Hysteresis in Asset Liquidity

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Abstract

Does the acceptability of an asset exhibit hysteresis? With a continuous-time New Monetarist model of asset liquidity dynamics, this paper answers yes. I describe asset acceptability as a slow-moving state variable—it increases as more sellers obtain the technology that allows them to accept the asset and decreases as some sellers lose their technology. With strategic complementarity, asset acceptability can exhibit hysteresis, which provides a novel explanation for the hysteresis in dollarization. The model explains the irreversible dollarization in Argentina from 1989 to 1998.

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1 Introduction

Hysteresis in dollarization, the fact that a temporary increase in inflation has long-lasting effects on the level of dollarization, has been documented in many countries.[1] The seminal work by Uribe (1997) explain the phenomenon by introducing an idea of "dollarization capital", which represents the society's cumulative experience in using dollar. Uribe (1997) shows that if there exists network externalities in the sense that as the dollarization capital accumulates, the cost of using dollars decrease, then dollarization can exhibit hysteresis. In this paper, I micro-found the notion of the dollarization capital through the usefulness of foreign currencies in transactions, and show that it may be the acceptability of dollars, or more generally, the acceptability of assets, that exhibits hysteresis.

What determines how acceptable an asset is in transactions? The work by Lester et al. (2012) (hereafter LPW) suggests that if accepting an asset requires a technology, which is costly, then there exists a tradeoff between the cost of the technology and the trade surplus, which determines how acceptable an asset is. In this paper, I build on LPW, but model asset acceptability as a slow-moving state variable—a dollarization capital. Underlying this are two assumptions. First, it takes time to acquire the technology.[2] Second, once acquired, the technology does not decay immediately.[3] This allows us to study the dynamics of asset acceptability, for example, how a new asset (e.g., Bitcoin) becomes widely used, and how an existing asset (e.g. cash, gold) become obsolete.

The dual-currency version of my model provides an explanation for the hysteresis in dollarization. In this version of the model, a perfectly liquid *home* currency coexists with a partially liquid *foreign* currency. A key driving force for the hysteresis is the strategic complementarity between buyers and sellers' decisions. Every individual seller's ability to use the technology contributes to the aggregate acceptability of the foreign currency, which determines its liquidity value. Higher liquidity value in turn incentivize sellers to invest more in acquiring the technology. This creates a feedback loop between the price of an asset and its liquidity, and gives rise to multiple steady states. However, not every steady state is equally reachable. By modeling asset acceptability as a state variable, equilibrium sets are path dependent. For example, in some cases, dollarization is always possible, but de-dollarization is possible only when the dollarization level is not too high. As a result, an economy that experienced hyperinflation may not be able to de-dollarize to where it used to be, even if domestic inflation returns to the initial level.

Apart from deterministic equilibria, I also consider cases where agents form self-fulfilling beliefs that the domestic currency may crash in the future. These pessimistic beliefs give rise to dollarized equilibria even if domestic inflation is low. When buyers believe that the domestic currency will eventually crash,

[1] For example, from 1984-1985 Bolivia experienced a period of hyperinflation, during which the country started to become highly dollarized. While the episode was quickly stabilized, Bolivia remained highly dollarized for more than two decades. Kehoe et al. (2019) documents that from 1987 to 2009, more than half of total deposits remained foreign currency deposits.

[2] For example, it takes time to learn about the asset, to complete an investment, or to acquire a license.

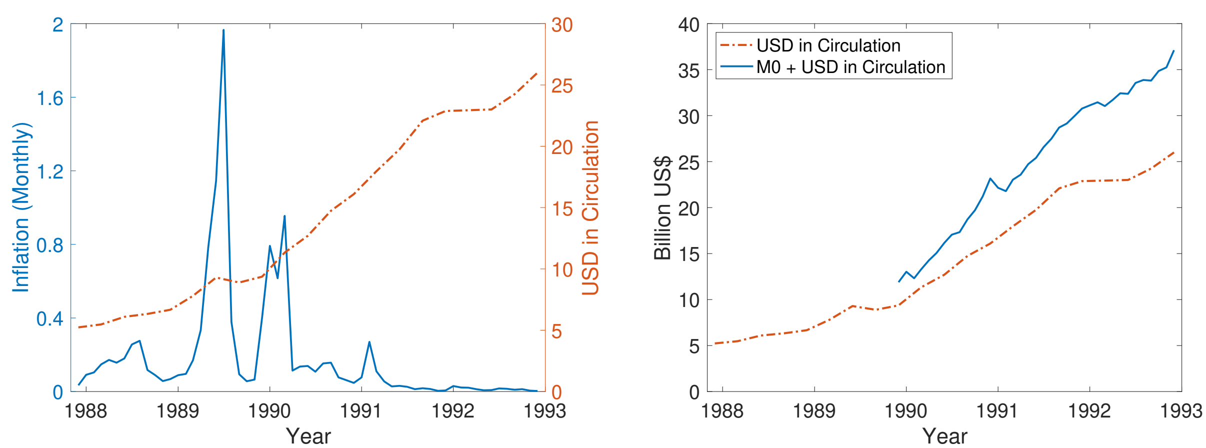
[3] For example, when a shop owner learns how to tell fake monies from authentic ones, the knowledge remains in their mind, allowing them to reject payments made in fake money, until the technology of counterfeiting improves. Alternatively, a point-of-sale system that allows a retailer to accept card payments depreciates only slowly over time.

the effective rate of return of the domestic currency decreases. As a result, buyers preemptively hold less domestic currency and more foreign currency, thereby putting downward pressure on the value of domestic currency. On the other hand, sellers, knowing that the foreign currency will eventually become the dominant one, react by preemptively investing more in technology acquisition. This rationalizes, for example, the persistent dollarization in Argentina in the 1990s, when the Argentine pesos were pegged to US dollars.

I conclude by studying how asset acceptability responds to monetary policy, and the implications on asset prices and monetary policy transmissions. For this purpose, I consider an economy where a perfectly liquid fiat money coexists with a partially liquid real asset. When there is a permanent increase in the money growth rate, the value of money decreases. However, sellers immediately respond by investing more in acquiring the technology that allows them to accept the asset. The reactions of asset acceptability mitigates the effects of monetary policy, especially in the long run.

1.1 The example of Argentina

Argentina is a prototypical example of a dollarization hysteresis. From 1989 to 1990, Argentina experienced a period of hyperinflation. In April 1991, Argentina adopted the currency board, after which a roughly 1:1 fixed exchange rate between Argentine pesos and US dollars was maintained for a decade (Buera and Nicolini, 2019). The currency board was effective in bringing down inflation. In Figure 1a, the blue, solid curve plots the monthly inflation rate in Argentina. It spiked between 1989 and 1990, and dropped to a very low level (comparable to the US) in mid 1991.



(a)

(b)

Figure 1: Dollarization in Argentina. Source: the Central Bank of the Argentine Republic, Kamin and Ericsson (2003), https://tradingeconomics.com.

However, the success in combating inflation did not lead to de-dollarization. The red, dash-dotted curve in Figure 1a represent the amount of US dollars circulating in Argentina. It shows that even after the hyperinflation was stabilized, the amount of US dollars held in Argentina continued increasing. Moreover,

Figure 1b shows that US dollars continued to take up a large proportion of the total amount of currency in circulation in Argentina. Indeed, the end of 1989, 78.8% of the money in circulation was US dollars. At the end of 1992, the number only dropped to 70.1%, suggesting that Argentina indeed experienced persistent dollarization even in a period extremely low domestic inflation.

In Section 6, I calibrate the model to Argentine data from the end of 1989 to the end of 1992, and show that the model can explain the persistent dollarization in Argentina during that period. Underlying the persistent dollarization was the fact that the acceptability of the foreign currency continued to increase even after domestic inflation was stabilized.

1.2 Related literature

Earlier work on the hysteresis in dollarization is surveyed by Calvo and Végh (1996).[4] Dornbusch et al. (1990) suggests that the hysteresis is due to financial adaptation, i.e., the emergence of new financial instruments creates alternatives to domestic currency, reducing the demand for domestic currency under any level of interest rate. Duffy et al. (2006) studies the role of worsening domestic financial development. My model is more related to Uribe (1997), who introduces the idea of *dollarization capital*, which measures a society's aggregate experience in using dollars. Uribe (1997) shows that with network externality—as individuals use the foreign currency, the cost of using the foreign currency decreases for everyone in the economy, dollarization can exhibit hysteresis. Network externality also plays an important role in this paper—as individual sellers invest in acquiring the technology that allows them to accept an asset, the asset becomes more acceptable for all buyers. In this sense, my model is a combination of Uribe (1997) and Lester et al. (2012). I micro-found the dollarization capital in Uribe (1997) with asset acceptability, and I model the asset acceptability in Lester et al. (2012) as a capital, allowing me to study hysteresis.

This paper contributes to the New Monetarist literature that studies the relationship between asset liquidity and asset prices. Some of those work, including Geromichalos et al. (2007), Lagos and Rocheteau (2008), and Lagos (2011), consider the coexistence of multiple assets, all of which are perfectly liquid. As a result, the rate of return of the assets must equalize. Lagos (2010) considers the differences in liquidity, but the liquidity of assets is assumed to be exogenous. Both Lester et al. (2012) and Li et al. (2012) endogenize asset liquidity through informational frictions, the former through endogenous acceptability, and the latter through the threat of fraud. While most of the literature do not think of asset liquidity as slow-moving, I model explicitly how asset liquidity evolves over time as a slow-moving state variable, or a capital.

Within the search-theoretic framework, earlier work on the emergence of international currencies dates back to Matsuyama et al. (1993). Using an evolutionary approach, they show that this can happen in a Kiyotaki-Wright model (Kiyotaki and Wright, 1989, 1993) of indivisible money and indivisible goods with two countries and two currencies. Wright and Trejos (2001) is the first to study dollarization and international

[4] I only focus on currency and/or asset substitution. For the role of financial dollarization, see Ize and Yeyati (2003) and Bocola and Lorenzoni (2017).

currency in the context of the Shi (1995) and Trejos and Wright (1995) models of indivisible money and divisible good. Subsequent work includes Trejos (2003), Head and Shi (2003), Craig and Waller (2004). Lester et al. (2012) is the first to consider the role of the endogenous acceptability of the foreign currency. Zhang (2014) extends the work by Lester et al. (2012) to a two-country economy and studies the emergence of an international currency. Lotz and Vasselin (2019) study the coexistence of fiat and e-money. This paper extends this literature by shedding light on the causes and consequences of the hysteresis in dollarization, or more generally, the in asset liquidity.

2 Environment

The environment is based on Choi and Rocheteau (2021).[5] Time is continuous and indexed by $t \in \mathbb{R}_+$. The economy is populated with two types of infinitely-lived agents: a unit measure of buyers and a unit measure of sellers. There are two types of non-storable goods. The first good, which is taken as the numeraire, is traded in a frictionless competitive market that is always open. The second good is produced and consumed in pairwise meetings between buyers and sellers.

The expected discounted lifetime utility of buyers is

$$\mathcal{U}^b = \mathbb{E}\left\{\sum_{n=1}^{\infty} e^{-\rho T_n} u\left[y\left(T_n\right)\right] + \int_0^{\infty} e^{-\rho t} dC(t)\right\}, \tag{1}$$

where $y(t)$ is the consumption in pairwise meetings at time t and $C(t)$ is the cumulative net consumption of the numeraire good.[6] The first term on the right hand side of (1) is the discounted sum of the utility from consuming in pairwise meetings, where T_n is the time at which the n-th pairwise meeting occurs. A buyer who consumes $y \in \mathbb{R}_+$ units of good in a pairwise meeting receives a utility of $u(y)$, where u is infinitely differentiable, strictly increasing, and strictly concave, with $u(0) = 0$ and $u'(0) = \infty$ is large. Furthermore, there exists a $y^* \in \mathbb{R}_+$ such that $u'(y^*) = 1$. The second term is the discounted linear utility from consuming or producing the numeraire good.

