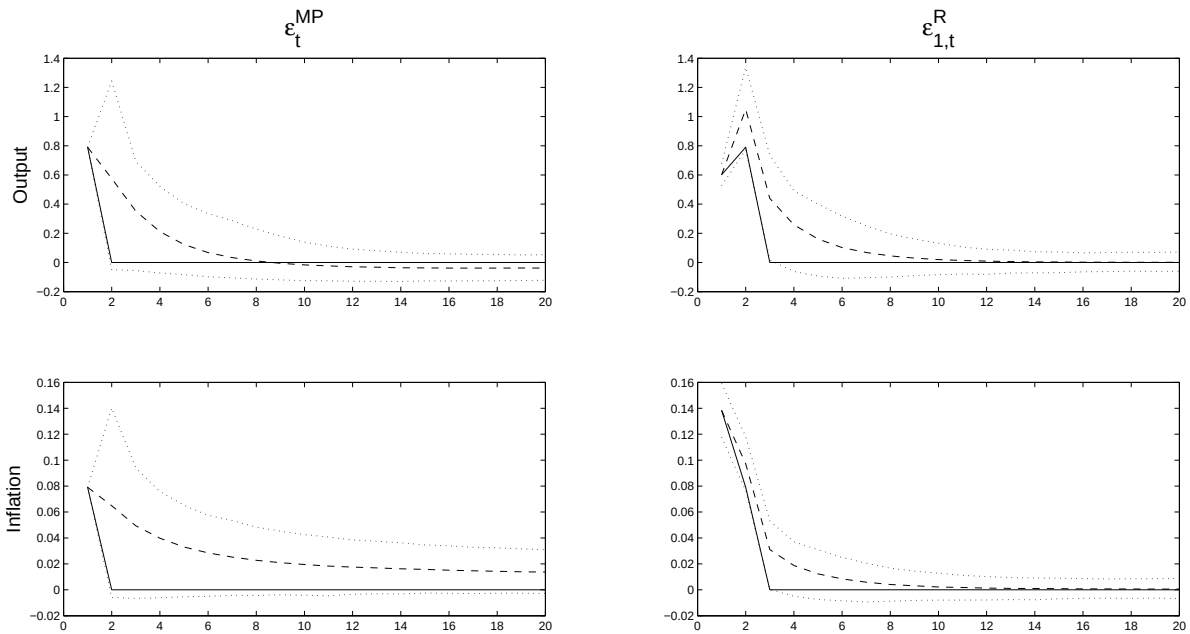

Figure 1: Impulse Responses of Endogenous Variables to Unanticipated and Forward Guidance Shocks. Solid Line: Rational Expectations; Dashed Line: CGL; Dotted Lines: 95% Confidence Bands.

second simulations over the final K periods. The process is then repeated for $5,000$ simulations and the mean impulse response across the $5,000$ simulations is calculated to arrive at the final impulse response trajectory. The impulse response function horizon is chosen to be twenty periods, that is, $K = 20$.

Impact–As seen in Figures 1 and 2, the initial response of the macroeconomic variables is approximately the same under both adaptive learning and rational expectations. This result is not surprising since Evans and Honkapohja (2001) state that CGL coefficients converge to a Normal distribution centered around its rational expectations counterparts. Thus, the initial impact under adaptive learning could be greater or less than the initial impact under rational expectations.

After Impact–Figures 1 and 2 also display the impulse responses after the forward guidance announcement is known to agents. From the household's perspective, they must optimally allocate consumption across time based on their expectations of future variables. Since they know that the interest rate will decrease in the future, a household changes its optimal consumption across time and increases current consumption. In addition, firms know they may not be able to change their price in the future regardless of the state of the economy. Thus, they take into account expectations of future variables as seen in equation (3). When the central bank announces that

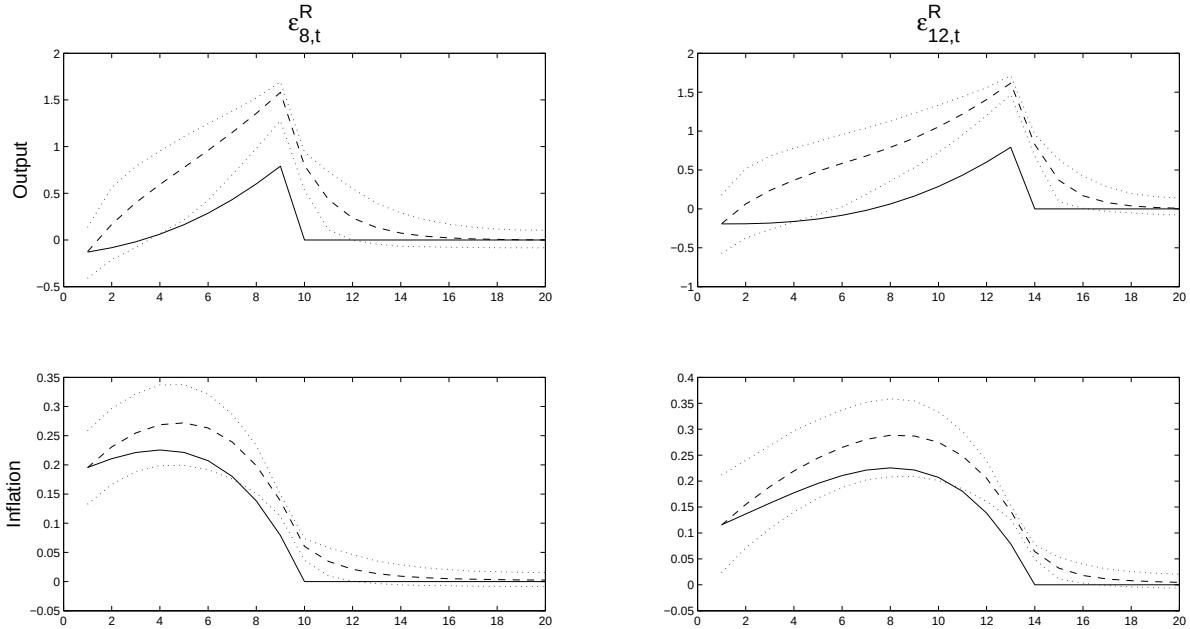

Figure 2: Impulse Responses of Endogenous Variables to Forward Guidance Shocks. Solid Line: Rational Expectations; Dashed Line: CGL; Dotted Lines: 95% Confidence Bands.

the interest rate will increase in the future, a firm knows that the future output gap and inflation will be affected, and thus, this action affects current pricing decisions. Furthermore, there exists a larger and more delayed effect on the economy under a forward guidance shock than under an unanticipated monetary policy shock. This result is similar to Milani and Treadwell (2012).

