BARGAINING IN THE SHADOW OF WAR:
WHEN IS A PEACEFUL RESOLUTION MOST LIKELY

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Abstract.

We construct game theoretic foundations for bargaining in the shadow of a war. Aggressor and defender both have noisy signals of the outcome and make simultaneous offers to settle. If the offers cross, they settle on the average offer; otherwise, both antagonists incur the additional cost of war. We obtain an essentially unique Nash equilibrium and characterize its conditional war probabilities and outcomes. Some of the results are intuitive. E.g., an increase in the cost of war (or a decrease in the range of possible outcomes) reduces the probability of a war. Other results defy ordinary intuition. For example, wars are possible even when both sides are jointly pessimistic, and when the cost of war is low, evenly balanced rather than unbalanced powers are more likely to settle. We show that this last result can be extended to a more general setting.
1. Introduction.

Early work on bargaining in the shadow of a war assumed that the participants had differential expectations. However, this work (1) did not explicitly model how each side would take into account that the other side had a different but equally valid estimate of the war outcome; (2) assumed that a settlement would take place if and only if the aggressor expectations were below the defender’s expectations; and (3) did not consider strategic behavior by the belligerents where one or both would trade off probability of a settlement for an increased share of the surplus. In this article, we provide a game-theoretic foundation for this intuition. In the process, we derive new results, the most startling being that when war costs are low, the probability of a war is highest when the probability of the aggressor winning is either very high or very low.

Most recent research on war employs only one-sided incomplete information (about costs, ability, or some other characteristic) and one-sided offers (see for example, Powell, 1999, and Slantchev, 2003), which seems highly unrealistic. Aggressors and defenders have access to different information and therefore have different expectations about the outcome of a war. While spying may reduce some of this differential in information, much information cannot or will not be credibly conveyed to the other side. Each side has reason to lie. Here, we model a symmetric situation where there is two-sided incomplete information. For example, the aggressor may know
more about its offensive capabilities than about the defender’s defensive capabilities (and vice versa for the defender). But the outcome of the war depends on both the offensive and defensive strengths.

Another way that this paper differentiates itself from the previous literature is the bargaining protocol. In virtually all of the literature, one side makes an offer and then the other side observes the offer and either accepts it or rejects the offer, in which case there is a war. This creates vastly different outcomes, depending on which side makes the (first) offer. Here, we present the following symmetric bargaining protocol. The aggressor submits its demand and the defender submits its offer to a third party. If the offer is greater than or equal to the demand, then there is a settlement half way between the two; otherwise, the aggressor goes to war.\(^2\) Of course, this is not a realistic characterization of the actual bargaining. But it is a way to get to the essential symmetry in most bargaining situations (when the threat to go to war is credible).

Our bargaining protocol can be seen as an extension of Chatterjee and Samuelson (1983). They consider a buyer and a seller, each having a private value for the good drawn from a uniform distribution. If the demand by the seller is less than the offer by the buyer, then there is a trade at the half-way point; otherwise, the seller keeps the item and the buyer keeps her money. The Chatterjee-Samuelson model plays a central role in understanding bargaining in markets even though few, if any, actual buyer-seller negotiations employ the C-S protocol. One could imagine all kinds of complicated, possibly unsolvable, “realistic” dynamic protocols, where some information may or may not be revealed in earlier rounds. The reason for the influence of the C-S protocol is that it provides a sensible reduced form for these more complicated models at the same time it provides the actual solution to the particular protocol.

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1 These aforementioned issues arise whether or not the belligerents share a common prior.
2 One might ask why the third party does not just announce a settlement at the average of the demands and offers even if there is not an overlap or undertake a more sophisticated analysis and figure out the true observations from the belligerents’ demands and offers and then announce a settlement at the average of their observations. Unfortunately that protocol provides perverse incentives, e.g., the aggressor should demand infinity.
Despite the similarities to Chatterjee and Samuelson, our paper differs in two fundamental ways. As noted earlier, we are concerned with common values: the true value depends on both signals. Second, the truth can be revealed through a costly war.

The Rubinstein offer and counter-offer models also been applied to war. However, these Rubinstein models give greater bargaining power to the side that makes the first offer. Here, we focus on games with symmetric bargaining power (although not necessarily an equal outcome – the defender may just have a choice between giving up 200,000 acres or 400,000 acres of land. Not surprisingly, the Rubinstein bargaining literature has focused on different issues from those considered here: the nature of the two-sided incomplete information is different (e.g., the cost of bargaining to the other party is unknown), and the concerns are different (e.g., the relative inefficiency of a repeated demand and offer game is compared to a one-time-only offer game). See Ausubel, Cramton and Deneckere, 2002, for a recent review and a proof that the Chatterjee-Samuelson one-time offer game is an upper bound on the efficiency of the alternating-offer game.

Our symmetric game has symmetric Nash equilibrium strategies that turn out to be piecewise linear and essentially unique. The simple equilibrium structure yields several useful results that were either not accessible or less intuitive in earlier work. For example, we derive exact expressions for the distribution of war settlements as a function of war costs, and characterize the conflicts that go to war.

Section 2 presents the simple conflict situation outlined above. Section 3 derives precise quantitative results for this model. Section 4 shows that the ordinal (qualitative) properties extend more generally. As long as the demand and offer curves are monotonically increasing and parallel, the results of theorem 4 hold. That is, symmetry, equal costs, a uniform distribution, and the particular bargaining protocol outlined above are not necessary for the results of theorem 4 to hold. Section 5 provides concluding remarks. Appendix A gives formal proofs of the propositions, and Appendix B presents derivations supporting the main arguments in Section 4.
2. The Model

The game has two players, referred to as the aggressor and the defender, and three stages. At the preliminary stage 0, the players have common knowledge about the structure of the game, including the payoff function and the distributions of signals. In particular, both players know that the lowest possible expected outcome of the war is \( L \geq 0 \) and the largest is \( U > L \). \( L \) will be have to be sufficiently large so that regardless of the signals, war is always a credible threat. For convenience, we will initially assume that \( L = 0 \) and \( U = 1 \) (but later we will shift \( L \) and \( K \) upwards). At stage 1, the aggressor privately observes a signal \( \theta_a \) of the expected war outcome drawn according to the cumulative distribution function \( F^A \), and chooses a demand \( a \). Simultaneously, the defender privately observes an independent signal \( \theta_d \) drawn from \( F^D \), and chooses an offer \( d \). At the final stage 2, the payoffs are determined as follows. (a) If \( a \geq p \), the conflict is settled at the average offer \( (a + d)/2 \). (b) If \( d < a \), the demand and offer are inconsistent and the conflict results in war. Each player then incurs cost \( c \geq 0 \), and the expected outcome of the war is \( (\theta_a + \theta_d)/2 \).