The expected discounted lifetime utility of the sellers is

$$\mathcal{U}^s = \mathbb{E}\left\{-\sum_{n=1}^{\infty} e^{-\rho T_n} y\left(T_n\right) + \int_0^{\infty} e^{-\rho t} dC(t)\right\}. \tag{2}$$

The first term on the right hand side of (2) is the disutility from producing y in the pairwise meetings, and the second term is the discounted linear utility of consuming or producing the numeraire good.

Buyers and sellers are matched bilaterally according to a Poisson process with arrival rate α. When in a pairwise meeting, agents do not have access to the technology to produce the numeraire good. Moreover, unsecured promises to repay loans are not credible due to lack of commitment and monitoring. These

[5] It is a continuous-time version of Lagos and Wright (2005) and Rocheteau and Wright (2005).

[6] The numeraire good can be consumed or produced both in flow and in discrete amounts; in the former case $dC(t) = c(t)dt$ and in the latter case $C(t^+) - C(t^-) \neq 0$. See Choi and Rocheteau (2021) for a more detailed discussion.

assumptions imply that the buyer of the good in pairwise meetings cannot finance y with the production of the numeraire good, thereby creating a need for a means of payment.

There is an asset that can serve as this means of payment. It is perfectly storable and durable, and takes the form of a continuous-time Lucas tree, i.e., a claim to a non-negative dividend flow d. If $d > 0$, the asset is intrinsically valuable, and is defined as a *real asset*. If $d = 0$, the asset is a *fiat money*, an intrinsically useless object. The supply of the Lucas tree in the economy at time t is M_t. In order to guarantee the existence of a steady state, I assume that the supply of the asset is fixed if $d \neq 0$, i.e., $M_t = M$ for all t. If $d = 0$, the money supply can grow (or shrink) at a rate $\gamma_t \equiv \dot{M}_t/M_t$. New money is injected into the economy as lump-sum transfers (taxes if $\gamma_t < 0$) to the buyers. In Section 3 and 3.1, I consider one asset economies. In the subsequent sections, I generalize the setup to a multiple asset economy. The asset is not fully acceptable to sellers. The probability that a random seller accepts the asset is χ, where χ will be endogenized later.

3 Equilibrium in a one-asset economy

Define dM/ρ the fundamental value of the asset. It is the discounted sum of future dividends. I focus on the case where $dM/\rho < p(y^*)$, i.e., the fundamental value of the asset is not enough to satisfy the maximum transaction need, or liquidity is scarce. In Appendix D, I study the case where liquidity is abundant. Let ϕ_t denote the price of the asset in terms of the numeraire. The expected rate of return of the asset is

$$r_t = \frac{d + \dot{\phi}_t}{\phi_t}. \tag{3}$$

It consists of two parts: dividend payment, and the changes in the value of the asset over time. Let $W_t(m)$ denote the value function of a buyer with real asset holdings equal to m. In appendix C.1, I show that $W_t(m) = m + W_t$ due to the linearity of buyers' preferences for the numeraire good.

The buyer's value function solves the following Hamilton-Jacobi-Bellman equation:

$$\rho W_t = \max_{m \geq 0} \left\{ -(\rho - r_t)m + \alpha \chi_t \Gamma(m) + \tau_t + \dot{W}_t \right\}, \tag{4}$$

where $\Gamma(m)$ is the buyer's surplus from a bilateral trade, which will be defined later. At any time t, a buyer chooses their optimal real asset holdings in order to maximize the sum of four terms on the right side of (4). The first term is the opportunity cost of holding the asset. It is the difference between the rate of time preference and the rate of return of the asset, multiplied by the buyer's real asset holdings. The second term is the buyer's expected surplus from a bilateral trade, which is the product of three terms, the arrival rate of the next pairwise meeting, the aggregate acceptability of the asset, and buyer's surplus from the trade. The third term is a lump-sum transfer (or tax). And the last term is the change of the value function over time.

We now turn to the bargaining problem in a pairwise meeting between a buyer and a seller. The outcome of the negotiation is a pair $(y, p(y))$ where y is the amount of good produced by the seller for the buyer and $p(y)$ is the payment from the buyer to the seller. The payment function, $p(y)$, is determined jointly by the buyers and the sellers. Here, I do not specify the exact form of the payment function; the only restrictions are: (1) $p(y)$ is infinitely differentiable, with $p'(y) > 0$ and $p''(y) < 0$ for all $y \in (0, y^*)$ and $p'(y) = 0$ for all $y > y^*$, (2) the buyer's surplus is increasing and concave in the buyer's real asset holdings, and (3) the seller's surplus is increasing in the buyer's real asset holdings. For example, if the payment function is determined according to the Kalai proportional bargaining, $p(y) = \theta y + (1-\theta)u(y)$ for some $\theta \in (0,1)$. If buyers and sellers trade according to gradual bargaining [7] (Rocheteau et al., 2021), then

$$p(y) = \int_0^y \frac{u'(x)}{\theta u'(x) + 1 - \theta} dx \quad \text{for all } y \leq y^*.$$

Given the payment function, the trade surplus of a buyer with real asset holdings m is

$$\Gamma(m) = \max_{y \geq 0} \left\{ u(y) - p(y) \ s.t.\ p(y) \leq \min \left\{ m, p(y^*) \right\} \right\}.$$

It is the difference between the utility from consumption and the payment, subject to the feasibility constraint that the payment cannot exceed the buyer's total liquid wealth. If a seller bargains with a buyer whose real asset holdings is $\tilde{m}$, the seller's surplus is

$$\Psi(\tilde{m}) = p\left[y\left(\tilde{m}\right)\right] - y\left(\tilde{m}\right).$$

where $y\left(\tilde{m}\right) = p^{-1}\left[\min\left(p(y^*), m\right)\right]$. The first-order condition for the choice of asset holdings, assuming interiority, is

$$\rho + \gamma - \frac{d + \dot{\phi}_t}{\phi_t} = \alpha \chi_t \left[\frac{u'(y_t)}{p'(y_t)} - 1 \right], \tag{5}$$

where $\gamma = 0$ if $d > 0$. The left hand side of (5) is the cost of holding the asset. The right hand side is the expected marginal liquidity value of the asset in a pairwise meeting. When market clears, $m_t = \phi_t M$, and we can rewrite the buyer's optimality condition (5) as

$$\rho + \gamma - \frac{dM + \dot{m}_t}{m_t} = \alpha \chi_t L(m_t), \tag{6}$$

where $L(m) \equiv u'\left[y\left(\tilde{m}\right)\right] / p'\left[y\left(\tilde{m}\right)\right] - 1$.

Endogenous acceptability In order to formalize the idea that it takes time to adopt an asset as a means of payment, I make the following assumptions. There is a technology that the sellers must be equipped with in order to be able to accept the asset. Sellers either possess this technology (*type 1*) or do not (*type 0*). Denote χ the fraction of sellers that are type 1. From the buyer's perspective, χ is the aggregate acceptability of the asset. A type 0 seller who wants to adopt the technology must choose some effort level $e \in \mathbb{R}_+$ to acquire it at some flow cost $\varphi(e)$. I assume that $\varphi(0) = \varphi'(0) = 0$, $\varphi'(e) > 0$, and $\varphi''(e) > 0$. I also assume that $\lim_{e \to \infty} e\varphi'(e) - \varphi(e) = \infty$. If the type 0 seller invests e then she transitions to type 1 at Poisson arrival rate e. The technology, however, is not permanent and is destroyed at Poisson arrival rate δ.

[7] The gradual Nash solution has several advantages in this environment (Rocheteau et al., 2021): it has axiomatic and strategic foundations ; it is strongly monotone, i.e., agents' surpluses increase with the value of their assets; gradual spending of real balances is optimal from the buyer's standpoint; it guarantees that a monetary equilibrium always exists provided that Inada conditions on preferences are imposed, $u'(0) = +\infty$; and it keeps the model tractable.

Sellers have no transactional motives to hold assets. The value function of a type 0 seller is denoted V_t^0. It solves the following HJB equation:

$$\rho V_t^0 = \max_{e \geq 0} \left\{ -\varphi(e) + e\left(V_t^1 - V_t^0\right) + \dot{V}_t^0 \right\}. \tag{7}$$

where V_t^1 is the value function of a type 1 seller. The type 0 seller does not trade, even if she meets buyers, since she cannot accept the asset. She invests some effort e at cost $\varphi(e)$ to acquire the technology, and transitions to type 1 at rate e, in which case she enjoys a gain in her expected lifetime utility equal to $V^1 - V^0$. The first-order condition for the optimal effort to learn about the asset is

$$\varphi'(e_t) = V_t^1 - V_t^0. \tag{8}$$

The marginal cost of the investment is equal to the capital gain from being able to accept it as means of payment. The value function of a type 1 seller solves the following HJB equation

$$\rho V_t^1 = \alpha \Psi(\tilde{m}_t) + \delta\left(V_t^0 - V_t^1\right) + \dot{V}_t^1, \tag{9}$$

At Poisson rate α the seller receives a trading opportunity with a buyer holding $\tilde{m}$ units of the asset (in real terms), and obtains a trade surplus of size $\Psi(\tilde{m})$. At Poisson rate δ the seller loses their technology and transitions to type 0. Let $\omega_t \equiv V_t^1 - V_t^0$ denote the gain from having the technology. From (7) and (9), ω_t solves

$$(\rho + \delta)\,\omega_t = \alpha \Psi(\tilde{m}_t) - \max_e \left\{-\varphi(e) + e\omega_t\right\} + \dot{\omega}_t.$$

When market clears, $\tilde{m}_t = m_t$ for all buyers. Substituting the optimality condition, $\varphi'(e_t) = \omega_t$, and applying the market clearing condition, the equation can be rewritten as

$$(\rho + \delta)\,\varphi'(e_t) = \alpha \Psi(m_t) + \varphi(e_t) - e_t \varphi'(e_t) + \varphi''(e_t)\dot{e}_t. \tag{10}$$

The measure of sellers who accept the asset evolves according to:

$$\dot{\chi}_t = e_t(1 - \chi_t) - \delta \chi_t. \tag{11}$$

It increases with the flow of type 0 sellers who become type 1, $e_t(1-\chi_t)$, and decreases with the measure of type 1 sellers who receive the idiosyncratic shock, $\delta\chi_t$.

An equilibrium is a list of time paths (m_t, e_t, χ_t) that solves the system of ODEs, (6), (10), (11), and the transversality condition

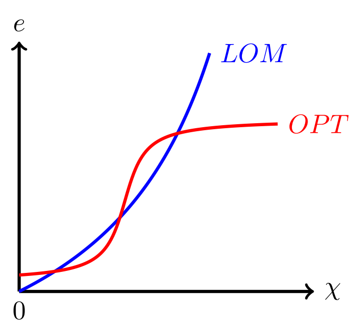
$$\lim_{t \to \infty} \mathbb{E}_t[e^{-\rho t} m_t] = 0, \tag{12}$$

given the initial condition χ_0. Equation (12) requires that the expected present value of the buyer's real asset holdings must approach zero as time goes to infinity.

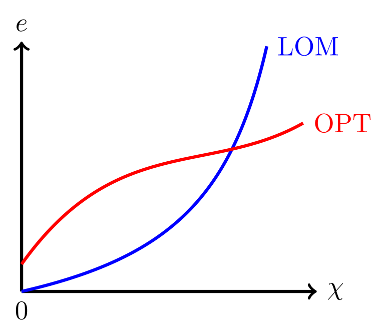
Before we proceed, let me define two concepts. First, a dynamical system exhibits hysteresis if it is path-dependent, i.e., if the equilibrium set differs depending on the initial condition. Second, a monetary equilibrium is one where the asset is used as a medium of exchange and priced above its fundamental value for a non-zero amount of time. When $d = 0$, there is a steady-state equilibrium that is non-monetary. The following proposition summarizes the set of monetary equilibrium.