The impulse responses show that adaptive learning agents fail to understand the precise effect an announcement to lower the future interest rate will have on the economy. Adaptive learning agents know the forward guidance announcement announced by the central bank. However, since they do not understand the precise effect this shock will have on the economy, adaptive learning agents are continually readjusting their forecasts each period causing the impulse responses to exhibit more persistence than under rational expectations. In addition, when the forward guidance shock has been realized upon the economy, there exists a greater substitution effect under adaptive learning than rational expectations. Adaptive learning agents substitute into more consumption than rational expectations agents. The former agents overshoot their rational expectations counterparts. This conclusion occurs because rational expectations agents precisely know how the anticipated changes in monetary policy will affect the endogenous variables at later dates. However, adaptive learning agents imprecisely understand how a commitment to lower the future interest rate will have on the economy since they do not know the true model of the economy.

After Shock Realized–The impulse response graphs of rational expectations and adaptive learning do not follow the same path after the shock is realized upon the economy. The impulse responses with rational expectations agents converge quicker to zero percentage deviation from the unshocked series. Rational expectations agents understand that the shock will not occur in the future and they quickly adjust their expectations. However, the impulse responses under adaptive learning exhibit more persistence than the impulse responses under rational expectations. This outcome is present because the dynamics of the impulse responses under adaptive learning are driven by adjustments in the beliefs of the agents. Adaptive learning agents revise their estimates of the parameters of the economy each period, while rational expectations agents fully understand the model's parameters. The impulse responses of a conventional monetary policy shock shown in the first column of Figure 1 also display the same difference in persistence.

The results coincide with the literature on adaptive learning. The outcomes match Eggertsson (2008) who found that temporary policy shifts do not have as large of an effect on the economy as permanent policy shifts under the assumption of rational expectations. The persistence results also coincide with Milani (2007a) who found that a DSGE model with constant-gain learning generates persistence in the macroeconomic variables.

To summarize, the message from this section is that adaptive learning agents fail to understand the precise effect a forward guidance announcement has on the economy. When the forward guidance shock is known to agents, the output gap and inflation rate under adaptive learning proceed in a different path than under rational expectations. After the shock has been realized, rational expectations agents quickly adjust their expectations to the knowledge that the shock is gone, while adaptive learning agents' beliefs are more persistent. These results are attributed to rational expectations agents precisely understanding the effects forward guidance has on the economy, while the beliefs of adaptive learning agents slowly adjust.

4.2.2 Policy Exercise

The results displayed by the impulse response functions showed that adaptive learning agents failed to understand the precise effects forward guidance has on the economy. This current section shows this conclusion through a different scenario. Specifically, the central bank would like to keep the interest rate fixed at a certain level $\bar{i}$ for $L + 1$ periods. The experiment is described next and is motivated by the policy exercise described in Del Negro et al. (2012).

Suppose at the beginning of period $T + 1$, the central bank implements forward guidance such that the interest rate will be fixed at $\bar{i} = 0$ in period $T + 1$ and L periods into the future. This

announcement corresponds to an unanticipated shock in period $T+1$ and news about the future interest rate $1, 2, \ldots, L$ periods into the future. In this scenario, the monetary policymaker's job is to choose ε_{T+1}^{MP} and $\eta_{T+1} = [\varepsilon_{1,T+1}^{R}, \ldots, \varepsilon_{L,T+1}^{R}]'$ such that the interest rate in periods $T+1$ to $T+L+1$ equals $\bar{i}$. The central bank also believes that agents hold rational expectations, which is a common assumption in macroeconomic literature. To show that adaptive learning agents respond differently to the same forward guidance information, the adaptive learning agents are given the same guidance on the interest rate as under rational expectations. Furthermore, the exercise is assumed to start in period $T+1$.[22] The model is then simulated from $T+1$ to the end of the forward guidance horizon $(T+L+1)$. The process is then repeated $5,000$ times and the mean across the $5,000$ simulations is calculated.

This policy exercise also assumes that the central bank is committed to its goal of $\bar{i}$ every period during the forward guidance horizon. Rational expectations agents precisely understand the central bank's guidance, and thus, the interest rate each period implied by rational expectations equals $\bar{i}$. Since the adaptive learning process is different than rational expectations, the same forward guidance will not give a model implied $\bar{i}$ during the forward guidance horizon. To model a commitment to $\bar{i} = 0$, the central bank chooses ε_t^{MP} each period over the forward guidance horizon to ensure the interest rate equals $\bar{i}$.[23]

Figure 3 compares the dynamics under rational expectations and adaptive learning for the output gap and inflation. The values of the output gap and inflation during the forward guidance horizon are averaged across simulations. The solid line represents rational expectations while the dashed line line displays the adaptive learning path. Under both expectations assumptions, forward guidance has an obvious stimulative effect on impact. Since interest rates are lowered, the output gap and inflation increase. As time elapses and the forward guidance horizon draws to an end, the stimulative effects of this central bank policy fade away. In addition, since adaptive learning agents' beliefs are distributed around its rational expectations counterparts, the initial effect is approximately the same under both series. However, the effect under adaptive learning could vary from rational expectations depending on adaptive learning agents' beliefs used at time $T+1$ to forecast future variables.