We make the following assumptions: the restriction to non-decreasing strategies, and (more substantively) with \( c \geq 0 \) and uniform signal distributions. That is, \( F^D(\theta) = F^A(\theta) = \theta \) and \( dF^D(\theta) = dF^A(\theta) = d\theta \) for all \( \theta \) in \([0, 1]\). The uniform distribution is very convenient because the objective functions take a simple form.

The objective of the aggressor is to maximize the expected return (net of any war costs that might be incurred), conditioned on the aggressor’s realized signal \( \theta_A \) and the defender’s strategy \( D \). The payoff function for the aggressor is

\[
(1) \quad \Pi^A(a, \theta_a, D, F^D) = 0.5 \int_{D^{-1}(a)} [a + D(x)]dx + 0.5 \int_0^{D^{-1}(a)} [\theta_a + x - 2c]dx
\]
Similarly, the objective of the defender is to minimize losses (including any war costs that might be incurred), conditioned on the defender’s realized signal \( \theta_d \) and the aggressor’s strategy \( A \). The loss function for the defender is

\[
\Pi^D(d, \theta_d, A, F^A) = 0.5 \int_0^{A^{-1}(d)} [d + A(y)]dy + 0.5 \int_{A^{-1}(d)}^{1} [\theta_d + y + 2c]dy.
\]

3. Basic Results

The first question one might ask is whether a Nash equilibrium exists. An affirmative answer is not automatic because we have restricted strategy sets that exclude mixed strategies. It turns out, however, that the basic war model has an equilibrium of a very simple form.

**Proposition 1.** The basic war model has a NE in the piecewise linear, continuous bid functions graphed in Figure 1. The functions are:

(3) \( A(\theta_a) = 2\theta_a/3 - 2c + 1/2 \), truncated above at \( \min\{1, 2c + 1/2\} \) and below at \( \max\{0, 2c - 1/6\} \), and

(4) \( D(\theta_d) = 2\theta_d/3 + 2c - 1/6 \), truncated above at \( \min\{1, 7/6 - 2c\} \) and below at \( \max\{0, -2c + 1/2\} \).

All proofs appear in the Appendix. The intuitive reason for \( \theta_a \) and \( \theta_d \) being preceded by a fraction is that each belligerent knows that its signal is only part of the truth and thus only partially responds to the signal. The intuitive reason for the lower truncation in Figure 1A is that the aggressor has no incentive to demand less than the lowest possible offer from the defender. Demanding less would not increase the likelihood of a settlement (it is already 100%), but would reduce what the aggressor gains in a settlement. Likewise, the upper truncation reflects the fact that the defender has no incentive to offer more than the aggressor’s highest demand.
The minimum and maximum values of \( d = D(\theta_d) \) are \( 2c - 1/6 \) and \( 2c + 1/2 \) when \( \theta_d = 0 \) and 1, respectively. So \( a = A(\theta_d) \) is never strictly greater than the largest value of \( D \) nor strictly below the smallest value. Similarly, \( d \) is bounded by the natural range of \( A \).

Figure 1B shows the perhaps more intuitive conflict where, at equal signals, the aggressor demands more than the defender offers. This conflict arises when \( A(x) = 2x/3 - 2c + 1/2 > D(x) = 2x/3 + 2c - 1/6 \). That is, when \( c < 1/6 \), the aggressor’s demand curve is above the defender’s offer curve. When \( c > 1/6 \), the aggressor’s demand curve is below the defender’s offer curve as in Figure 1A. The intuitive explanation for the aggressor demand curve being above the defender offer curve when the cost of war is low and below the defender offer curve when the cost of war is high is that the aggressor wants to receive as much as possible and the defender wants to pay as little as possible. Thus each side wants to extract the surplus for itself. When cost of war is low, the issue of surplus extraction is paramount and thus the aggressor’s demand curve is above the defender’s offer curve. As the cost of going to war increases, each side will be more willing to settle. Although the one side’s greater willingness to settle increases the intransigence of the other side, this effect is less than the direct effect of the increased cost on the other side. So the net effect of an increase in the cost of war is a shift downward of the aggressor’s demand curve and a shift upward of the defender’s offer curve.
Note that \((A + D)/2 = \theta_a/3 + \theta_d/3 + 1/6 > (\theta_a + \theta_d)/2\) if and only if the average of the observations is less than 1/2. That is, the distribution of the average of the demand and offer is less extreme than the distribution of the war outcome were the conflict to go to war. Again, this is because the belligerents temper their offers relative to the signals.

It is useful to compare these demand and offer curves to those generated by other approaches. First consider, the differential expectations approach. Here there is no point estimate of the demand and offers, only upper and lower bounds, so we can only provide minimal demand and maximal offer curves. For the aggressor, the minimal demand would be \(-c + \theta_a\); for the defender the maximal offer would be \(c + \theta_d\). Note that neither belligerent takes into account that that the other side may have a different signal. In the one-sided asymmetric information one-side offer conflict, only one side, say the defender, has a signal. When the aggressor makes the demand, then the defender’s offer curve is \(c + \theta_d\). The aggressor does not have a signal in this conflict, but one could draw a horizontal line to represent the demand by the aggressor.

One might ask whether there are other NE. Of course, there are trivial NE, where all conflicts end in war because both the aggressor and the defender make offers certain to be rejected, e.g., \(A(\theta) = 1\) and \(P(\theta) = 0\) for all signals \(\theta\).\(^3\) There are also variations on strategies (3) and (4) that are inessential in that they induce the same outcomes (i.e., the same mapping from signals to payoffs). To illustrate, suppose that \(c = 1/12\). Then by (4) the maximum defender offer is 2/3, and by (3) the aggressor demand curve is higher and the conflict will go to war with probability 1 whenever \(\theta_a > .5\). Of course, the war outcome depends only on the signal, not the demand. Hence the outcome will always be the same if we replace (3) by any aggressor demand function that coincides with (3) for \(\theta_a \leq .5\) and has arbitrary values in \([2/3, \infty)\) for \(\theta_p > .5\). Thus some of the truncations in Figure 1 are inessential. However, they keep the graphs of the functions within the unit square, which simplifies later calculations.

\(^3\) We will drop the subscripts from \(\theta_a\) and \(\theta_d\) when the meaning is clear.
Our uniqueness result covers equilibria that are symmetric in the sense that corresponds to the symmetry of game. For example, suppose that when the defender observes the signal 1/4 she offers 1/3. Symmetry would imply that when the aggressor observes 3/4 he demands 2/3. With this in mind, we make the following

**Definition.** The strategies \( A \) and \( D \) of the simple war model are symmetric if for all \( \theta \) in \([0,1]\) we have \( A(\theta) = 1 - D(1 - \theta) \) or, equivalently, \( D(\theta) = 1 - A(1 - \theta) \).