Proposition 1 ***(Monetary steady states in a one-asset economy)*** *There exist an odd number of monetary steady states, ranked by the value of χ^*, i.e., $\chi_1^* < \chi_2^* < \cdots < \chi_{2k-1}^*$. The odd-indexed steady states are saddle points, and the even-indexed steady states are unstable spirals.*

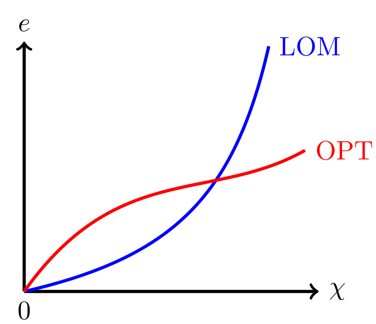
The possibility of multiplicity is a result of the strategic complementarity between buyers and sellers' decisions. Indeed, combining the m and the e nullclines (that is, $\dot{m} = 0$ and $\dot{e} = 0$), we obtain an increasing mapping from χ to e. The increasing relationship is a result of the strategic complementarity between the buyers and the sellers' decisions – a higher χ encourages buyers to hold more assets, making the asset more valuable, which incentivizes the sellers to invest more in acquiring the technology. In Figure 2, this mapping is represented by the red curves in (labeled "OPT" for "optimization"). The χ nullcline (that is, $\dot{\chi} = 0$) is represented by the blue curves (labeled "LOM" for "law of motion"). It guarantees that the proportion of sellers that possess the technology stays constant. While both curves are increasing, LOM ranges from 0 to infinity and OPT ranges from a non-negative number to a positive number. Therefore, the two curves intersect an odd number of times interiorly. In Lemma 4 in Appendix B, I provide a sufficient condition for the uniqueness of monetary steady state.



(a) $d > 0$, three steady states



(b) $d > 0$, one steady state



(c) $d = 0$, two steady states: one non-monetary & one monetary

Figure 2: Steady state(s) in a one asset economy.

Corollary 1 *The dynamical system exhibits hysteresis only if there are multiple monetary steady states.*

For the rest of this section, I illustrate, with a numerical example, that there is no hysteresis when there is a unique monetary steady state. We will focus more on the hysteresis case in Section 4. The model is parameterized as follows: $u(y) = \log(y + b) - \log(b)$ with $b = 0.0001$, $p(y) = \int_0^y u'(x)/[\theta u'(x) + 1 - \theta]dx$ with $\theta = 0.5$, $\varphi(e) = \kappa e^2$ $\kappa = 5$, $\alpha = 0.5$, $\rho = 0.03$, $\delta = 0.06$, $M = 1$, and $d = 0.01$. The red, blue and green surfaces represent the m, e and χ isoclines, respectively. The three isoclines intersect once and only once at a unique the steady state, $(m^*, e^*, \chi^*) = (1.27, 0.13, 0.68)$. The green curve represents the unique stable manifold of the steady state. Starting from any initial condition $\chi_0 \in [0,1] \setminus \{\chi^*\}$, there is a unique non-stationary equilibrium where (m_t, e_t, χ_t) converges to the unique steady state along the green curve. Along the equilibrium path, m_t, e_t, and χ_t move in the same direction. For example, if the initial acceptability of the asset is lower than the steady state, then sellers and buyers choose the optimal effort and real asset holdings such that the acceptability of the asset increases over time. Moreover, the increase in acceptability creates a positive reinforcement on the sellers' willingness to invest more and the buyers' willingness to hold more asset, which brings the acceptability of the asset up further. The process continues over time until the steady state is reached, where the marginal cost of investment begins to become too high for the sellers to be willing to increase their investment.

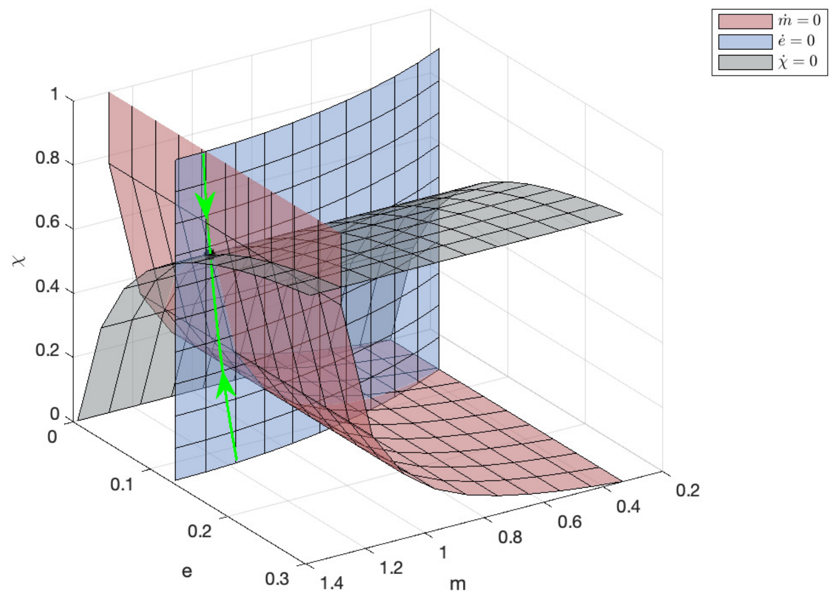


Figure 3: Numerical example: a one-asset economy

3.1 Acceptability as a control v.s. state variable

A key difference between this paper and Lester et al. (2012) is that here, acceptability is modeled as a state variable, while in Lester et al. (2012), acceptability is a control variable chosen directly by sellers. In order to highlight how this difference affects the equilibrium set, I study a version of this model where asset acceptability is a control, instead of state variable. It is similar to a one-asset, continuous-time version of Lester et al. (2012) with endogenous information.

I assume that there is only one type of sellers. At any point in time, all sellers choose $\varkappa \in [0,1]$ at cost $\psi(\varkappa)$ in order to be able to use the technology that allows them to accept the asset with probability $\varkappa$.[8]

[8] An alternative way to model it is to assume, as in Lester et al. (2012), that the cost of the technology is linear, but is heterogeneous across agents according to some distribution. The case where the cost is linear and homogeneous across agents

The cost function $\psi : [0, 1] \to \mathbb{R}_+$ is increasing and convex, with $\psi(0) = \psi'(0) = 0$ and $\psi(1) = \infty$. The value function of a seller now solves

$$\rho V_t = \max_{\varkappa \in [0,1]} \left\{ -\psi(\varkappa) + \alpha \varkappa \Psi(m_t) + \dot{V}_t \right\},$$

where m_t is the buyers' degenerate asset holdings at time t. The seller's first order condition for the optimal choice of $\varkappa$ is

$$\varkappa_t = \psi'^{-1} \left[\alpha \Psi(m_t) \right]. \tag{13}$$

Equation (13) implies that at any time t, given m_t, all sellers choose the same $\varkappa$ that equates the marginal cost of using the technology, $\psi(\varkappa)$ and the marginal benefit of the technology, $\alpha\Psi(\varkappa)$. The right hand side of (13) is an increasing function of m_t. As buyers' real asset holdings increase, sellers are more willing to accept the asset at a higher rate and a higher cost, The aggregate acceptability of the asset is

$$\chi_t = \int_0^1 \varkappa_t di = \psi'^{-1} \left[\alpha \Psi(m_t) \right]. \tag{14}$$

Combining (14) and the buyer's first order condition, (6), we obtain

$$\rho - \frac{dM + \dot{m}_t}{m_t} = \alpha \psi'^{-1} \left[\alpha \Psi(m_t) \right] L(m_t), \tag{15}$$

An equilibrium is a time path, $\{m_t\}$ that solves (15).

A steady state is a m^* that solves

$$\rho - \frac{dM}{m^*} = \alpha \psi'^{-1} \left[\alpha \Psi(m^*) \right] L(m^*), \tag{16}$$

The left panel of Figure 4 illustrates the determination of the steady-state equilibria. The red curve plots the left side of (16), which represents the cost of holding the asset. It increases from 0 to ρ as m goes from dM/ρ to ∞. The blue curve plots the right side of (16), which represents the liquidity value of the asset. It is determined jointly by $\psi'^{-1} [\alpha \Psi(m)]$, the equilibrium asset acceptability, and $L(m)$, the buyers' marginal gains from trade once the asset is accepted. When the asset is priced at its fundamental value, its liquidity value is positive. When $m = p(y^*)$, the liquidity value of the asset becomes zero. This suggests that a steady state m^* must exist.

Now consider possibilities for non-stationary equilibrium. The law of motion of m follows

$$\frac{\dot{m}_t}{m_t} = \rho - \frac{dM}{m_t} - \alpha \psi'^{-1} \left[\alpha \Psi(m_t) \right] L(m_t). \tag{17}$$

The right panel of Figure 4 plots $\dot{m}/m$ as a function of m in the case of a unique steady state. It is negative when $m < m^*$ and positive when $m > m^*$. Therefore, if the initial value of the asset, m_0 is less than m^*, then m decreases over time and will eventually violate the constraint that m cannot be priced below dM/ρ, its fundamental value. On the other hand, if $m_0 > m^*$, then m grows at rate ρ as $m \to \infty$, which violates

has been studied in the working paper version of Rocheteau (2023).

the transversality condition. Therefore, if there is a unique steady state, then there is no non-stationary equilibria. The only equilibrium is the steady-state equilibrium.

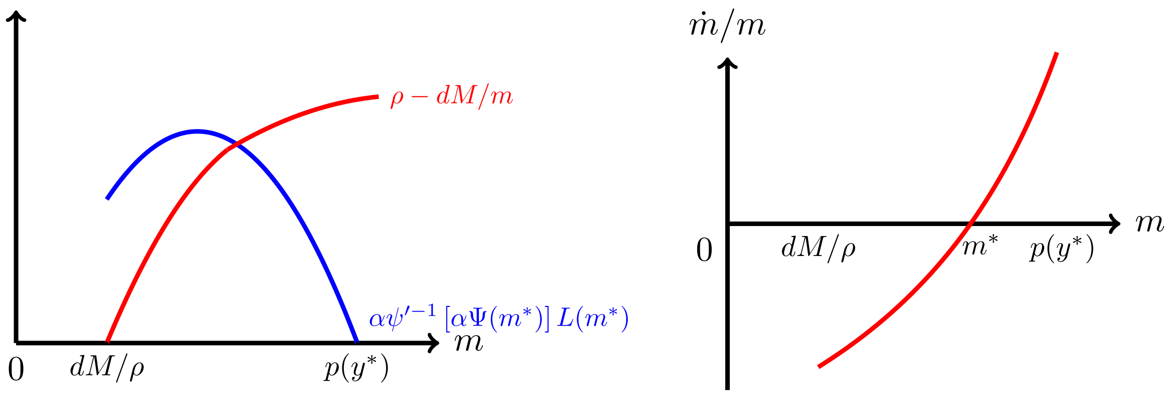


Figure 4: Equilibria when χ is a control variable

This reveals a key difference between the equilibrium set studied in Section 3 and the one in Lester et al. (2012). Conditional on there being a unique steady state, Lester et al. (2012) suggests that the equilibrium value of the asset, as well as its acceptability, always jumps to the steady state regardless of initial conditions. In contrast, in the model considered in Section 3, asset acceptability cannot jump. Instead, it transitions to the steady state slowly over time. In Section 4, I show that this feature allows us to provide a novel explanation for the hysteresis in dollarization.