Figure 3 also shows that adaptive learning agents fail to understand how the same forward guidance commitments made under rational expectations will impact the economy under learning. This results in larger variation in both the output gap and inflation and a slower speed back

[22] T is chosen to be a large number so that the adaptive learning coefficients converge to its stationary distribution.

[23] This adjustment seems fair since agents' expectations in real life about the future interest rate might not respond as exactly as the central bank would want, and thus, the interest rate might not equal a model implied $\bar{i}$.

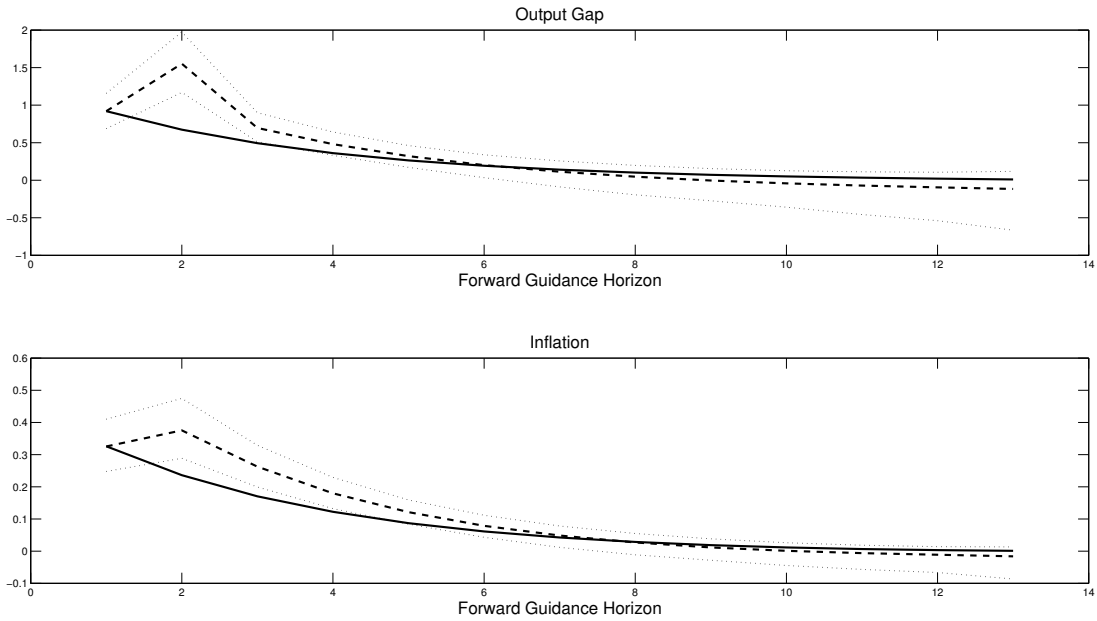

Figure 3: Dynamics of the Output Gap and Inflation in Response to Forward Guidance. Solid Line: Rational Expectations; Dashed Line: CGL; Dotted Lines: 95% Confidence Bands.

to long-run equilibrium under adaptive learning than under rational expectations. The adaptive learning agents observe the unanticipated lowering of the interest rate in period $T + 1$. In the next period, they adjust their expectations of the output gap and inflation upwards due to this previous information. The adaptive learning path then continues at a downward path quicker than under rational expectations. Furthermore, the effect from central bank forward guidance results in more pessimism under adaptive learning than under rational expectations at longer horizons. By having only partial information about the true model of the economy, adaptive learning agents fail to foresee the precise positive impact the forward guidance information has on the dynamics of the output gap and inflation. Thus, this aspect leads to a period of undershooting on the part of adaptive learning agents. The output gap and inflation under adaptive learning fall short of the paths of rational expectations at longer horizons. Rational expectations agents, however, precisely understand the effects of forward guidance on the output gap and inflation. They understand the stimulative effect forward guidance has on the economy, and thus, the output gap and inflation is higher than under adaptive learning over this latter period.

The source of this difference between the two paths regards the assumptions made under rational expectations and adaptive learning. The rational expectations agents know the true model and aggregate probabilities. They can infer the precise effect forward guidance has on the economy. However, the expectations of adaptive learning agents do not respond in the same way. Adaptive

learning agents do not know the true model of the economy, and thus, cannot infer the proper aggregate probabilities and expectations. Even though they know the changes in the future path of interest rates implemented by the central bank, adaptive learning agents imprecisely understand how that guidance impacts the economy. Since they readjust their forecasts each period, adaptive learning agents overshoot and undershoot the paths implied by rational expectations. Moreover, this result shows a consequence of the decision of monetary policymakers. If the central bank assumes agents have rational expectations, which is standard in the macroeconomic literature, the predicted outcomes seem to be misleading. The more realistic assumption of adaptive learning displays a different path than what would occur under rational expectations.

4.3 Economic Crisis

In response to the 2007-2009 Great Recession, forward guidance was implemented by central banks around the world. With that event in mind, this section builds upon the previous subsection's exercise by considering forward guidance during an economic recession. The economy is assumed to start after T periods have passed, that is, after a period of economic stability (corresponding to say the period before the recent Great Recession).[24] The model is then simulated from $T+1$ to the end of the forward guidance horizon $(T+L+1)$. As in the previous subsection, the central bank implements forward guidance by choosing the unanticipated monetary policy and anticipated forward guidance shocks such that the nominal interest rate equals zero from periods $T+1$ to $T+L+1$. To capture features from the recent Great Recession, a large negative demand shock impacts the economy in period $T+1$, and causes a recession. A sequence of five more negative demand shocks follows so that the recession lasts six periods.[25] In the following periods, the shocks are drawn from a normal distribution. Thus, the forward guidance horizon spans a recession and normal times. The process is then repeated $5,000$ times and the mean across the $5,000$ simulations is calculated.