It is easy to see that the trivial NE strategies mentioned above are symmetric. So are the piecewise linear strategies (3-4). Our next result is that the equilibrium is essentially unique in its class.

**Proposition 2.** All non-trivial piecewise-linear symmetric Nash equilibria of the basic war model induce the same outcome as strategies (3-4).

In the rest of this section, “equilibrium” refers to the outcome generated by strategies (3-4). We focus on the probability of war and the distribution of conflicts that go to war, because for empirical work war data are much easier to collect than data on settlements. The first comparative statics result shows that the probability of a war decreases nonlinearly in the war cost \( c \).

**Proposition 3.** The equilibrium probability of war is \( 1 - 18c^2 \) for \( 0 \leq c \leq 1/6 \); is \( 2(1 - 3c)^2 \) for \( 1/6 \leq c \leq 1/3 \); and is 0 for \( c \geq 1/3 \).

Inspection of (4-4) and Figure 1 provides much of the intuition. A war is certain if the highest defender offer \((2/3 + 2c - 1/6)\) is below the lowest aggressor demand \((-2c + 1/2)\); that is, if \( c \leq 0 \). A war will never occur if the lowest defender offer \((2c - 1/6)\) exceeds (or equals) the highest aggressor demand \((7/6 - 2c)\); that is, if \( c \geq 1/3 \). For intermediate costs, the conflict goes to war when \( a = (2/3)\theta_a - 2c + 1/2 > (2/3)\theta_d + 2c - 1/6 = d \), i.e., when \( \theta_a - \theta_d > 6c - 1 \). The last inequality is more likely to be satisfied, and thus the conflict is more likely to go to war, when \( c \) is...
smaller. Although the result that an increase cost of war will increase the probability of a settlement is not surprising, the non-linear relationship may be. The change in expressions at $c=1/6$ is associated with the change in Figure 1 from conflict a (with $D$ above $A$) to conflict b (with $A$ above $D$). The Appendix derives the exact expressions.

Now consider the probability of a war conditional on the value of the potential war expected outcome of the war $W = (\theta_a + \theta_d)/2$. Initial intuition suggests that conflicts with extreme expected outcome of the wars are more likely to be settled, and conflicts with expected outcome of the wars near the average of 0.5 are more likely to go to war. Our next result confirms this intuition for high war costs, but reverses it for low costs.

**Proposition 4.** The equilibrium probability of a war increases in $|W - 0.5|$ when $c < 1/6$ and decreases in $|W - 0.5|$ when $c > 1/6.$

That is, the farther away the war outcome will be from the median war outcome, the more likely that there will be a war when war costs are low and the less likely there will be a war when war costs are high. The intuition can be extracted from Figure 2. Recall from the discussion of Proposition 3 that a war will take place if and only if $\theta_a - \theta_d > 6c - 1$, i.e. if and only if the signal combination $(\theta_a, \theta_d)$ lies Northwest of the lines of slope +1 labeled $6c - 1 = K$. Consider first the conflict $K > 0$, i.e., $c > 1/6$; say $K = 0.5$ as in Figure 2. The war region now is the triangle to the northwest of the dotted line. Lines of given value $W = (\theta_a + \theta_d)/2$ have slope -1. For $W$ near zero (and for $W$ near 1) the negatively sloped $W$ line does not intersect the war triangle. Hence the war probability conditioned on such $W$ is zero. For a value of $W$ closer to 0.5, such as that shown in Figure 2, the $W$ line does intersect the war triangle. Since signals are independent and uniform, their joint distribution is uniform over the unit square. Hence the desired conditional war probability is the length the $W$ line segment inside the triangle as a fraction of its length inside the unit square. This is clearly maximized when the $W$ line meets the corner of the triangle, i.e., when $W = 0.5$. 
To see the less intuitive conflict, suppose $c < 1/6$ so $K < 0$ as in the unlabeled upward sloping line below the diagonal in Figure 2. The settlement region now is the triangle southeast of this line, while the war region is the rest of the unit square, northwest of the line. In this conflict, lines with extreme values of $W$ do not intersect the settlement region, and the war probability is 1. As $W$ moves towards 0.5, a larger fraction of the $W$ segment lies in the settlement region and the war probability decreases.

The geometry comes from the fact that when $c$ is small, the aggressor’s demand curve in Figure 1b is above the defender’s offer curve. When the potential expected outcome of the war $W$ is very high (or very low), the belligerents’ signals must be fairly similar because $W$ is the average of the signals. When the belligerents’ signals are similar and $c$ is small, the aggressor’s demand $a$ will tend to be above the defender’s offer $d$ even if $\theta_a$ is slightly smaller than $\theta_d$. As a result, the conflict goes to war. However, when $W$ is close to .5, it is possible that the defender observed a signal close to 1 and the aggressor observed a signal close to 0. In such situations, the aggressor’s demand will be below the defender’s offer even though the aggressor’s demand curve is above the defender’s offer curve. That is, there is more scope for signals that will lead to settlement at intermediate levels of the potential expected outcome of the war. If we interpret $\theta$ as probability of the aggressor winning the war (and assume that the spoils of war are fixed at, say, one trillion
dollars), then this result says that those conflicts where the aggressor has a 50 percent chance of winning are the least likely to go to war.

Of course, when the cost of war is high, then the aggressor’s demand curve is below the defender’s offer curve (as in Figure 1a). As a consequence, small differences that arise when the war outcome is either close to 0 or 1 will not be sufficient to make the aggressor’s demand above the defender’s offer. So a war will not take place. On the other hand, when $W$ is near .5, it is quite possible for $\theta_d$ to be sufficiently larger than $\theta_d$ so that $a$ is greater than $d$, resulting in a war.

Turning our attention to the one-sided asymmetric information one-sided offer model, we get a contrary result. If the defender knows the probability of winning and the aggressor makes the offers, then only when the defender has a high probability of winning will the defender reject the aggressor offer and the conflict go to war. So that model predicts that wars sample from extreme conflicts, where the probability of the defender winning is high (but not when the probability is very low).

As we have just seen, our model predicts that the larger the cost of war, the more that wars are concentrated at the center of potential expected outcome of the wars. Assuming that the belligerents do not know the probability of the aggressor winning, this means that an increase in the cost of a war will result in the probability of the aggressor winning in those wars actually fought will converge to 50%. This is in contrast to the asymmetric model discussed above where higher costs of war lead to higher probabilities of the defender winning. That is, the probability of the aggressor winning wars that are actually fought converges to 0. The defender rejects settlement offers only in those instances where the war outcome is most likely to be in favor of the defender. As the cost of going to war increases, the bar is raised for going to war so that the probability of the defender winning at war increases.
4. Extensions.