4 Hysteresis in dollarization

In this section, I study a dual-currency economy where a partially liquid foreign currency (dollars) coexists with a fully liquid domestic currency (pesos). Consider a small open economy where two assets can serve as means of payment: the home currency (h) and the foreign currency (f). The home currency is an intrinsically useless object that is issued exclusively by the domestic central bank. The foreign currency is issued by a foreign country that is not modeled explicitly. Both currencies are perfectly storable and durable. I assume that all sellers are able to accept the domestic currency, while only a fraction χ of sellers can accept the foreign currency, where χ is determined endogenously by sellers' investment decisions. The buyer's value function now solves the following Hamilton–Jacobi–Bellman equation:

$$\rho W = \max_{m^h, m^f \geq 0} \left\{ -(\rho - r^h)m^h - (\rho - r^f)m^f + \alpha\chi\Gamma(m^h + m^f) + \alpha(1-\chi)\Gamma(m^h) + \dot{W} \right\}, \tag{18}$$

where m^h and m^f are real balances of the domestic and foreign currencies, respectively, and r^h and r^f are the rate of return of holding the home and the foreign currencies, respectively. Buyers and sellers are matched randomly. When a buyer is matched with a seller at rate α, with probability χ, the seller is type 1, in which case the output is $y^2 = \min\{p^{-1}(m^h + m^f), y^*\}$. With probability $1 - \chi$, the seller is type 0, in which case only the home currency is accepted, and $y^h = \min\{p^{-1}(m^h), y^*\}$. The rate of return of the foreign asset, r^f, is taken as given, while the rate of return of the domestic asset, r^h, is determined endogenously.

The buyer's optimal conditions are:

$$\rho - r^h \geq \alpha\chi L(m^h + m^f) + \alpha(1-\chi)L(m^h) \quad \text{“} = \text{” if } m^h > 0, \tag{19}$$

$$\rho - r^f \geq \alpha\chi L(m^h + m^f) \quad \text{“} = \text{” if } m^f > 0. \tag{20}$$

The left sides of (19)-(20) are the flow costs of holding the real balances, measured by the difference between the buyer's rate of time preference and the rate of return of money. The right sides are the expected marginal revenues, measured by the product of the frequency of trading opportunities and the expected marginal match surplus. Assuming $\rho - r^h < \alpha L(0)$, one of the two inequality must hold as an equality, i.e., $m^h + m^f > 0$.

The sellers' optimal conditions solve the following ODE:

$$\dot{e}\varphi''(e) = (\rho + \delta + e)\,\varphi'(e) - \alpha\left[\Psi(m^h + m^f) - \Psi(m^h)\right] - \varphi(e). \tag{21}$$

The right side is the difference in trade surpluses between the type 1 and the type 0 sellers. Given any initial condition χ_0, an equilibrium is a list of time paths, $(m_t^h, m_t^f, e_t, \chi_t)$ that solves (19), (20) (21), (11), and the transversality condition

$$\lim_{t\to\infty} \mathbb{E}_0 \left[e^{-\rho t}\left(m_t^h + m_t^f\right)\right] = 0. \tag{22}$$

In the following, I define a *dollarization steady state* as a steady state equilibrium where the domestic residents hold a positive amount of the foreign currency, i.e., when $m^{f*} > 0$. A *non-dollarization* steady state is defined as a steady state where $m^{f*} = 0$.

4.1 Deterministic equilibrium under an inflation-targeting monetary policy

I start by considering the case where r^h is determined by an inflation-targeting monetary policy, so that $r^h = -\pi^h$ at all t, where π^h is determined by the central bank. In Section 4.2, I endogenize π^h. With r^h given, both m_t^h and m_t^f are pinned down by χ_t. This reduces the dimensionality of the dynamical system and allows us to focus on the dynamic relationship between e and χ. The following lemma studies the relationship between the rate of return of the two currencies and the buyers optimal choice of real balances.

Lemma 1 *Suppose $r^h < r^f$ and $\alpha L(0) > \rho - r^h$, then there exist a pair*

$$(\underline{\chi}, \overline{\chi}) = \left(\frac{\rho - r^f}{\rho - r^h}, 1 - \frac{r^f - r^h}{\alpha L(0)}\right)$$

such that when $\chi \in [0, \underline{\chi}]$, $m^h > 0 = m^f$; when $\chi \in (\underline{\chi}, \overline{\chi})$, $m^h, m^f > 0$, and $m^f/(m^h + m^f)$ strictly increases in χ.

Lemma 1 states that buyers do not hold the foreign currency if χ is too low, and do not hold the domestic currency if χ is too high. Moreover, if χ is neither too high nor too low, then if all other exogenous variables are the same, the dollarization ratio, $m^f/(m^h + m^f)$, strictly increases as χ increases, suggesting that χ can be viewed as a proxy for dollarization.

We start by studying the set of steady states. A steady state is a pair (e^*, χ^*) that solves

$$(\rho + \delta + e)\varphi'(e) - \varphi(e) = \alpha \left[\Psi(m^h + m^f) - \Psi(m^h)\right], \tag{23}$$

$$e(1 - \chi) = \delta\chi, \tag{24}$$

where m^h and m^f are given by (19) and (20). We define a dollarization (resp. non-dollarization) steady state as one where the foreign currency is (resp. is not) used in transactions. The following lemma summarizes the sets of equilibria under an inflation-targeting monetary policy.

Lemma 2 *(Set of steady states in a dual-currency economy under inflation targeting)*

1. *If $\pi^h > -r^f$, there exist an odd number of steady states.*
2. *When δ is sufficiently small, there exists a $\tilde{\pi} \in \mathbb{R}$ such that when $\pi^h > \tilde{\pi}$, there exists multiple steady states. When $\pi^h < \tilde{\pi}$, there exists a unique steady state that is non-dollarization.*

Lemma 2 states that the existence of a dollarization steady state depends on domestic inflation. When inflation is low, then the only possible steady state is the non-dollarization one. Once domestic inflation is sufficiently high, the model starts to admit multiple dollarization steady states where the foreign currency is accepted and used in transactions. The relationship between π^h and the set of steady states is illustrated graphically in Figure 5. In both panels, the red curves represent equation (23), the e nullcline. The blue curves represent equation (24), the χ nullcline. In the top left panel of Figure 5, domestic inflation is sufficiently high, and the two nullclines intersect three times, including one non-dollarization steady state where $(\chi, e) = (0, 0)$, and two non-dollarization steady states where $(\chi, e) \in (0, 1) \times \mathbb{R}_{++}$. In the top right panel, domestic inflation is sufficiently low, and the two curves intersect at a unique non-dollarization steady state where $(\chi, e) = (0, 0)$.

Now consider the full set of perfect foresight equilibria. A non-stationary equilibrium under an inflation targeting monetary policy is a time path $(e_t, \chi_t, m_t^h, m_t^f)$ that solves the two-dimensional ODE system (11) and (21), given (19), (20), the transversality condition, and the initial condition χ_0. The following lemma studies the local stability around the steady states.

Lemma 3 *(Local stability in a dual-currency economy under inflation targeting)*
If there are multiple steady states, ranked by the value of χ^*, i.e., $\chi_1^* < \chi_2^* < \cdots < \chi_{2k-1}^*$, then the odd-indexed steady states are saddle points, and the even-indexed steady states are unstable spirals.

Figure 5 illustrates the results in Lemma 3. In the left panel, the lower, non-dollarization steady state is a saddle point. There is a saddle path (represented by the green curve) leading towards this steady state.

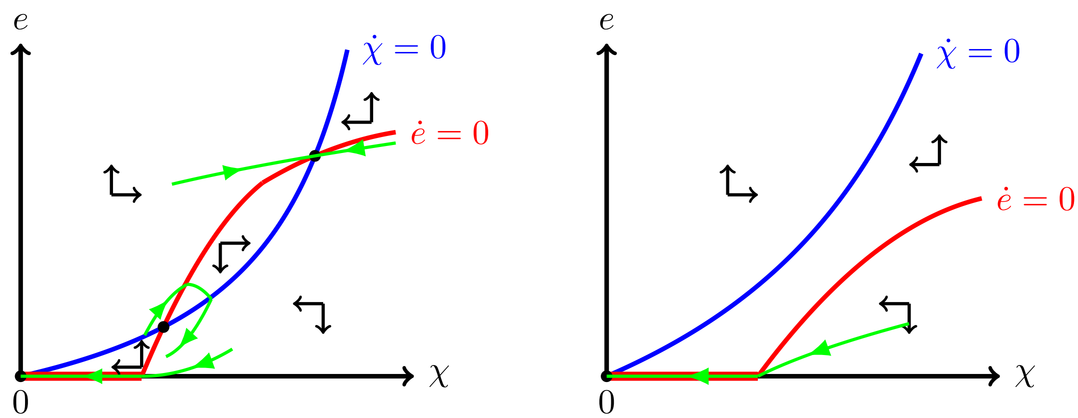


Figure 5: Phase diagrams. Left: when π^h is large. Right: when π^h is small

When $\chi < \underline{\chi}$, the saddle path coincides with the horizontal axis, meaning that the sellers do not exert any effort when χ is too small. The middle steady state is a source, around which there is an equilibrium trajectory spiraling outward. The high steady state is also a saddle point, around which there exists a saddle path that leads toward it. The right panel of Figure 5 illustrates the phase diagram of the case where there is only one steady state, the non-dollarization one. The steady state is a saddle point. There exists a unique saddle path leading towards the non-dollarization steady state.

Global dynamics In the following, I illustrate how the global dynamics depend on domestic inflation.[9] In Figure 6, domestic inflation is low . There is a unique steady state, the non-dollarization one, which is a saddle point. The green curve represents the out-of-steady-state equilibrium path. For any initial state $\chi_0 \in (0, 1]$, the only equilibrium is the one where e jumps to the green path and the economy dollarize until χ approaches 0 asymptotically.

[9]Here I show the set of global dynamics that are most relevant to hysteresis. It also shows up most frequently in numerical examples.

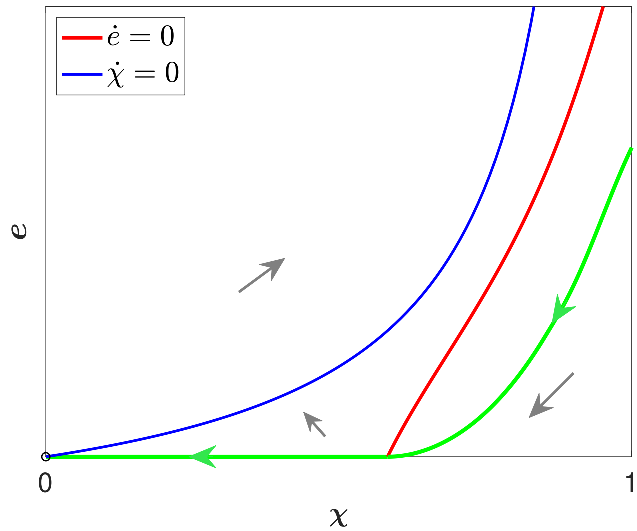


Figure 6: Case I: low inflation

In Figure 7, domestic inflation is high, but not too high. In this case, the model admits three steady states. The low and the high steady states are saddle points while the medium one is an unstable spiral. When the initial acceptability, χ_0, is sufficiently low, the equilibrium is unique and approaches the non-dollarization steady state. Similarly, when χ_0 is sufficiently high, the equilibrium is unique and approaches the high steady state. When χ_0 is in between, there exist multiple equilibria that approaches either direction, depending on peoples' beliefs.

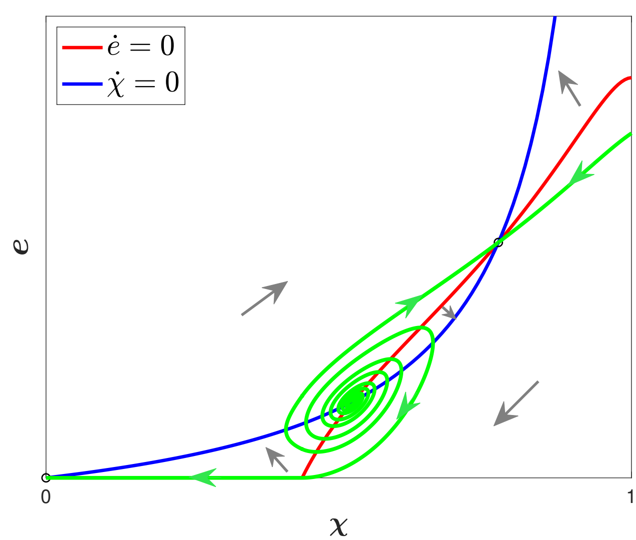


Figure 7: Case II: intermediate inflation

In Figure 8, domestic inflation is sufficiently high. The number of steady states, as well as their local stability, are the same as the previous case, but the global dynamics are different. If χ_0 is sufficiently small, then there exist multiple out-of-steady-state equilibria, one approaching the high steady state, others approaching the non-dollarization steady state. However, if χ_0 is sufficiently large, then the only perfect foresight equilibrium is the one that leads to the high steady state.