Figure 4 displays the macroeconomic effects of forward guidance during an economic recession. The graph shows the value of the output gap under adaptive learning minus the value of the output gap under rational expectations.[26] Under both expectations formation assumptions, the negative demand shocks cause the output gap to drop below its steady state value. However, the positive effects of forward guidance are overstated under the assumption of rational expectations relative

[24] This strategy also ensures the adaptive learning coefficients converge to its stationary distribution.

[25] This length is based on the duration of the Great Recession as defined by the National Bureau of Economic Research.

[26] A positive value on Figure 4 indicates the output gap under adaptive learning is higher than under rational expectations. A negative value implies the output gap under rational expectations is higher than under adaptive learning.

to adaptive learning. Throughout the forward guidance horizon, the value of the output gap under rational expectations is higher than under adaptive learning. The former agents know the economy is in a recession and precisely understand how forward guidance will alleviate the economy as their expectations are based on the true model of the economy. However, adaptive learning agents observe the economic downturn, but fail to completely understand the positive effects of forward guidance. They must *estimate* the effects of forward guidance on the economy as their forecasts are based on an econometric model. Thus, adaptive learning agents are slower to understand how forward guidance will alleviate the downturn in the economy.

Additional intuition for the results of this section is displayed in Figure 5, which shows the values of adaptive learning minus rational expectations of the discounted long-run expectations of the output gap, inflation and the interest rate across the forward guidance horizon. The adaptive learning agents are more pessimistic about the future output gap and inflation as their long-run expectations are lower than under rational expectations. The former are overestimating the ramifications of the downturn in the economy and their estimates of the effects of forward guidance are not strong enough to overcome this negative reaction. However, rational expectations agents understand the effects of the economic downturn and how forward guidance will precisely alleviate the economy.

The results in Figure 5 also relate to the empirical findings of Del Negro et al. (2012). Their model, which was solved under the assumption of rational expectations, produced an exceedingly large reaction of the macroeconomic variables to forward guidance statements. Del Negro et al. (2012) argued that the source of the excessive responses was an unusually large drop of the long-run interest rate to forward guidance statements relative to the data. In this current paper, the bottom panel of Figure 5 shows a comparable result: the long-run expectation of the interest rate is lower under rational expectations than adaptive learning. This produces larger responses of long-run expectations of the output gap and inflation, and consequently, the current output gap and inflation under rational expectations than adaptive learning. Thus, this result suggests two additional takeaways. A forward guidance model better matches the data under the assumption of adaptive learning than rational expectations. In addition, unusually large responses of the macroeconomic variables to forward guidance found in Del Negro et al. (2012) could be due to the way in which expectations are modeled.

Overall, the results suggest a main finding for policymakers. If monetary policy is based on a model with rational expectations, which is the standard assumption in the macroeconomic literature, the results may be misleading. This section shows that the assumption of rational expec-

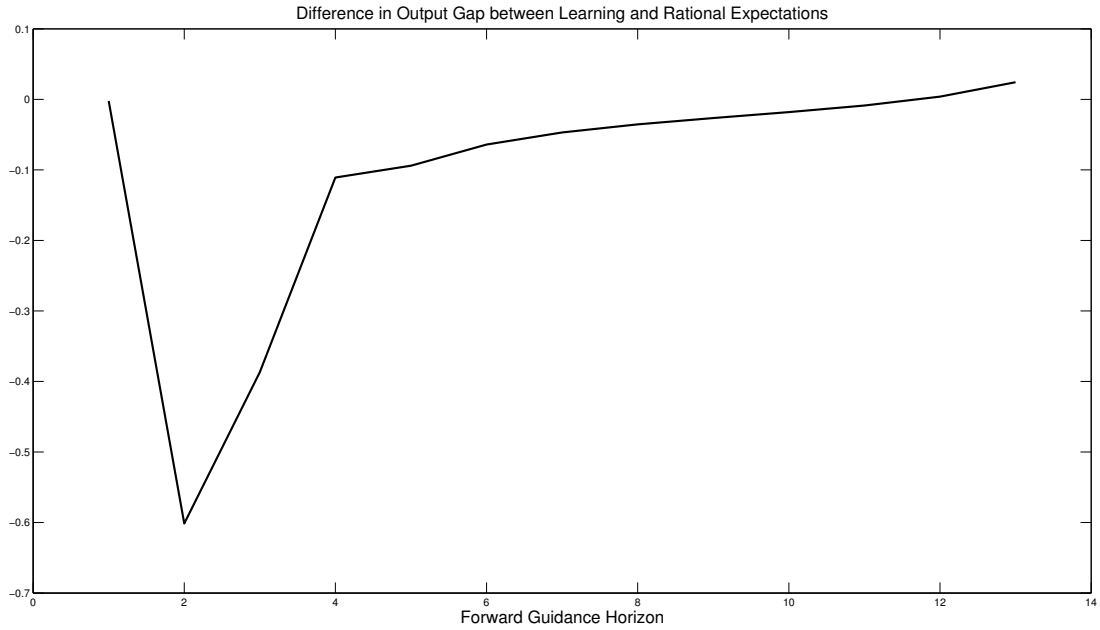

Figure 4: Macroeconomic Effects of Forward Guidance during an Economic Crisis

tations overstates the effects of forward guidance relative to adaptive learning during an economic recession. The adaptive learning results also match the data better than rational expectations.

5 Extensions

5.1 Alternative Parameterization

The results of this paper are investigated under different values of σ, the intertemporal elasticity of substitution parameter. σ measures the effect current and future real interest rates have on current consumption and output. This parameter is important since forward guidance involves statements about future nominal interest rates, and consequently, the real interest rate. Furthermore, this paper investigates the outcomes of the model when $\sigma = 0.15$, $\sigma = 1$, and $\sigma = 1.5$.[27] These results are displayed via adaptive learning impulse responses of the output gap and inflation to negative one unit forward guidance shocks similar to Section 4.2.1.