Additional insights can be gleaned from relaxing the assumptions of the basic model. First, consider shifting the distribution of war outcomes by adding or subtracting a constant $M$ to the upper and lower endpoints $U$ and $L$ of the range of possible outcomes. In the model, the probability of war depends on the differential in expectations, not the level of these expectations. Hence such shifts in the outcome range have no effect, other things equal. However, thinking carefully about such shifts leads to a more subtle issue concerning the way the war is specified. We shall return to this issue shortly.

Next, consider increasing the width $U - L$ of the war outcome range, holding constant the cost $c \geq 0$, and maintaining the assumption that signals are independent and uniform on $[L, U]$. We also assume for simplicity (a weaker assumption will do) that $L$ does not increase, and refer to such a stretching of possibilities as an outcome spread. We have a sharp comparative statics result.

**Proposition 6.** An outcome spread increases the equilibrium probability of a war.

Here equilibrium still refers to outcomes generated by the Nash equilibrium strategies (3-4), except that the variables are linearly transformed to apply to $[L, U] \neq [0, 1]$. The normalized war cost $c$ has $U - L$ in the denominator, hence the relative cost of a war falls in an outcome spread. In Figure 2, this is represented by a downward (or Southeasterly) shift in the $K$ line, hence an increase in the area of the war region (as a fraction of the area of the entire square). Appendix A shows that the desired result turns out to be a corollary of Proposition 3, which tells us that a (relative) cost decrease indeed increases the war probability.

What happens if we relax the assumption of independent uniformly distributed signals? Signal dependence *per se* does not appear to introduce any interesting new issues. The signal can be decomposed into a common component that shifts the outcome midrange, and idiosyncratic components that are independent. As long as the belligerents are risk neutral, it seems that only the idiosyncratic components matter.
The more important generalizations are to independent signals from a distribution that may not be uniform, to war costs that may not be symmetric, and to bargaining protocols that might not split the difference equally. The ordinal properties of our main propositions survive quite well, and extend the intuition of the basic model. There are three parts to the argument.

First, an increase in own war cost induces more moderate offers and demands. This result is intuitive, but the generality of the setting requires some extra work. Appendix B derives the direct effect $d_c$, the rate at which the defender’s best response offer increases as its cost increases, holding constant the aggressor’s demand function. It also derives the indirect effect $d_s$, the rate at which its offer increases as the aggressor’s demand shifts up. The aggressor has an analogous direct effect $a_c$ for an increase in its cost, and indirect effect $a_s$ for a shift in the defender’s offer function. Appendix B shows that that the direct effects are always towards moderation: $d_c > 0 > a_c$ in the relevant region. The direct effects cause shifts in the best response functions that reverberate via the indirect effects. The total effect of a change in own war cost is the sum $d_c/(1-d_s a_s)$ of a geometric series for the defender. Similarly, $a_c/(1-d_s a_s)$ is the total effect for the aggressor. Thus in the usual conflict $d_s a_s < 1$, the total effect has the same sign as the direct effect.

In the basic war model the effects turn out to be $d_c = a_c = 4/3$ and $d_s = a_s = -1/3$. Hence a unit increase in defender’s cost only will increase defender’s offer by $(4/3)/(8/9) = 3/2$ and reduce the aggressor’s demand by $(-1/3)(3/2) = -1/2$. Similarly, a unit increase in aggressor’s cost only will reduce the aggressor’ demand by $-3/2$ and increase aggressor’s offer by $1/2$. Thus we confirm the equilibrium cost coefficients (for equal shifts in both belligerents’ costs) in equations (5-6) of $3/2 + 1/2 = 2$ and $-1/2 - 3/2 = -2$, and now see that 75% of the impact is due to own cost and 25% to the other player’s cost.

The second part of the argument notes the consequences for extreme signal values. Figure 4 illustrates two key cases. Suppose that in equilibrium $D$ is above $A$, as in panel A. For reasons given earlier, $A$ will never be below the minimum value of $D$; hence $A(\theta_a) = D(L)$ for $\theta_a \leq \theta^{CL}_a$. 
Similarly, $D$ will never be above the maximum value of $A$; hence $D(\theta_d) = A(H)$ for $\theta_d \geq \theta_a^{CH}$.

Panel B shows the opposite case where $A$ is above $D$, and again each belligerent’s function has a flat segment over a range of extreme signal values. The same thing happens if the functions were to intersect in the interior. Indeed, there is always a flat segment at both extremes unless (a) both untruncated functions are strictly increasing and (b) $D(L) = A(L)$ and/or $D(H) = A(H)$. But the first part of the argument shows that ties as in (b) are exceptional and will be broken by slightly increasing or decreasing either belligerent’s war cost. The second issue is that there are bounds in the outcome (probabilities of winning cannot be less than 0 or greater than 1; and gains to the aggressor are limited to the extent of the defender’s wealth. Hence we can safely assume a flat segment in one of the belligerent’s functions at both extremes.

The third and last part of the argument is the same as for Proposition 4, and so is the intuition. When $W$ is either very large or very small, the observations by the belligerents must be similar. If the demand curve is below the offer curve, this means that the conflict will be settled peacefully; if the demand curve is above the offer curve, this means that the conflict lead to war. More specifically, in Case A (where $D$ is above $A$), a settlement is certain for all expected outcome of the wars in the more extreme half of the flat segments ($W < (L + \theta_a^{CL})/2$ and $W > (H + \theta_a^{CH})/2$ in Panel A, where $L = 0$), and in general the conflicts that go to war over-sample the middling expected outcome of the wars.
By the first part of the argument, case A is to be expected when war costs are high. By the same token, when war costs are low, we expect to see case B (where A is above D). Now war conflicts over-sample the more extreme expected outcome of the wars, and the most extreme expected outcome of the wars ($W < (L + \theta_d^{CI})/2$ and $W > (H + \theta_a^{CI})/2$ in Panel B) are certain to go to war. Hence our main qualitative results appear to be quite robust. Note that this result depends on the relative position of the demand and offer curves, which need not represent symmetric bargaining, symmetric costs, or even the specific protocol.

To counter a bias we perceive in much of the literature, we have modeled bargaining as entirely symmetric. We see no reason to suppose informational or bargaining asymmetries based on who makes the offer, but there is a natural strategic asymmetry: the aggressor can always decide not to go to war. The model presented in section 2 does not allow for this asymmetry, but it may now be worth a brief discussion.

The key question is when the aggressor would prefer to not to go through with the war after its demand has been rejected. A numerical example will help fix ideas. Suppose that $\theta_a = 0$, then $a = 1/2 – 2c$ according to equation (3). If the aggressor’s demand is rejected, he can infer $(2/3)\theta_d + 2c – 1/6 < 1/2 – 2c$, i.e., $\theta_d < - 6c + 1$. If the conflict goes to war, he expects to get $.5[0] + .5[1 – 6c] – c = .25 – 2.5c$. Hence $c < .1$ implies that the threat of a war is always credible in the basic war model. For $c > .1$ in more general models, the threat of a war is still credible if $c < L$. Assuming $c < L$ is the standard approach used in the literature to insure that the aggressor’s threat of going to war is always credible.