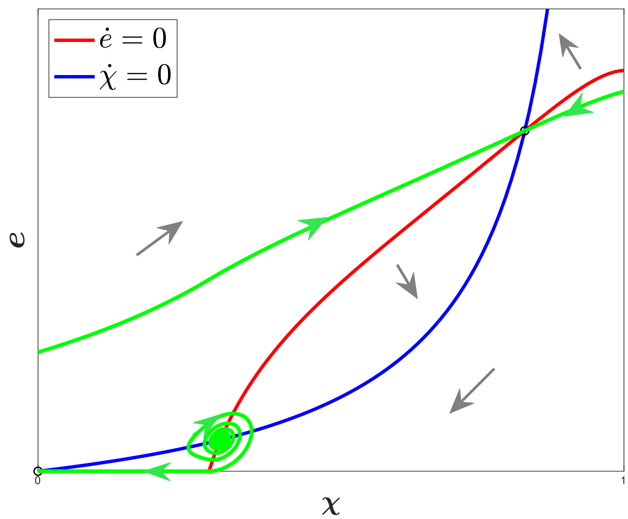


Figure 8: Case III: high inflation

The global dynamics above reveals a key feature of acceptability the dual-currency model—its sensitivity to initial conditions. In the high inflation case, when the aggregate acceptability of the foreign currency is sufficiently low, the economy may dollarize, de-dollarize, or fluctuate between the two. However, as acceptability exceeds a certain threshold, de-dollarization is no longer possible. In the intermediate inflation case, dollarization is not possible when the initial acceptability of the foreign currency is low. It is only when inflation is sufficiently low that the economy de-dollarizes regardless of initial conditions. The results are summarized in Figure 9. The red curves plot the mapping from π^h to the set of steady-state χ. The red shaded area represents the area where a spiral exist. The arrows represent the directions of χ in all possible equilibria. The three regions: low π^h, intermediate π^h, and high π^h, corresponds to the three types of global dynamics discussed above.

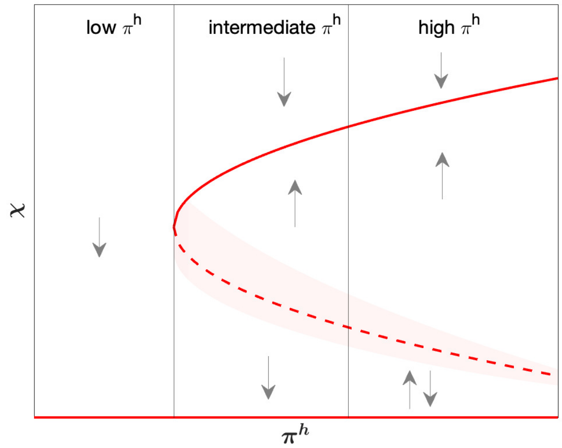


Figure 9: Inflation and the dynamics of χ

Hysteresis The left panel of Figure 10 illustrates the hysteresis in dollarization. Suppose the economy starts at steady state S_1, which lies within the intermediate π^h region, and therefore dollarization is not possible. When an unexpected increase in inflation brings π^h to a sufficiently high level, dollarization becomes possible, and the economy shifts slowly to S_2, a new steady state. Now consider an unexpected decrease in inflation that brings π^h back to its original value, then χ decreases slowly over time, until steady state S_3 is reached. Note that S_3 is a dollarization steady state. The right panel of Figure 10 plots the time path of χ for such an example.[10] The hyperinflation episode begins at $t = 5$ and ends at $t = 10$, during which the acceptability of the foreign currency grows from 0 to close to 1. However, after $t = 10$, even if domestic inflation has returned to the initial level, the foreign currency decreases only slightly, remaining highly acceptable from $t = 10$ onward.

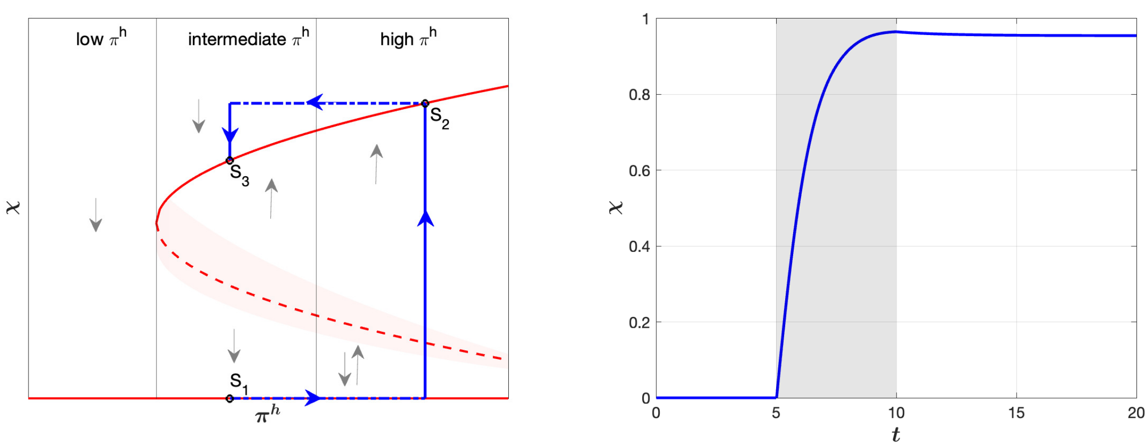


Figure 10: Hysteresis in dollarization

[10]The parameterization of the model as follows. $u(y) = 2\sqrt{y}$; $p(y) = \int_0^y \{u'(x)/[\theta u'(x) + 1 - \theta]\}dx$ with $\theta = 0.5$. $\varphi(e) = \kappa e^2$ with $\kappa = 0.06$. $\rho = 0.02$, $\alpha = 1/3$, $\delta = 0.02$, $r^h = 0.0225$, and $r^f = -0.015$. During the hyperinflation episode $r^h = 0.15$.

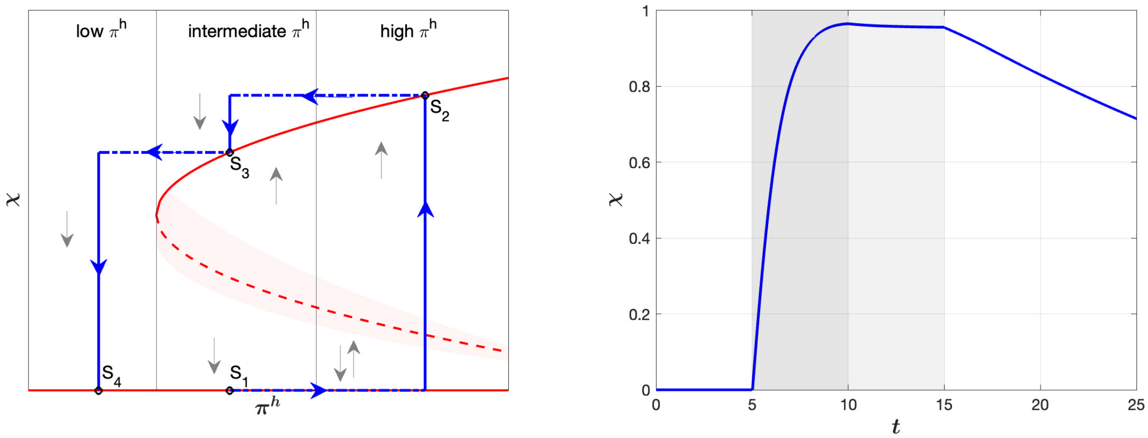


Figure 11: De-dollarization

A full de-dollarization is possible only if π^h is further decreased. In Figure 11, I plot such an example. The difference between this example and the previous one is that at time $t = 15$, r^h decreases from 0.0225 to 0.02. This causes the economy to shift into the low π^h region, and the acceptability of the foreign currency decreases slowly over time until the economy is fully de-dollarized. Note that the speed of de-dollarization can be significantly lower than the speed of dollarization, if the cost of acquiring the technology, as well as the separation rate δ, are sufficiently low.

4.2 Deterministic equilibrium under the money growth rule

In Section 4.1, the rate of return of the domestic currency is pinned down by an inflation-targeting monetary policy. In this section, I study the case where the monetary policy is implemented through a money growth rule, where the supply of the domestic currency grows at a constant rate γ.

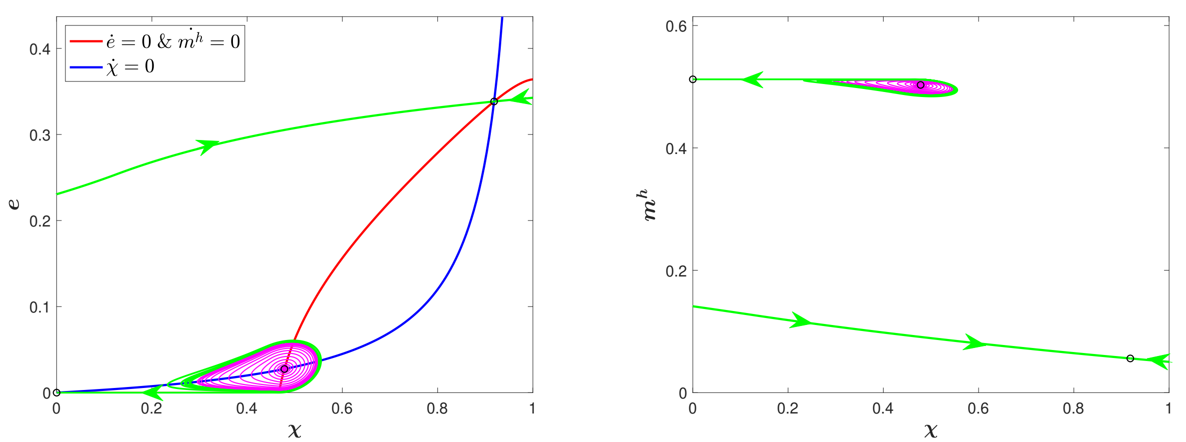
Assuming interiority, the buyer's optimal choice of the domestic currency becomes

$$\rho + \gamma - \frac{\dot{m}_t^h}{m_t^h} = \alpha \chi_t L(m_t^h + m_t^f) + \alpha(1 - \chi_t) L(m_t^h). \tag{25}$$

The rest of the equilibrium conditions are the same as in Section 4.1. An equilibrium thus solves the three dimensional ODE system (25), (21) and (11), given (20) and χ_0.

I illustrate the equilibrium with a numerical example. The parameterization of the model is as follows: $u(y) = 2\sqrt{y}$; $p(y) = \int_0^y \{u'(x)/[\theta u'(x) + 1 - \theta]\} dx$ with $\theta = 0.1$. $\varphi(e) = 1.5e^2$, $\rho = 0.02$, $\alpha = 1/2$, $\delta = 0.03$, $r^f = -0.015$, and $\gamma = 0.06$. Figure 12a projects the phase diagram onto the $\chi - e$ plane, and Figure 12b projects the equilibrium trajectories onto the $\chi - m^h$ plane. The red curve in 12a is the combination of the e and the m^h nullclines. The blue curve is the χ nullcline. The two curves intersect at three steady states, one non-dollarized and two dollarized, across which χ and e are positively correlated, and χ and m^h are negatively correlated. The non-dollarized and the highly dollarized steady states are saddle points, and the middle steady state is a sink. In both Figure 12a and 12b, the green curves represent the stable manifolds (saddle paths) of the saddle points, and the pin curves represent one trajectory that leads to the sink. There is an unstable limit cycle around the middle steady state.

As in Section 4.1, the set of equilibrium depends on the initial condition. If the acceptability is sufficiently low, dollarization, de-dollarization, and non-monotonic equilibria are all possible, depending on agents' beliefs. However, when the initial acceptability is sufficiently high, the only equilibrium is to dollarize. In short, although with slight difference, the equilibrium set here is similar to the one described in Figure 8. The rest of the analysis follows through in the same manner.