The results displayed in Figure 6 show that higher values of σ produce greater forward guidance effects than lower values of σ. As σ increases, the output gap responds more to current and future real interest rates. Thus, since forward guidance involves information about future nominal interest rates, demand responds more to news that the interest rate will decrease in the future. As σ decreases, the output gap does not respond as much to changes in current and expected

[27] $\sigma = 1$ is the baseline case used in this paper. For illustrative purposes, this paper chooses the two other values of σ to be 0.15 and 1.5.

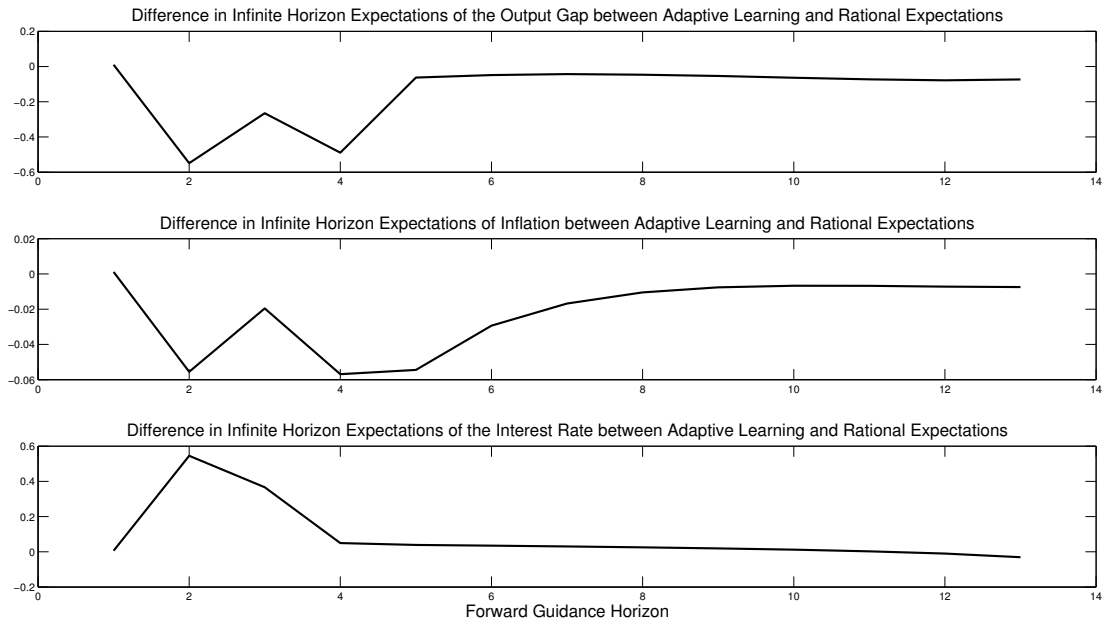

Figure 5: Difference between Adaptive Learning and Rational Expectations Infinite Horizon Expectations of the Macroeconomic Variables. A positive value indicates the value under adaptive learning is higher than under rational expectations. A negative value indicates the variable's value under adaptive learning is lower than under rational expectations.

future interest rates. Therefore, the impact of policy shocks on the economy is less pronounced. Overall, the impulse responses of the output gap and inflation are not as responsive to forward guidance news in comparison to results under a higher value of σ.

5.2 Alternative Constant Gains

In this section, a robustness exercise is simulated to examine the effects of forward guidance policy when adaptive learning agents vary the degree in which they discount previous observations. Specifically, higher and lower values of the gain parameter, $\bar{\tau}$, are used. In addition to $\bar{\tau} = 0.02$, the other constant gains assumed are $\bar{\tau} = 0.01$ and $\bar{\tau} = 0.05$.

The results in Figure 7 show that the responses of the macroeconomic variables to forward guidance under adaptive learning depend on the value of $\bar{\tau}$. From the time of the forward guidance announcement to when the shock is realized, agents with higher constant gains seem to misvalue more the effects of forward guidance than agents with lower constant gains. Under higher values of $\bar{\tau}$, agents place more weight on new information, and thus, exhibit a stronger reaction to forward guidance news. Each period's estimates and beliefs should vary more from the previous period's estimates. However, under lower values of $\bar{\tau}$, agents do not misvalue the effects of forward guidance