5. CONCLUDING REMARKS

We obtained a nontrivial Nash equilibrium for the basic war model and showed it was essentially unique in its class (Propositions 1-2). In this equilibrium, a war takes place if and only if $\theta_a - \theta_d > 6c – 1$. It is instructive to compare this result to the traditional divergent expectations story, where a war takes place if and only if the differential in the belligerents’ draws are greater than the total cost of going to war, i.e., a war takes place if and only if $\theta_a - \theta_d > 2c$. In our model
the traditional condition is neither necessary nor sufficient. If \( c < 1/4 \), then \( 6c - 1 < 2c \) and a war is possible in equilibrium even though the condition \( \theta_a - \theta_d > 2c \) is violated. For example, if \( c = 1/6 \), then, inconsistent with the traditional divergent expectations story, there will be a war whenever \( \theta_a \) is between \( \theta_d \) and \( \theta_d + 1/3 \). Similarly, if \( c > 1/4 \) then \( 6c - 1 > 2c \) and there will be equilibrium settlements inconsistent with the traditional story. For example, if \( c = 1/3 \), there will always be a settlement, while the divergent expectations story would predict a war whenever \( \theta_a - \theta_d > 2/3 \). In a nutshell, the traditional divergent expectations explanation is misleading. More intuitively, the basic problem with the simple divergent expectations approach is that it does not take into any explicit account one side’s expectations about the other side’s observations and behavior. In a common values setting, which characterizes most war conflicts where the loss and gain are in dollar amounts, each side must take into account that his/her observations are only part of the picture. The simple divergent expectations approach does not do this. Furthermore, the simple divergent expectations approach does not consider the other side’s strategic behavior – it just says that if the signal for the aggressor minus the signal for the defender is less than the cost of the war, then there will be a settlement. And if this is not the case, then there will be no settlement. Even if the setting were not common values, each side needs to think strategically. Each side rationally is willing to reduce the probability of settlement in order to extract more surplus from the settlement (a point made by Powell)

We derived several comparative static results regarding the probability of a war. We showed that an increase in \( c \) (or a decrease in the range of possible outcomes) reduces the probability of a war (Propositions 3, 6).

We also showed that when the war cost is relatively high, wars are more apt to come from the conflicts with potential expected outcome of the wars near the median (Propositions 4, 5). To compare this result with that of balance of power work, it is again useful to characterize the payoff as being one trillion dollars if the aggressor wins, and to interpret \( \theta_a \) and \( \theta_d \) as the aggressor’s and defender’s estimates of the aggressor’s probability of winning. With this interpretation, our result is that conflicts are less likely to go to war when the aggressor’s probability of winning is either
very low or very high. Hence our result here is consistent with the argument put forth by balance of power theories that wars tend to select from conflicts with a 50% chance of winning.

However, when the war cost is relatively low, wars are more apt to come from the conflicts with potential expected outcome of the wars at the extremes (Propositions 4, 5), contrary to some balance of power theories.

Because information on settlements is often not available, much research is based on war data. As we have shown, the inferences on settlements drawn from war data depend greatly on the cost of wars. One needs the models presented here to get a more accurate understanding of all conflicts – both tried and settled – whenever the data relies only on conflicts that went to war.

More important, there is now a new set of ordinal comparative statics results that can be tested indirectly. It will be particularly interesting to see whether the low-war-cost result can account for observed data, since it contradicts previous intuition and conventional wisdom.

Finally, we believe that the application of the common-values framework to war is under-exploited relative to the one-sided asymmetric model. We hope that this paper will promote further applications in this area.
Proposition 1. The basic war model has a NE in the piecewise linear, continuous bid functions graphed in Figure 1. The functions are:

\( A(\theta_a) = 2\theta_a/3 - 2c + 1/2 \), truncated above at \( \min\{1, 2c + 1/2\} \) and below at \( \max\{0, 2c - 1/6\} \), and

\( D(\theta_d) = 2\theta_d/3 + 2c - 1/6 \), truncated above at \( \min\{1, 7/6 - 2c\} \) and below at \( \max\{0, -2c + 1/2\} \).

Proof.

We need only verify that (A1) is a best response to (A2) and vice versa. Note first that it is impossible for both the demand and offer to equal 1. \( A(\theta_a) = 2\theta_a/3 - 2c + 1/2 \geq 1 \) only if \( c \leq 1/12 \). But if \( c \leq 1/12 \), then \( D(\theta_d) = 2\theta_d/3 + 2c - 1/6 \leq 2/3 \). A similar exercise shows that both the demand and offer cannot equal 0. Thus the 0, 1 truncations are inessential because when the truncations are operative, the conflict would go to war whether there was a truncation or not, and the demands and offers do not affect the war outcome. The truncation of \( A(\theta_a) \) above at \( 2c + 1/2 \) and the truncation of \( D(\theta_d) \) below at \( -2c + 1/2 \) are also inessential for similar reasons.

The defender minimizes its expected payment

\[
\Pi^D(d, \theta_d, A, F^p)_s = 0.5 \int_{0}^{A^{-1}(d)} (d + A(y))dy + 0.5 \int_{A^{-1}(d)}^{1} (\theta_d + y + 2c)dy.
\]

Differentiating this expression with respect to \( d \), one obtains the first order condition:

\[
\Pi^D_{d} = d/A'(A^{-1}(d)) + .5A^{-1}(d) - [.5A^{-1}(d) + .5\theta_d + c]/A'(A^{-1}(d)) = 0.
\]

Multiplying (A4) through by \( A'(A^{-1}(d)) \), we get the more convenient expression:
We first look at the conflict where \( c < 1/6 \). In this situation the aggressor’s demand curve is above the defender’s offer curve and the truncations are inessential; so we can ignore them. We are checking the defenders’ best response to \( A(\theta_a) = 2\theta_a/3 - 2c + 1/2 \), so we substitute \( A^{-1}(d) = 3d/2 + 3c - 3/4 \) and \( A' = 2/3 \) into (A5) to obtain

\[
(A6) \quad 0 = d - 3d/12 - 3c/6 + 3/24 - .5\theta_d - c = .75d - 1.5c + 1/8 - .5\theta_d
\]

The unique solution is \( d = 2\theta_d/3 + 2c - 1/6 \), as desired. Moreover, the second derivative of the objective function is

\[
(A7) \quad \Pi^D_\theta = 1.5/A'(A^{-1}(d)) - .5/[A'(A^{-1}(d))]^2 = 1.5/(2/3) - .5/(4/9) = 9/4 - 9/8 = 9/8 > 0,
\]

so we are indeed at a minimum. Hence we have verified the best response when \( c < 1/6 \).