(a) Equilibrium: projected onto the $\chi - e$ plane

(b) Equilibrium: projected onto the $\chi - m^h$ plane

Figure 12: Equilibrium paths under the money growth rule

4.3 Self-fulfilling risk and sunspot equilibrium

In Sections 4.1 and 4.2, a necessary condition for dollarization is that the rate of return of the foreign currency is higher than the domestic currency. In practice, however, dollarized economies like Argentina can remain highly dollarized even during periods where the inflation rate was comparable to the US. In this section, I address the phenomenon with sunspot equilibrium: if agents believe that the domestic currency may crash in the future, dollarization is possible even when domestic inflation is sufficiently low. The setup is similar to Section 4.2 except for one modifications: there is an extrinsic shock (that is, the shock is uncorrelated with economic fundamentals such as preferences or technology) that occurs at Poisson arrival rate λ.

Let T denote the time at which the shock realizes. Consider an equilibrium where, after the realization of the extrinsic shock, agents believe that the domestic currency is worthless, i.e., $m^h = 0$, in which case the domestic currency is not useful in transactions, and the foreign currency becomes the only medium of exchange. The economy reduced to a one-asset economy with $d = 0$, as in Section 3, except that the rate of return of the asset (that is, the foreign currency) is exogenously determined. For simplicity, let's focus on the case where there is a unique steady state, and thus a unique equilibrium given the χ_T.

Before the realization of the expectation shock, the expected rate of return of the domestic currency is

$$r_t^h = \frac{\dot{\phi}_t - \lambda\phi_t}{\phi_t} = \frac{\dot{\phi}_t}{\phi_t} - \lambda = \frac{\dot{m}_t^h}{m_t^h} - (\gamma + \lambda).$$

Therefore, λ decreases the rate of return of the domestic currency. Assuming interiority, the buyer's optimal choice of the domestic currency now becomes

$$\rho + \gamma + \lambda - \frac{\dot{m}_t^h}{m_t^h} = \alpha\chi_t L(m_t^h + m_t^f) + \alpha(1 - \chi_t)L(m_t^h). \tag{26}$$

The type 0 seller's value function now solves:

$$\rho V_t^0 = \max_{e \geq 0} \left\{ \alpha\Psi(m_t^h) - \varphi(e) + e\left(V_t^1 - V_t^0\right) + \lambda\left(\widehat{V_t^0} - V_t^0\right) + \dot{V}_t^0 \right\}. \tag{27}$$

$$\rho V_t^1 = \alpha\Psi(m_t^h + m_t^f) + \delta\left(V_t^0 - V_t^1\right) + \lambda\left(\widehat{V_t^1} - V_t^1\right) + \dot{V}_t^1, \tag{28}$$

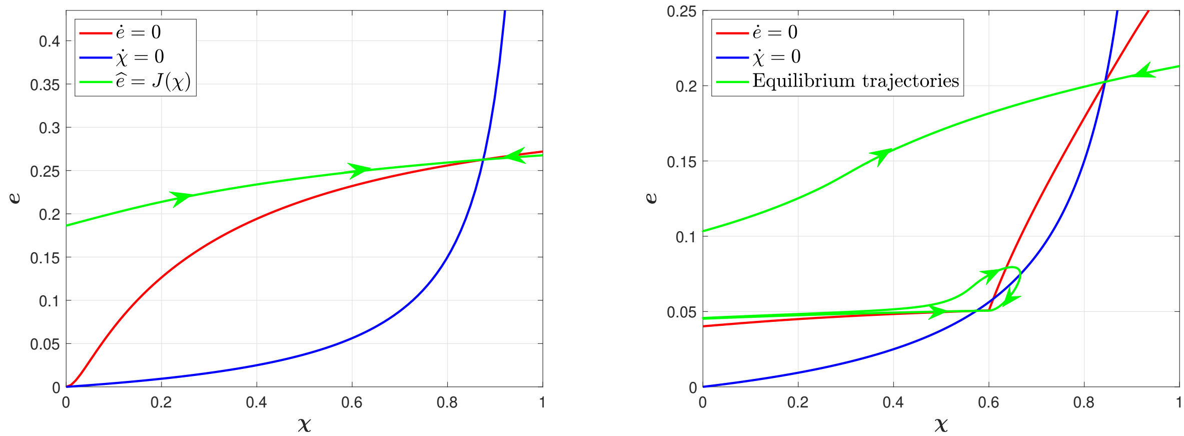
where $\widehat{V_t^0}$ (resp. $\widehat{V_t^1}$) is the continuation value of the type 0 (resp. 1) seller if the domestic currency crashes at time t. The fourth (resp. third) term on the right side of (27) (resp. (28)) is new compared to the deterministic case. It states that at rate λ, the extrinsic shock is realized, and seller's life time utility switches from V_t^0 to $\widehat{V_t^0}$ (resp. from V_t^1 to $\widehat{V_t^1}$). First order condition gives $\varphi'(e_t) = V_t^1 - V_t^0$.

Combining (27)-(28), we obtain

$$(\rho + \delta + e_t)\,\varphi'(e_t) = \alpha\left[\Psi(m_t^h + m_t^f) - \Psi(m_t^h)\right] + \varphi(e_t) + \lambda\left[\varphi'(\widehat{e}_t) - \varphi'(e_t)\right] + \varphi''(e_t)\dot{e}_t. \tag{29}$$

where $\widehat{e}_t$ is the new optimal e that the sellers will choose if the expectation shock occurs at time t. From Section 3, we know that $\widehat{e}_t$ is a function of χ_t. An equilibrium trajectory before the realization of the shock thus solves (26), (20), (29) and (11) given χ_0.

I illustrate the equilibrium with a numerical example. The parameterization of the model is as follows: $u(y) = 2\sqrt{y}$; $p(y) = \int_0^y \left\{u'(x)/[\theta u'(x) + 1 - \theta]\right\}dx$ with $\theta = 0.1$. $\varphi(e) = 1.5e^2$, $\rho = 0.015$, $\alpha = 1/3$, $\delta = 0.0375$, $r^f = -0.015$, and $\gamma = 0.015$. In a deterministic equilibrium, dollarized steady states do not exist. However, if we set $\lambda = 0.02$, then dollarization is possible, and in fact, inevitable. Figure 13a plots the equilibrium after the realization of the shock. There is a unique monetary steady state which is a saddle point, and a unique saddle path, represented by the green curve, that leads towards it. Denote $\widehat{e} = J(\chi)$ the saddle path. When the shock realizes at T, e jumps immediately to $\widehat{e}_T = J(\chi_T)$. In Figure 13b, I plot the equilibrium trajectories before the realization of the shock, projected from the $\chi - e - m^h$ space onto the $\chi - e$ plane. The equilibrium set is similar to Sections 4.1 and 4.2, except that the non-dollarized steady state is now replaced by a dollarized one.



(a) After the realization of the shock

(b) Before the realization of the shock (Projection of the equilibrium onto the $\chi - e$ plane)

Figure 13: Equilibrium after and before the realization of the shock

This exercise reveals that dollarization, and consequently, the hysteresis of dollarization, is possible even when money growth rate is under control, and domestic inflation is low. When agents believe that there is a possibility that the domestic currency may crash eventually, the buyers preemptively hold more foreign currency and less domestic currency, and the sellers preemptively invest more in acquiring the technology that allows them to accept the foreign currency. As a result, persistent dollarization can be rational even with low inflation.

5 Microfoundation

So far, the acceptability of an asset is determined by two key factors: technology acquisition and technology depreciation. In this section, I provide a microfoundation for the two processes. I interpret the technology as some knowledge that allows sellers to distinguish between authentic and fake foreign cash or to operate the point-of-sale system that accepts foreign currency denominated cards. The knowledge spreads within the population through imitation. As in Lucas Jr and Moll (2014), knowledge accumulates when sellers learn from each other. The depreciation of the technology is interpreted as the death of existing sellers. I show that (1) acceptability exhibits hysteresis only if imitation is costly—if sellers pay to meet other sellers; (2)

To formalize this, I assume that sellers meet each other randomly at rate β. When a type 0 seller meets a type 1 seller, she learns the knowledge immediately. Upon a meeting, the probability of the other seller being type 1 is χ. Therefore, learning happens more frequently if the foreign currency is more acceptable. Moreover, once a seller becomes type 1, she remains type 1 for the rest of her life.

Let δ be the rate at which the sellers die. When an existing seller (the parent) dies, she is replaced by a new seller (the child). If a seller is type 1 when she dies, the child inherit the technology from the parent with probability q. For now, I assume q is exogenous. The child of a type 0 seller is also type 0.

Solving the sellers' maximization problems following the same logic as in Section 3, we obtain

$$\dot{\Delta}_t = (\rho + \delta + \beta\chi_t)\Delta_t - \alpha\chi_t \left[\Psi(m_t^h + m_t^f) - \Psi(m_t^h)\right], \tag{30}$$

where $\Delta_t \equiv V_t^1 - V_t^0$. The buyers' demand for m^h and m^f solves equations (19)-(20). The law of motion of χ_t is now

$$\dot{\chi}_t = \beta\chi_t(1-\chi_t) - \delta(1-q)\chi_t. \tag{31}$$

Different from Section 4, the only endogenous variable on the right hand side of equation (31) is χ. Therefore, the equilibrium path of χ is independent of the other variable. Suppose without loss of generality that $\beta > \delta q$. At $\dot{\chi}_t = 0$, equation (31) becomes $\chi_t \in \left\{0, 1 - \frac{\delta(1-q)}{\beta}\right\}$. For any $\chi \in (0,1] \setminus \left\{1 - \frac{\delta(1-q)}{\beta}\right\}$, it approaches $1 - \frac{\delta(1-q)}{\beta}$ asymptotically.

Equation (31), (19), and (20) can be described as an independent system of two ODEs in terms of m^h and χ. The left panel of Figure 14 plots the phase diagram of the $\chi - m^h$ system. There is a unique steady state where the foreign currency is used and is acceptable.[11] For any $\chi \in (0,1]$ there is a unique equilibrium path, represented by the green curve, that leads to the steady state. Along the equilibrium path m^h and χ move in the opposite directions—as the foreign currency becomes more acceptable, buyers hold less domestic currency. This suggests that the right side of equation (30) as a function of Δ_t and χ. The right panel of Figure 14 plots the phase diagram of the $\chi - \Delta$ system. There is, again, an interior steady state and a unique saddle path leading towards the steady state.

The exercise suggests that if knowledge spreads within the population through imitation, which is costless, and if the rate of inheritance of the knowledge is exogenous, then acceptability does not exhibit hysteresis. In the following two subsections, I show that if learning is costly, or if inheritance is endogenous, acceptability can exhibit hysteresis.

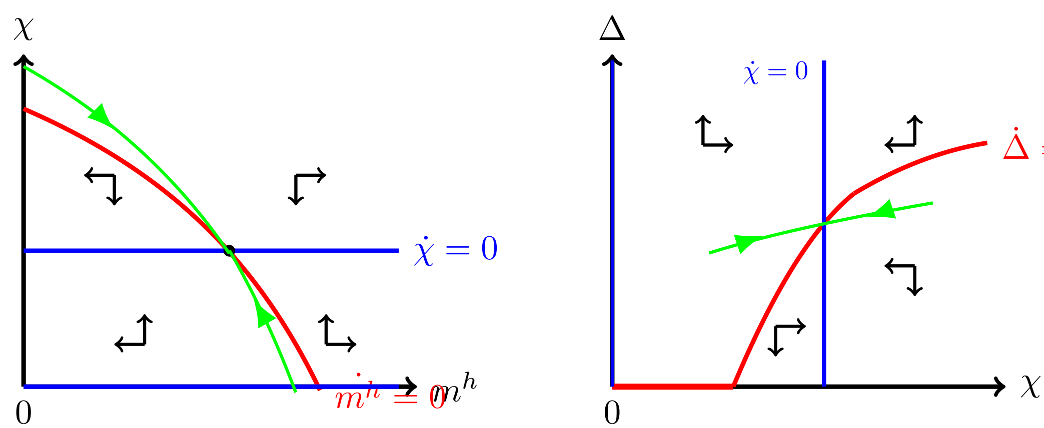


Figure 14: Phase diagrams. Left: the $\chi - m^h$ system; right: the $\chi - \Delta$ system

[11] Note that there is one additional steady state where the χ, e, and m^f are all zeros, in which case the foreign currency is not used in transactions and not valued.