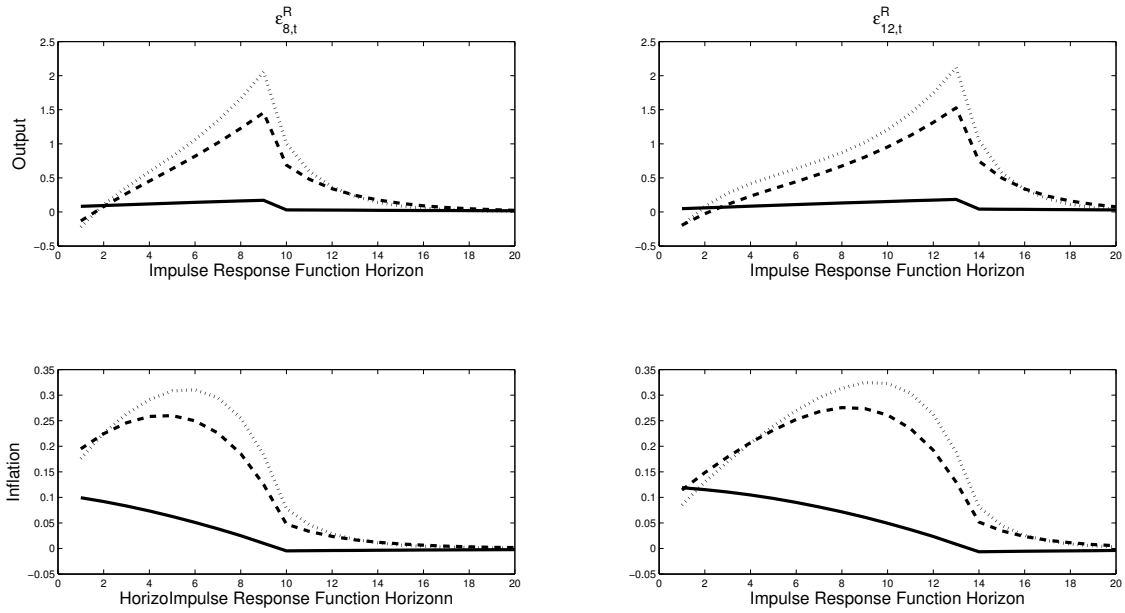

Figure 6: Impulse Response Functions to Forward Guidance Shocks Under Different Values of σ. Solid Line: $\sigma = 0.15$; Dashed Line: $\sigma = 1$; Dotted Line: $\sigma = 1.5$

as much as under higher values of $\bar{\tau}$. They do not exhibit as strong of a reaction to forward guidance news as agents with a higher value of $\bar{\tau}$. Moreover, after the shock is realized on the economy, agents with a higher $\bar{\tau}$ are quicker to realize the shock is not present as they weight previous observations more than agents with a lower $\bar{\tau}$. Thus, the impulse responses under higher values of $\bar{\tau}$ are quicker to return to zero percentage deviation from the unshocked series than under lower values of $\bar{\tau}$.

6 Conclusion

In order to combat the effects of the 2007-2009 global financial crisis, central banks around the world have instituted forward guidance. Because the effectiveness of forward guidance hinges on how expectations respond to forward guidance, it is of interest to investigate the link between expectation assumptions and forward guidance. The standard way to model expectations in the macroeconomic literature is the rational expectations hypothesis. However, if agents form expectations using a more plausible theory of expectations formation (e.g. adaptive learning), the forward guidance results are different.

This paper presents an infinite horizon New Keynesian model with forward guidance and compares the results under two types of expectation assumptions. Under the assumption of rational expectations, Evans and Honkapohja (2001) state agents form expectations based on the true model of the economy. However, adaptive learning agents do not know this information, and instead, act

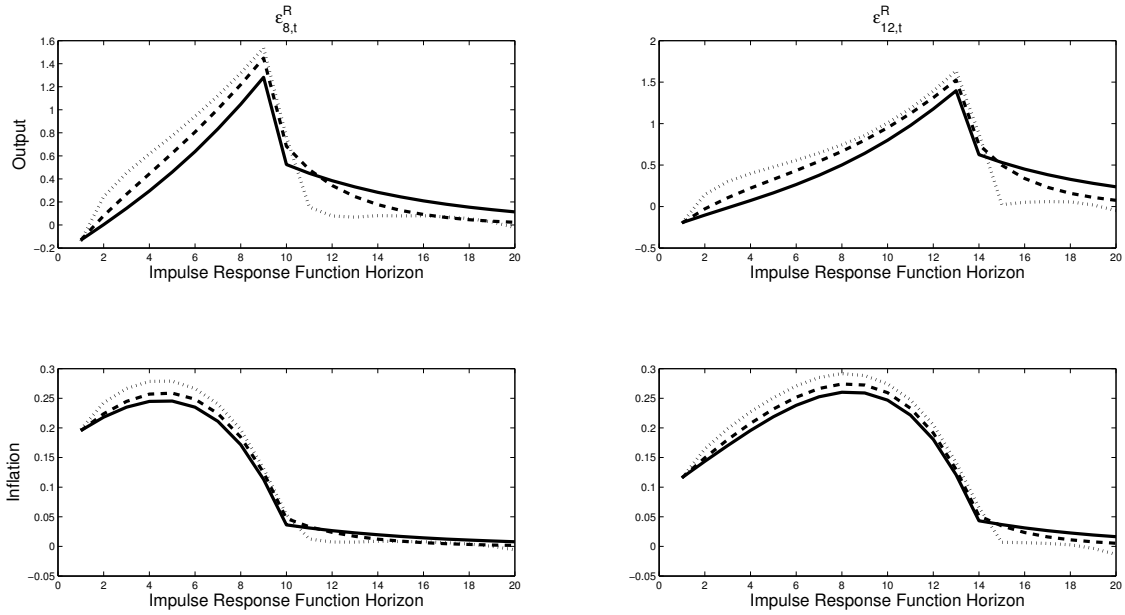

Figure 7: Impulse Response Functions to Forward Guidance Shocks under Different Values of $\bar{\tau}$. Solid Line: CGL with $\bar{\tau} = 0.01$; Dashed Line: CGL with $\bar{\tau} = 0.02$; Dotted Line: CGL with $\bar{\tau} = 0.05$.

as real-life economists and construct their expectations using standard econometric techniques. The results of this paper show that the desired effect of forward guidance depends on the manner in which agents form their expectations. When the central bank gives the same forward guidance information such that the interest rate equals zero for an extended period of time to both types of agents, the adaptive learning paths of the output gap and inflation overshoot and undershoot the ones of rational expectations. In addition, the impulse responses of the macroeconomic variables show that adaptive learning agents miss the precise response to forward guidance shocks. The impulse responses under adaptive learning display a more persistent effect than its rational expectations counterparts. During a period of economic crisis (e.g. a recession), the effects of forward guidance under rational expectations are overstated relative to adaptive learning. Specifically, the output gap is larger under the assumption of rational expectations than adaptive learning. These results occur because rational expectations agents precisely understand the effects forward guidance has on the economy as they form their expectations from the true model of the economy. However, adaptive learning agents must *estimate* the effects of forward guidance on the economy as they do not know the true model of the economy. Furthermore, these latter results have implications for policymakers. If the effects of forward guidance are based on a model with rational expectations, which is the standard assumption in the macroeconomic literature, the results may be misleading

relative to a more plausible theory of expectations formation (e.g. adaptive learning).