Next suppose that \( c \geq 1/6 \) so that the defender offer curve is on or above the aggressor demand curve and the demand and offer functions involve essential truncations as in Figure 1A. As explained in the text, the truncation does not reduce the probability of a settlement, but does make the settlement more favorable for the belligerent. The computations above for \( c < 1/6 \) still hold, but we still need to check the truncations. It suffices to show that even when \( \theta_d = 0 \), the defender will still want to settle. Here \( d = 2c - 1/6 \) and there will be a settlement for that amount. If the defender were to raise its offer, he would clearly be worse off. If the defender were to reduce its offer, then there would be a war. Recall that the aggressor will demand \( 2c - 1/6 \) for all values of \( \theta_a \leq 6c - 1 \). Hence, the expected cost of the war to the defender would be \(.5(0) + .5[.5(6c - 1)] + c = 2.5c - .25 \). The term in brackets is the expected value of \( \theta_a \) given that \( \theta_a \leq 6c - 1 \). When is the expected cost of a war, \( 2.5c - .25 \), greater than the cost of a settlement, \( 2c - 1/6 \)? When \( .5c > 1/12 \); equivalently, when \( c > 1/6 \) – the condition for the aggressor’s demand curve to be below the defender’s offer curve.
The game being symmetric, a similar argument verifies that the given piecewise-linear Aggressor strategy is a best response to the defender’s given strategy. ///

**Proposition 2.** All non-trivial piecewise-linear symmetric Nash equilibria of the basic war model induce the same outcome as strategies (3-4).

**Proof.** We will now make use of the symmetry conditions to rewrite (A5) as a function of $D$ rather than $A$. We first explicitly consider the following symmetry relationships.

Let $y = A(z) = 1 - D(1 - z)$. Then $z = A^{-1}(y)$ and $D^\dagger(1 - y) = 1 - z$. Equivalently, $z = 1 - D^\dagger(1 - y)$. So $A^{-1}(y) = 1 - D^{-1}(1 - y)$ and $A'(A^{-1}(d)) = D'(1 - A^\dagger(d)) = D'(1 - 1 + D^\dagger(1 - d)) = D'(D^\dagger(1 - d))$.

Substituting these relationships into (A5) we get:

\[
(A8) \quad d + .5[1 - D^\dagger(1 - d)]D'(D^\dagger(1 - d)) - .5\theta_d - c - .5[1 - D^\dagger(1 - d)] = 0
\]

\[
= d + .5[1 - D^\dagger(1 - d)] [D'(D^\dagger(1 - d)) - 1] - .5\theta_d - c = 0
\]

$D$ is assumed to be piecewise-linear. Let $d = a_d + b_d c + e_d \theta_d$, then $\theta_d = (d - a_d - b_d c)/e_d$ and $D' = e_d$. Equation (A8) can be rewritten as follows:

\[
(A9) \quad d + .5(e_d - 1) - .5(e_d - 1)(1 - a_d - b_d c)/e_d - .5\theta_d - c = 0
\]

Equivalently,

\[
(A10) \quad d(1.5e_d - .5)/e_d = - .5(e_d - 1)(1 - 1/e_d) - .5(e_d - 1)a_d/e_d - .5(e_d - 1)(b_d c)/e_d + c + .5\theta_d
\]

or

\[
(A11) \quad d = - .5(e_d - 1)(1 - 1/e_d)e_d / (1.5e_d - .5) - .5(e_d - 1)a_d / (1.5e_d - .5)
\]

\[
- .5(e_d - 1)(b_d c)/ (1.5e_d - .5) + ce_d / (1.5e_d - .5) + .5\theta_d e_d / (1.5e_d - .5)
\]
Now the coefficient of $\theta_d$ is $e_d$. So from (A11), we have the following relationship:

(A12) $e_d = .5e_d/(1.5e_d - .5) = e_d/(3e_d - 1)$

The solutions are $e_d = 2/3, 0$. Clearly, the pair (A1-A2) satisfies these conditions. The inessential truncations need not hold because, as shown in Proposition 1, the probability of a war is 1 regardless. So the outcome is not changed as long as the slope is greater than or equal to zero once the point of truncation is reached (hence the word “piecewise”).

The only question remaining is whether there is another piecewise-liner function with this set of slopes but in a different combination. We first show that $A = 1/2 + 0\theta_a, D = 1/2 + 0\theta_d$ is not a NE for all $0 < c < 1/3$. Suppose that $\theta_p = 1$, then the expected outcome if the conflict goes to war is 3/4. So the aggressor will raise its demand above .5 if $c < .25$.

Every other horizontal line where $D$ is below $A$ always results in war and therefore is trivial. The reverse can never be an equilibrium as all conflicts would be settled and the aggressor would want to increase its demand and the defender would want to decrease its offer.

There are two other families of possibilities:

(i) The 2/3-slope line is broken up by one or more 0 slope lines.
(ii) The 2/3-slope line is still in the middle, but the 0 slope line(s) start or stop at a different place.

We will focus on (i). The argument for (ii) is a blend of the previous arguments.

Let us consider a horizontal portion between two line segments with 2/3 slope. Moving along the horizontal portion, the defender’s loss from going to war strictly increases continuously as $\theta_d$ increases. Furthermore, we know that the probability of war is less than 1 as the defender increases its offer for still larger values of $\theta_d$ to reduce the probability of war. Thus, the
defender should continuously strictly increase its offer to continuously reduce the probability of a war. But a 0 slope says otherwise. Hence we are led to a contradiction. ///

**Proposition 3.** The equilibrium probability of war is $1 - 18c^2$ for $0 \leq c \leq 1/6$; $2(1 - 3c)^2$ for $1/6 \leq c \leq 1/3$; and 0 for $c \geq 1/3$.

**Proof.** A conflict goes to war when $a = (2/3)\theta_a - 2c + 1/2 > (2/3)\theta_d + 2c - 1/6 = d$, i.e., when $\theta_a - \theta_d > 6c - 1$. Clearly, as $c$ increases the probability of a war decreases. More specifically, the probability of a war is 0 if the lowest defender offer ($2c - 1/6$ in equilibrium) is weakly above the highest aggressor demand ($7/6 - 2c$ in equilibrium); so in equilibrium the condition reduces to $c \geq 1/3$. In this situation, the aggressor will demand the minimum defender offer and the defender will offer the maximum aggressor demand. That is, $A(\theta) = D(\theta) = .5$ (remember that we are restricting our analysis to non-trivial symmetric equilibria).