5.1 Endogenous search intensity

Now suppose that in order to meet other sellers, sellers need to pay a flow cost. Indeed, it requires more effort for a shop owner to contact other sellers while maintaining normal operation of their business. I assume that a seller who pays a flow cost $\varphi(e)$ can meet a random seller at rate e. As χ increases, the probability of meeting a type-1 seller increases, and thus type 0 sellers are more wiling to invest. For simplicity, I assume, as in Section 4.1, that the monetary policy is implemented through a constant nominal interest rate. Equation (31) now becomes

$$(\rho+\delta+e\chi)\varphi'(e)-\chi\varphi(e)=\alpha\chi\left[\Psi(m_t^h+m_t^f)-\Psi(m_t^h)\right]+\varphi''(e)\dot{e}-\frac{\varphi'(e)}{\chi}\dot{\chi}. \tag{32}$$

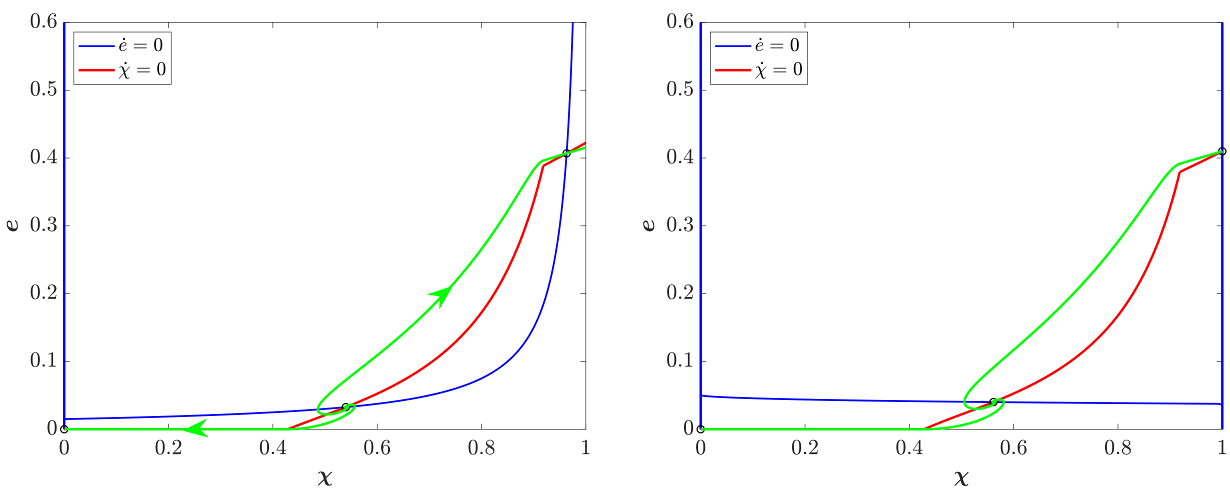
The law of motion of χ_t is

$$\dot{\chi}_t=e_t\chi_t(1-\chi_t)-\delta(1-q)\chi_t. \tag{33}$$

And therefore, when $\chi\neq 0$, (32) can be rewritten as

$$(\rho+\delta q+e)\varphi'(e)-\chi\varphi(e)=\alpha\chi\left[\Psi(m_t^h+m_t^f)-\Psi(m_t^h)\right]+\varphi''(e)\dot{e}. \tag{34}$$

Figure 15 illustrates the equilibrium with a numerical example. The model is parameterized as follows: $u(y)=A\left[(y+b)^{1-\sigma}-b^{1-\sigma}\right]/(1-\sigma)$ with $b=0.0001$ and $\sigma=0.8$, $\varphi(e)=2e^2$, and $p(y)=\theta y+(1-\theta)u(y)$ with $\theta=1/2$, $\alpha=1$, $\delta=0.05$, $\rho=0.05$, $r^f=-0.02$, $r^h=-0.1$, and $q=0.7$. I plot the equilibrium in Figure 15a. Similar to Section 4.1, the equilibrium set is sensitive to initial conditions, suggesting that the system exhibits hysteresis.



(a) $q=0.7$ (b) $q=\chi^{3/4}$

Figure 15: Equilibrium: endogenous search intensity

The results hold through even if the rate at which new sellers inherit the knowledge from their parents, q is endogenously determined. Indeed, whether some knowledge can be passed on to the next generation is often times endogenously determined by, for instance, how common this knowledge is, and how often it is used, etc. Therefore, in Figure 15b, I consider the case where q is an increasing function of χ, using $q = \chi^{3/4}$ as an example. Note that as $\chi \to 1$, $q \to 1$, suggesting that the the knowledge is perfectly passed on across generations if the foreign currency is fully acceptable. As a result, the highly dollarized steady state now has $\chi = 1$, i.e., all sellers are able to accept the foreign currency.

6 Quantitative analysis

The dual currency model shows that a temporary inflationary episode can result in persistent dollarization. In this section, I quantify the effect by calibrating the model to Argentina data from the end of 1989 to the end of 1992, and show that the model can explain the persistent dollarization in Argentina.

6.1 Calibration

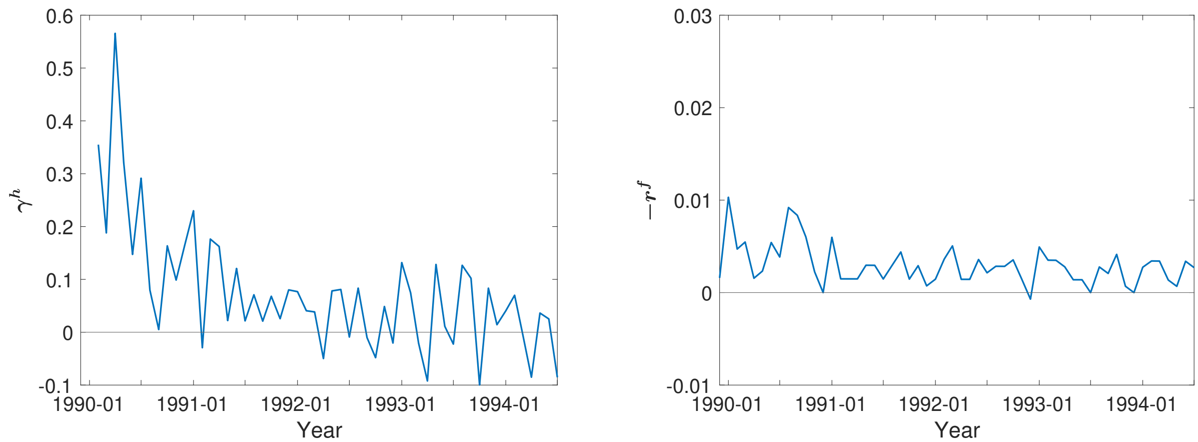
The version of the model I use is the one in Section 4.3 with one modification to accommodate for the establishment of the currency board. Until April 1991, I take the growth rate of peso supply, γ^h, as given; starting May 1991, I assume that the rate of return of the Argentine pesos is pinned down by the US inflation rate. Consider the following functional forms: $u(y) = A\left[(y+b)^{1-\sigma} - b^{1-\sigma}\right]/(1-\sigma)$, $\varphi(e) = \kappa e^{\eta}$, and $p(y) = \int_0^y \left\{u'(x)/[\theta u'(x) + 1 - \theta]\right\} dx$. There are ten parameters to calibrate: $(A, \alpha, \sigma, \theta, \delta, \kappa, \eta, \lambda, \rho, b)$. Two parameters, ρ and b, are set directly. Following Caravello et al. (2023), I set the rate of time preference ρ to 0.0033 (a unit of time is one month), so that the annual discount rate is approximately 4%. The parameter b is set to 0.0001 in order to guarantee $u(0) = 0$. The remaining eight parameters are calibrated jointly.

Data The sample period is December 1989 to December 1992. I use five time series and two data points in this calibration. The five time series are: (1) The aggregated Currency and Monetary Instrument Reports (CMIR) data from 1988 to the end of 1992 reported in Kamin and Ericsson (2003); (2) the monthly inflation rate in Argentina from the Central Bank of Argentina; (3) the monthly inflation rate in the United States from Federal Reserve Economic Data (FRED); (4) the nominal exchange rate (monthly average of daily rates) between Argentine Pesos and US dollars from FRED; and (5) the monthly M0 in Argentina from December 1989 to December 1992 from tradingeconomics.com, who collected the data from the Central Bank of Argentina.[12] The two data points are: the real GDP in Argentina in 1992 (from FRED), and (2) the average markup in Argentina in 1992 from De Loecker and Eeckhout (2018).

Two exogenous variables are obtained directly from the data. The growth rate of domestic currency, γ^h is proxied by the growth rate of M0 in Argentina. The rate of return of the foreign currency, r^f, is proxied by the negative of the inflation rate in the United States during the sample period. Figure 16a and 16b plot the two time series. Since the US inflation rate was stable and low throughout the sample period, for simplicity I assume that r^h is constant at -0.00286, the negative of the average US inflation rate from January 1985

[12] Unfortunately, the M0 data is not available before December 1989.

to December 1995. From May 1991 to December 1992, I assume that $r^f = r^h$ due to the monetary board. From December 1989 to April 1991, I assume that agents are able to perfectly foresee the future path of the monetary policy rules until April 1991, and believe that starting May 1991 the exchange rate will be fixed forever.



(a) M0 growth rate (monthly, Argentina), a proxy for γ^h

(b) Inflation rate (monthly, US), a proxy for $-r^h$

Figure 16: Exogenous variables

Measure of dollarization I define the dollarization ratio (hereafter DR) as follows:

$$DR \equiv \frac{m^h}{m^h + m^f} = \frac{FCC}{DCC + FCC},$$

where FCC stands for foreign currency in circulation, and DCC stands for domestic currency in circulation. FCC is taken from the CMIR data in Kamin and Ericsson (2003). To estimate FCC, they estimate the stock of US dollars circulating in Argentina based on the flow of US dollars between Argentina and the US through the Currency and Monetary Instrument Reports (CMIRs) collected and aggregated by the US Treasury Department. DCC is proxied by the M0 in Argentina.