There are other modifications to the model presented in this paper that are worth noting. For instance, this paper allows agents to know the end date of forward guidance. Another type of forward guidance policy allows the central bank to link the expiration date of forward guidance to economic conditions. For instance, the unemployment rate is a criterion that the Federal Reserve has used to link to its forward guidance policy. The RLS formula also could be altered to include a gain parameter that changes based on recent forecast errors as discussed in Milani (2007b) and Marcet and Nicolini (2003). This formation of the gain parameter allows agents to better track structural breaks in the economy. In addition, agents can be assumed to have heterogeneous expectations as in Branch and McGough (2009). Branch (2004) uses survey data and shows evidence that respondents have heterogeneous expectations. Overall, the role of expectations formation is especially crucial to understand the effects of forward guidance.

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Appendix

A Model

This section describes the derivation of equations (1) and (3), which follows Preston (2005).

A.1 Households

There exists a continuum of households indexed by i. Households maximize expected future discounted utility

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[U(C_T^i; \xi_T) - \int_0^1 v(h_T^i(j); \xi_T) dj \right] \tag{A.1}$$

where β is the discount factor. Utility depends on C_T^i, which is consumption by household i of goods in the economy. Households also receive a disutility when supplying labor, $h_T^i(j)$, for the production of each good j. ξ_T denotes an aggregate shock. $\hat{E}_t^i$ denotes (potentially) non-rational expectations that satisfy standard probability laws, such as $\hat{E}_t^i \hat{E}_{t+1}^i = \hat{E}_t^i$. Beliefs are assumed to be homogeneous across agents, but agents do not know this fact.

A household is subject to a budget constraint that takes the following form

$$M_t^i + B_t^i \leq (1 + i_{t-1}^m) M_{t-1}^i + (1 + i_{t-1}) B_{t-1}^i + P_t Y_t^i - T_t - P_t C_t^i \tag{A.2}$$

where T_t denotes lump-sum taxes and transfer, M_t^i is money holdings, and i_t^m denotes interest paid on money balances. Asset markets are assumed to be incomplete such that household's can transfer wealth between periods through a one-period riskless bond B_t^i. Accordingly, i_t is the interest paid on bonds. Y_t^i is household i's real income. P_t is the aggregate price index, and $P_t Y_t^i$ denotes household i's nominal income which is given by

$$P_t Y_t^i = \int_0^1 [w_t(j) h_t^i(j) + \Pi_t(j)] dj \tag{A.3}$$

A household receives wages $w_t(j)$ for hours worked towards the production of good j, $h_t^i(j)$. Since each household owns an equal part of each firm, it receives profits from the sale of good j ($\Pi_t(j)$). Note that money does not show up in the utility function, even though it shows up in the budget constraint. It is assumed that money does not relieve any transactional frictions. However, as shown in the budget constraint, a household may choose to hold money because it can act as a financial asset that could provide a return.

The aggregate variables C_t^i and P_t are assumed to be defined by the Dixit-Stiglitz constant-elasticity-of-substitution aggregator

$$C_t^i \equiv \left[\int_0^1 c_t^i(j)^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \tag{A.4}$$

$$P_t \equiv \left[\int_0^1 p_t(j)^{1-\theta} dj\right]^{\frac{1}{1-\theta}} \tag{A.5}$$

where $\theta > 1$ is the elasticity of substitutions across differentiated goods, $c_t^i(j)$ describes household i's consumption of good j, and $p_t(j)$ is the price of good j.

By log-linearizing the intertemporal budget constraint and Euler equation, the following results are obtained

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{C}_T^i = \bar{w}_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{Y}_T^i \tag{A.6}$$

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma(\hat{i}_t - \hat{E}_t^i \hat{\pi}_{t+1}) + g_t - \hat{E}_t^i g_{t+1} \tag{A.7}$$

where $\sigma \equiv \frac{-U_c}{U_{cc}\bar{C}}$ defines the intertemporal elasticity of substitution, $g_t \equiv \sigma \frac{U_{c\xi}\xi_t}{U_c}$ denotes a preference shock, and $\bar{w}_t^i \equiv \frac{W_t^i}{P_t \bar{Y}}$ is share of real wealth ($W_t^i \equiv (1 + i_{t-1})B_{t-1}^i$) as a fraction of steady-state income. The " ˆ " symbol over variables denotes log deviations from steady state. By solving (A.7) backwards from date T to t, taking expectations at time t, plugging the result into (A.6), and integrating over i, the following equation for aggregate consumption emerges

$$\hat{C}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\hat{Y}_T - \beta\sigma(\hat{i}_T - \hat{\pi}_{T+1}) + \beta(g_T - g_{T+1})\right] \tag{A.8}$$

Note that $\int_i \bar{w}_t^i di = 0$ since bonds are in zero net supply from market clearing. $\hat{E}_t = \int_i \hat{E}_t^i di$ denotes the average expectations operator. By imposing the market equilibrium condition $\hat{Y}_t = \hat{C}_t$ and defining the resulting equation in terms of the output gap $\hat{x}_t \equiv \hat{Y}_t - \hat{Y}_t^n$, the following equation emerges

$$\hat{x}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} [(1-\beta)\hat{x}_{T+1} - \sigma(\hat{i}_T - \hat{\pi}_{T+1}) + r_T^n] \tag{A.9}$$

where $\hat{Y}_t^n$ is the natural rate of output, that is, output prevailing under flexible prices, and $r_t^n \equiv (\hat{Y}_{t+1}^n - g_{t+1}) - (\hat{Y}_t^n - g_t)$.