Since signals are independent and uniformly distributed, the equilibrium war probability is the area of the war region, i.e., the subset of signal combinations in the unit square where $\theta_a - \theta_d > 6c - 1$. For $1/6 \leq c \leq 1/3$, the war region is the isosceles triangle Northwest of the line of slope $+1$ with $\theta_a$ - intercept $6c - 1$. Hence the triangle has height $h = 1 - (6c - 1) = 2 - 6c$ and area $0.5h^2 = 2(1 - 3c)^2$. For $0 \leq c \leq 1/6$, the war region excludes only the isosceles right triangle Southeast of the line of slope $+1$ with $\theta_d$ - intercept $1 - 6c$. That triangle has height $h = 6c$ and area $0.5h^2 = 18c^2$, and so the war probability is the remaining area of the unit square, or $1 - 18c^2$. ///

**Proposition 4.** The equilibrium probability of a war increases in $|W - 0.5|$ when $c < 1/6$ and decreases in $|W - 0.5|$ when $c > 1/6.

**Proof.** For fixed $W \in [.5, 1)$, consider the set of signals on the line $(\theta_a + \theta_d)/2 = W$. Since $\theta_a = 2W - \theta_d$ and $W \in [.5, 1)$, the maximum value of $\theta_d$ is 1, in which conflict $\theta_a = 2W - 1$.

Similarly, the maximum value of $\theta_d$ is 1, in which conflict $\theta_d = 2W - 1$. The $W$ line has slope $\Delta \theta_d / \Delta W$. //
Δθ_d equal to -1. Hence the difference \( K = \theta_a - \theta_d \) is uniformly distributed over the interval \((2W - 2, 2 - 2W)\).

Recall that a war only takes place when \( \theta_a - \theta_d > 6c - 1 \). Hence the probability \( T(W) \) of a war is the length of the subinterval where \( K > 6c - 1 \) divided by the length of the interval. The interval is empty and \( T(W) = 0 \) if \( 6c - 1 > 2 - 2W \), i.e., if \( c > 0.5 - W/3 \). Otherwise,

\[
(A13) \quad T(W) = \frac{(2 - 2W) - (6c - 1)}{(2 - 2W) - (2W - 2)} = 0.25 \frac{3 - 2W - 6c}{1 - W}.
\]

To find the effect of an increase in \( W \) (the expected war outcome) on the probability of a war \( (T) \), we take the derivative of \( T \) with respect to \( W \) over the relevant interval, and simplify to obtain

\[
(A14) \quad T'(W) = -2(0.25)/[1 - W] + 0.25 \frac{3 - 2W - 6c}{(1 - W)^2}
\]

\[
= 0.25 \frac{2W - 2 + 3 - 2W - 6c}{(1 - W)^2}
\]

\[
= 0.25[1 - 6c]/[1 - W]^2.
\]

The sign of \( T' \) therefore is the sign of \( 1 - 6c \). When \( c > 1/6 \), the probability of a war decreases, but when \( c < 1/6 \) the probability of a war increases as the expected outcome goes from \(.5\) to \(1\). So, the proposition is proven for \( W \geq .5 \). The analysis is similar when \( W < .5 \). ///

**Proposition 6.** An outcome spread increases the equilibrium probability of a war.

**Proof.** Transform a war model on a nontrivial interval \([L, U]\) to a basic war model on \([0, 1]\) using the mapping \( x = \frac{x}{x + L}(U - L) \). It is easy to check that independent uniform signal distributions retain those properties under the transformation. Note that the transformed war cost is \( c = (C + L)/(U - L) \) if the non-normalized cost is \( C \). A spread is defined as an increase in \( U - L = V \), so we consider the partial derivative of \( c = (C + L)/V \) with respect \( V \). It is easily calculated to be \( (VL' - C - L)/V^2 \leq - (C + L)/V^2 < 0 \), where the weak inequality comes from the proviso that \( L \) does not
increase in $V$. Since the normalized cost decreases, we apply Proposition 3 to conclude that the equilibrium war probability increases.///
APPENDIX B

We analyze the general effect of cost shifts when defender has an offer function $d = D(\theta_d)$ and aggressor has a demand function $a = A(\theta_a)$. (The notation $d$ distinguishes the offer from the symbol $d$ for a derivative.) We assume that belligerents’ signals are independent, and the distribution functions $F^A$ and $F^D$ have strictly positive densities $f^A$ and $f^D$ that are differentiable almost everywhere on $[L, H]$. By Lemma 1 (not in this paper), we can assume that $A', D' > 0$ and have inverses with positive derivatives in the regions of interest.

Let $Z = A^{-1}(d)$. Then $A(Z) = d$ and $Z' = A^{-1} > 0$. The defender seeks to minimize

\[
2 \Pi^D(d, \theta_d, c, A, F^A) = \int_L^Z [d + A(y)]dF^A(y) + \int_Z^H \left[ \theta_d + y + 2c \right] dF^A(y).
\]

First order conditions for an interior solution:

\[
2 \Pi^D(d, \theta_d, c, A, F^A) = F^A(Z) + [d + A(Z)]f^A(Z)Z' - [\theta_d + Z + 2c]f^A(Z)Z' = 0.
\]

Since $F^A(Z), f^A(Z)Z' > 0$, equation (B2) implies that

\[
[\theta_d + Z + 2c - 2d] = \frac{F^A(Z)}{f^A(Z)Z'} > 0.
\]

Second order conditions for a minimum:

\[
2 \Pi^D_{dd}(d, \theta_d, A, F^A) = 3f^A(Z)Z' - [\theta_d + Z + 2c - 2d][f^A(Z)Z'^2 + f^A(Z)Z'''] - f^A(Z)Z'^2.
\]
$$= 3f^A(Z)Z' - \frac{F^A(Z)}{f^A(Z)Z'} [f^{A'}(Z)Z'^2 + f^A(Z)Z''] - f^A(Z)Z'^2 > 0$$

The inequality might not be strict at a set of isolated points, which we can safely ignore.

Equation (B4) is equivalent to

$$(B5) \frac{F^A(Z)}{[f^A(Z)Z']^2} [f^{A'}(Z)Z'^2 + f^A(Z)Z''] < 3 - Z'.$$

Let $\psi = \frac{1}{\ln(F^A(Z))} = \frac{F^A(Z)}{f^A(Z)Z'}$. Then by (B3) we have

$$(B6) \psi = \theta_d + Z + 2c - 2d.$$  

Using the definition of $\psi$ and (B5), we obtain the following relationship:

$$(B7) \psi' = 1 - \frac{F^A(Z)[f^{A'}(Z)Z'Z' + f^A(Z)Z'']}{[f^A(Z)Z']^2} > Z' - 2.$$  

In the basic war model (BLG), $Z'' = 0 = f^{A'}$, and thus $\psi' = 1$.

We now compute the direct effect of an increase in the defender’s cost on the demand, $d_c$, holding constant the aggressor’s demand function, $A(\theta_a)$. Implicitly differentiating (B6), we get:

$$(B8) \psi'd_c = Z'd_c + 2 - 2d_c$$  

Thus the direct effect is
(B9) \( d_c = 2/(\psi' + 2 - Z') \).