A.2 Firms

Firms in the model operate in a monopolistically competitive environment and are subject to a Calvo (1983) pricing scheme where a fraction $(1-\alpha)$ of firms are allowed to readjust their price

every period. Households supply labor to firms for use in production. A good is produced following the production function $y_t(i) = A_t f(h_t(i))$ where A_t is a technology shock. The demand curve for good i is given by $y_t(i) = Y_t(p_t(i)/P_t)^{-\theta}$. The following Dixit-Stiglitz aggregate price index is assumed

$$P_t = \left[\alpha P_{t-1}^{1-\theta} + (1 - \alpha p_t^{*1-\theta}\right]^{\frac{1}{1-\theta}} \tag{A.10}$$

A firm maximizes its expected present discounted value of profits

$$\hat{E}_t^i \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T}[\Pi_T^i(p_t(i))] \tag{A.11}$$

where $Q_{t,T}$ describes the stochastic discount factor showing how firms value its future stream of income. The stochastic discount factor is given by

$$Q_{t,T} = \beta^{T-t} \frac{P_t}{P_T} \frac{U_c(Y_T, \xi_T)}{U_c(Y_t, \xi_t)} \tag{A.12}$$

The profit function is defined by

$$\Pi_T^i(p_t(i)) = Y_t P_t^{\theta} p_t(i)^{1-\theta} - w_t(i) f^{-1}(Y_t P_t^{\theta} p_t(i)^{-\theta}/A_t) \tag{A.13}$$

Maximizing (A.11) with respect to $p_t(i)$ yields the following first order condition

$$\hat{E}_t^i \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} Y_T P_T^{\theta} [\hat{p}_t^*(i) - \mu P_T s_{t,T}(i)] = 0 \tag{A.14}$$

where $\mu = \frac{\theta}{\theta-1}$, and $s_{t,T}$ is the firm's real marginal cost function. Furthermore, by substituting in the stochastic discount factor and real marginal costs into the firm's first order condition and then log linearizing around a zero inflation steady state, the following result is produced

$$\hat{p}_t^*(i) = \hat{E}_t^i \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{1-\alpha\beta}{1+\omega\theta}(\omega + \sigma^{-1})\hat{x}_T + \alpha\beta\hat{\pi}_{T+1}\right] \tag{A.15}$$

where ω defines the elasticity of a firm's real marginal cost function with respect to its output. Note also that log linearizing (A.10) yields

$$\hat{\pi}_t = \hat{p}_t^*(1-\alpha)/\alpha \tag{A.16}$$

Integrating over i and plugging (A.16) into (A.15) yields the following equation for inflation

$$\hat{\pi}_t = \kappa\hat{x}_t + \hat{E}_t \sum_{T=t}^{\infty})(\alpha\beta)^{T-t}[\kappa\alpha\beta\hat{x}_{T+1} + (1-\alpha)\beta\hat{\pi}_{T+1} + \mu_T] \tag{A.17}$$

where $\kappa = \frac{1-\alpha}{\alpha}\frac{1-\alpha\beta}{1+\omega\theta}(\omega + \sigma^{-1})$.[28]

[28] As in Preston (2006), a supply shock μ_t is added.

B Rational Expectations Solution

By following Sims (2002), the model consisting of equations (2), (4), (5), (6) – (9), (14), and (15) can be solved to yield the solution

$$\widetilde{Y}_t = \widetilde{C} + \xi_1 \widetilde{Y}_t + \xi_2 \epsilon_t \tag{B.1}$$

where

$$\widetilde{C} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]' \tag{B.2}$$

$$\xi_1 = \begin{bmatrix}
0 & 0 & 0 & 0.57 & -0.35 & -0.77 & -0.55 & -0.35 & -0.18 & -0.03 & 0.08 & 0.16 & 0.21 & 0.23 & 0.24 & 0.23 & 0.21 & 0 & 0 \\
0 & 0 & 0 & 0.11 & 0.21 & -0.08 & -0.13 & -0.16 & -0.18 & -0.18 & -0.17 & -0.16 & -0.13 & -0.11 & -0.08 & -0.06 & -0.04 & 0 & 0 \\
0 & 0 & 0 & 0.27 & 0.32 & 0.77 & -0.30 & -0.33 & -0.34 & -0.32 & -0.29 & -0.25 & -0.21 & -0.16 & -0.11 & -0.07 & -0.03 & 0 & 0 \\
0 & 0 & 0 & 0.50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.29 & -0.07 & 0 & -0.77 & -0.55 & -0.35 & -0.18 & -0.03 & 0.08 & 0.16 & 0.21 & 0.23 & 0.24 & 0.23 & 0 & 0 \\
0 & 0 & 0 & 0.06 & 0.04 & 0 & -0.08 & -0.13 & -0.16 & -0.18 & -0.18 & -0.17 & -0.16 & -0.13 & -0.11 & -0.08 & -0.06 & 0 & 0
\end{bmatrix} \tag{B.3}$$

$$\xi_2 = \begin{bmatrix}
1.15 & -1.78 & -0.77 & -0.55 & -0.35 & -0.18 & -0.03 & 0.08 & 0.16 & 0.21 & 0.23 & 0.24 & 0.23 & 0.21 & 0.18 \\
0.23 & 1.03 & -0.08 & -0.13 & -0.16 & -0.18 & -0.18 & -0.17 & -0.16 & -0.13 & -0.12 & -0.08 & -0.06 & -0.04 & -0.02 \\
0.54 & 1.59 & 0.77 & -0.30 & -0.33 & -0.34 & -0.32 & -0.29 & -0.25 & -0.21 & -0.16 & -0.11 & -0.07 & -0.04 & -0.01 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0.57 & -0.35 & 0 & -0.77 & -0.54 & -0.35 & -0.1 & -0.03 & 0.08 & 0.16 & 0.21 & 0.23 & 0.24 & 0.23 & 0.21 \\
0.11 & 0.21 & 0 & -0.08 & -0.13 & -0.16 & -0.18 & -0.18 & -0.17 & -0.16 & -0.13 & -0.12 & -0.08 & -0.06 & -0.04
\end{bmatrix} \tag{B.4}$$