Hence \( d_c > 0 \) if the denominator is greater than 0. But this follows from (B7). Thus the direct effect of an increase in \( c \) on \( D \)’s offer is to increase \( d \).

We next compute the effect of a shift in \( D \)’s offer on \( A \)’s demand.

Let \( W = D^{-1}(a) \), then \( D(W) = a \). \( W' > 0 \) because \( D' > 0 \). (Note that this is not the same \( W \) as before.) The aggressor seeks to maximize

\[
\Pi^A(a, \theta, c, D, F^D) = \int_w [a + D(x)dF^D(x)] + \int_L [\theta + x - 2c]dF^D(x).
\]

The first order conditions for an interior solution are then:

(B11) \( 2\Pi^A_a(a, \theta, c, D, F^D) = 1 - F^D(W) + [\theta + W - 2c - 2a]F^D(W)W' = 0. \)

Since \( 1 - F^D(W), f^D(W)W' > 0 \), equation (B11) implies that

(B12) \[ \theta + W - 2c - 2a = \frac{F^D(W) - 1}{f^D(W)W'} < 0. \]

Second order conditions for a maximum:

(B13) \( 2\Pi^A_{aa}(a, \theta, c, D, F) = -3F^D(W)W' + [\theta + W - 2c - 2a][f^D(W)W'^2 + f^D(W)W'''] + f^D(W)W^2 \)

\[ = -3F^D(W)W' + \frac{F^D(W) - 1}{f^D(W)W'} [f^D(W)W'^2 + f^D(W)W'''] + f^D(W)W^2 < 0 \]

In the BLG this reduces to \(-9/2 + 0 + 9/4 < 0\).
Equation (B13) is equivalent to

\[
(B14)\quad \frac{F^D(W) - 1}{[f^D(W)W']^2} \left[ f^D(W)W'^2 + f^D(W)W'' \right] < 3 - W'
\]

Let \( \xi = \frac{F^D(W) - 1}{f^D(W)W'} \). Then

\[
(B15)\quad \xi = \theta_p + W - 2c - 2a \text{ by (B13)}.
\]

\( W \) is the cognate of \( Z \) and \( \xi \) is the cognate of \( \psi \). That is, the proofs are parallel.

\[
(B16)\quad \xi' = 1 - \frac{[F^D(W) - 1][f^D(W)W'W' + f^D(W)W'']}{[f^D(W)W']^2} > W' - 2 \text{ by (14}).
\]

Differentiating (B15) with respect to \( c \), we obtain

\[
(B17)\quad a_c = \frac{-2}{\xi'W' + 2}
\]

The denominator is positive by (B16); so the expression is negative.

Now consider a shift in the defender’s offer function, \( D \), by a small amount \( s \) (in either direction) and derive the sensitivity, \( a_s \). At the margin \( a = D(\theta_p) + s \), so now set \( W = D^{-1}(a - s) \), and the first order condition is:

\[
(B18)\quad \xi(a - s) = \theta_a + W(a - s) - 2c - 2a.
\]
Holding $c$ constant, we take the derivative of this expression with respect to $s$ and obtain an implicit equation for the desired effect $a_s$.

\[ (B19) \xi'(a-s)[a_s-1] + 2a_s = W'(a-s)[a_s-1]. \] In turn this implies

\[ (B20) a_s = \frac{\xi'-W'}{\xi'-W'+2}. \] The denominator is positive by (B16).

We next compute the indirect effect, $d_s$, of a shift upwards in the aggressor’s demand on the defender’s offer. At the margin $d = A(\theta_d) + s$, so now set $Z = A^{-1}(d - s)$, and the first order condition is:

\[ (B21) \psi'(d - s) = \theta_d + Z(d - s) + 2c - 2d. \]

Holding $c$ constant, we take the derivative of this expression with respect to $s$ and obtain

\[ (B22) \psi'(d - s)[d_s-1] + 2d_s = Z'(d - s)[d_s-1]. \] In turn this implies

\[ (B23) d_s = \frac{\psi'-Z'}{\psi'-Z'+2}. \]

The total effect of the defender’s cost increase on the defender’s offer includes

(i) the direct effect of an increase in cost on the defender’s offer ($d_c$); plus

(ii) the first round indirect effect on the defender’s offer: the effect of the change in the defender’s offer on the aggressor’s demand which in turn has an effect on the defender’s offer ($d_c a_s d_s$); plus

(iii) the second round indirect effect on the defender’s offer due to the shift at the end of the first round ($d_c[a_s d_s]^2$); etc.
Hence the total effect is

\[(B24) \ d_c + d_c a_s d_s + d_c [a_s d_s]^2 \ldots = d_c [1 + a_s d_s + \{a_s d_s\}^2 \ldots] = \frac{d_s}{1 - d_s a_s} \]

This expression is positive if \( a_s d_s < 1 \). Inspection of (B20) and (B23) shows that \( 0 < a_s < 1 \) when \( \xi' - W' > 0 \) and that \( 0 < d_s < 1 \) when \( \psi' - Z' > 0 \). In this conflict the indirect effect magnifies the direct effect of an increase in \( c \). When either \( \xi' - W' < 0 \) or \( \psi' - Z' < 0 \), but not both, then \( a_s d_s < 0 \) and the indirect mitigates, but does not change the sign of the direct effect. And if both \( a_s \) and \( d_s \) are negative but the product is less than 1, then the indirect effect will again magnify the direct effect.

The total effect (B24) has a different sign from the direct effect only in the conflict (a) \( \xi' - W' < 0 \), (b) \( \psi' - Z' < 0 \), and (c) the geometric mean of the absolute indirect effects exceeds 1, so \( a_s d_s > 1 \). We have not yet been able to find an example.

A similar analysis shows that the total effect of a change in the aggressor’s war costs is

\( a_c \frac{1}{1 - d_s a_s} \). Since the denominator is the same as for defender’s total effect, the same conflicts determine when the direct effect is magnified, mitigated or reversed.

In the basic war model, \( \psi' = \xi' = 1 \) and \( Z' = W' = 3/2 \). So \( d_c = 2/(\psi' + 2 - Z') = 4/3 \) and \( d_s = \frac{\psi' - Z'}{\psi' - Z' + 2} = -1/3 = a_s \). Hence, \( \frac{a_c}{1 - d_s a_s} = (4/3)(9/8) = 3/2 \). That is, following a unit increase in its own cost, the defender raises its offer (over the relevant range) by \( 3/2 \). The impact on the Aggressor is the indirect effect times this total effect, or \( (-1/3)(3/2) = -1/2 \). By symmetry, if the Aggressor also experiences a unit cost increase, then he reduces its demand by \( 3/2 \), inducing an in
increase in the defender’s offer by 1/2. Hence, if both experience a unit cost increase, then the total effect is 2, as seen in the equilibrium bid functions.