Calibration strategy I calibrate the eight parameters, $(\alpha, A, \sigma, \kappa, \eta, \delta, r^f, \lambda)$, jointly by minimizing the following objective function:

$$\min_{\Theta} \sum_t \left[DR_t^M(\Theta) - DR_t^D\right]^2 + \left[\frac{\mu_{1992}^M - \mu_{1992}^D}{\mu_{1992}^D}\right]^2 + \left[\frac{Y_{1992}^M - Y_{1992}^D}{Y_{1992}^D}\right]^2 + \left[\frac{m_{f,1992}^M - m_{f,1992}^D}{m_{f,1992}^D}\right]^2, \tag{35}$$

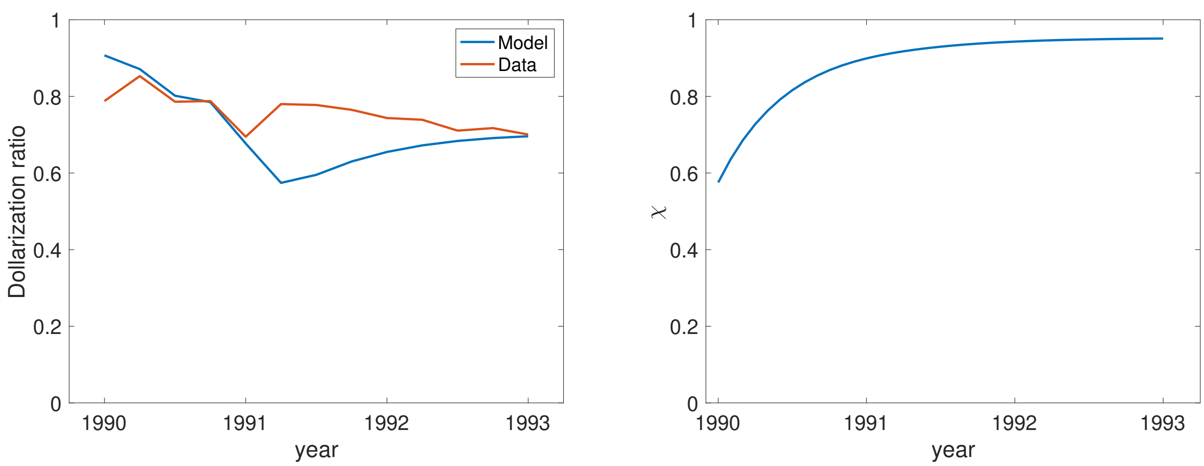
where $\Theta = (\alpha, A, \sigma, \kappa, \eta, \delta, r^f, \lambda)$. The objective function is the sum of four terms. The first term is the sum of squared deviation of the quarterly dollarization ratio from the data. The second term is the squared percentage deviation of the sellers' markup at the end of 1992 from the data. The second term is the squared percentage deviation of the real GDP in 1992 from the data. And the fourth term is the squared percentage deviation of the USD holdings in Argentina at the end of 1992 from the data. Moreover, whenever there are multiple equilibria, we choose the one that leads to the highly dollarized steady state (if feasible).

The following table lists the targets I use:

Parameter	Value	Definition	Target	Definition	value	Source
A	6.7766	Scaling factor of the utility function		Quarterly dollarization ratio		
α	0.4061	Matching rate	DR^D	from the end of 1990 to the	See KE	KE
σ	0.5962	CRRA coefficien		end of 1992		
θ	0.4703	Buyer's bargaining powe				
δ	0.0075	Rate of technology depreciation				
k	560.3002	Scaling factor of the cost function	Y^D_{1992}	Real GDP in 1992	\$228.78 bn	FRED
η	3.567	Power of the cost function	μ^D_{1992}	Average markup in 1992	1.48	DE
λ	0.0134	Perceived rate of collapsion	$m^D_{f,1992}$	USD holdings at the end of 1992	\$26 bn	KE

Table 1: Calibration: targets and results

Figure 17a compares the dollarization ratio in the model and in the data. The model is able to capture the persistence of dollarization. The biggest deviation is around the time the currency board was established, where the model predicts a lower dollarization ratio than the data, possibly because the agents in the model have perfect information on the monetary policy in the future whereas in reality this information may not be complete. Figure 17b reveals why dollarization was persistent during the sample period: despite all the effort put in pegging the exchange rate, the acceptability of US dollars continued increasing.



(a) Dollarization ratio: model v.s. data

(b) Implied acceptability of US dollars

Figure 17: Calibration result

7 Asset liquidity and monetary policy transmissions

In this section, I study the role of asset liquidity as a channel through which monetary policy affects the real economy. Consider an economy where a fiat currency, m, coexists with a Lucas tree, a, which pays a positive dividends d. The supply of the Lucas tree is fixed at A, with $dA/\rho < p(y^*)$, and the supply of the fiat money grows at a constant rate γ, i.e., $\gamma = \dot{M}_t/M_t$. Assume that all sellers are able to accept the fiat money, but only a fraction χ of sellers are equipped with the technology to accept the real asset.

Let ϕ^a and ϕ^m denote the value of the Lucas tree and the fiat currency, respectively. Therefore, the rate of return of holding the real asset and the fiat money are

$$r^a = \frac{d + \dot{\phi^a}}{\phi^a}, \;\; r^m = \frac{\dot{\phi^m}}{\phi^m}.$$

The buyer's value function now solves:

$$\rho W_t = \max_{a,m\geq 0} \left\{ -\left(\rho - r_t^a\right) a - \left(\rho - r_t^m\right) m + \alpha \left[\chi_t \Gamma\left(a+m\right) + \left(1-\chi_t\right) \Gamma\left(m\right)\right] + \tau_t + \dot{W}_t \right\}. \tag{36}$$

where a and m are a buyer's real asset holdings and real money balances, respectively. The first two terms on the right hand side are the opportunity cost of holding a and m. The buyer is matched randomly with a seller at rate α. With probability χ, the seller is type 1, in which case both a and m are accepted. With probability $1-\chi$, the seller is type 0, in which case only m is accepted. When market clears, $a_t = A\phi_t^a$ and $m_t = M_t\phi_t^m$. Under the market clearing conditions, the first order conditions are:

$$\rho - \frac{dA + \dot{a}}{a} \;\; = \;\; \alpha\chi L(m+a), \tag{37}$$

$$\rho + \gamma - \frac{\dot{m}}{m} \;\; \geq \;\; \alpha\chi L(m+a) + \alpha(1-\chi)L(m), \text{ “}='' \text{ if } m > 0. \tag{38}$$

The left side of (37)-(38) is the flow cost of holding the real asset and the fiat money under market clearing. The right side of (37) is the expected marginal benefit of holding the asset, measured by the product of α, the frequency of trade, χ, the probability of meeting a type 1 seller, and the liquidity premium. Similarly, the right side of (38) is the expected marginal benefit of holding the money, measured by the expected liquidity premium from two types of meetings where the fiat money is used: meetings with type 1 sellers and meetings with type 0 sellers. In the latter case the buyer cannot make payment offers that exceed m.

The seller's optimization problem reduces to the following ODE:

$$\varphi''(e)\dot{e} = (\rho + \delta + e)\varphi'(e) - \varphi(e) - \alpha\left[\Psi(m+a) - \Psi(m)\right], \tag{39}$$

where the terms on the ride side between the brackets is the increase in trade surplus from acquiring the knowledge and becoming a type 0 seller. The law of motion of χ solves equation (11). Given an initial state χ_0, an equilibrium is a list of time paths (a_t, m_t, e_t, χ_t) that solves (37), (38), (39), (11), and the transversality condition

$$\lim_{t\to\infty} \mathbb{E}_0[e^{-\rho t}(m_t + a_t)] = 0. \tag{40}$$

Consider a passive monetary policy, where the money authority changes the rate of money growth, γ. I study numerically the effects of a monetary policy shock. The model is parameterized as follows. The utility function is $u(y) = 2\sqrt{y}$. I assume proportional bargaining, $p(y) = \theta y + (1-\theta)u(y)$, with $\theta = 0.5$. $\varphi(e) = 4e^2$. $\rho = 0.05$, $\alpha = 1$, $dA = 0.01$, and $\delta = 0.02$. Initially, the money growth rate $\gamma_0 = 0.01$, and the economy is at the steady state, with

$$(m_0^*, \; a_0^*, \; e_0^*, \; \chi_0^*) = (1.13, \; 0.28, \; 0.027, \; 0.57).$$

Define y_2 (resp. y_m) the output in a meeting between a buyer and a type 1 (resp. type 0) seller. Initially,

$$(y^*_{2,0},\ y^*_{m,0}) = (0.91,\ 0.65),$$

which implies that the initial expected output is

$$\mathbb{E}(y^*_0) = \chi^*_0 y^*_{2,0} + (1 - \chi^*_0) y^*_{m,0} = 0.80.$$

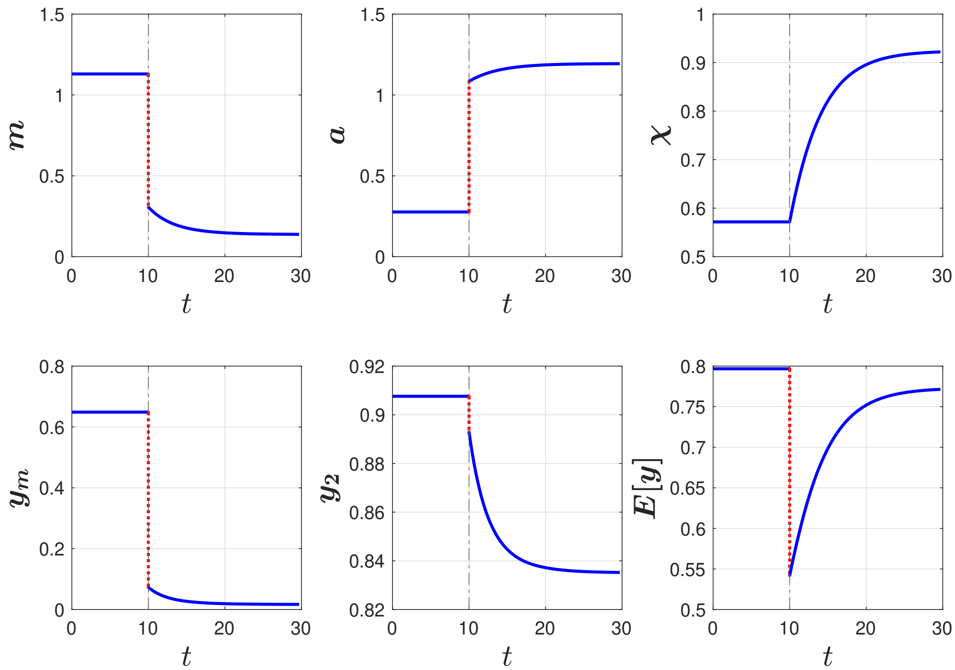


Figure 18: The effects of an unexpected increase in γ (Red dotted parts represent discrete jumps.)

At $t = 10$, γ jumps to 0.05 unexpectedly and permanently. Figure 18 plots the responses of m, a, χ, y_2, y_m, and $\mathbb{E}(y)$, to the increase in γ.[13] At the time of the shock, the real money balances m jumps down to 0.31 immediately, while the market capitalization of the asset, a, jumps up to 1.08 immediately, suggesting that the asset now becomes more desirable as the money becomes more costly to hold. The trends continue in the long run, with m reaching a new and lower steady state, 0.14, and a reaching a higher steady states, 1.20. The acceptability of the asset, χ, adjusts slowly and evolves over time to a higher new steady state, i.e., more agents become familiarized with the real asset. In terms of output, y_m is pinned down by m and responds to the monetary policy shock in a similar way, jumping from 0.65 down to 0.073 immediately and slowly transitions to the new steady state, 0.017. In contrast, y_2 is determined by $m + a$, the total liquidity. The decrease in m and the increase in a partially cancel out. As a result, y_2 does not respond as much to the shock. When the shock hits, y_2 jumps from 0.91 to 0.89, and continuously decreases over time until it

[13] The response of e is not plotted in Figure 18, but e's response is qualitatively similar to a, i.e., at $t = 0$, e jumps immediately from 0.027 to 0.22, and then evolves slowly over time to the new steady state, 0.24.

reaches the new steady state, 0.83. The bottom right panel of Figure 18 plots the response of $\mathbb{E}[y]$ to the shock, which is non-monotone. When the shock hits at $t = 10$, $\mathbb{E}[y]$ jump down from 0.80 to 0.54. However, after $t = 10$, $\mathbb{E}[y]$ starts recovering, moving up slowly over time until it reaches the new steady state at 0.77.

This exercise suggests that the effects of monetary shocks are mitigated by the the reactions of the liquidity of assets. When the central bank increases the money growth rate, and thereby targeting a higher inflation rate, money becomes more costly to hold, and real balances drop. In an economy where the fiat money is the only medium of change, an unexpected increase in the money growth rate may create a significant drop in output in the long run. However, this model suggests that when money becomes more costly to hold, more sellers will be willing to invest in acquiring the technology that allows them to accept alternative means of payments, and the liquidity of other assets increases. Over time, output recovers, and the long-run negative effect of inflation can be much lower than the case where only money can be used in transactions.

8 Conclusion

In this paper, I present a continuous-time New Monetarist model of the dynamics of asset acceptability. I model asset acceptability as a capital, which formalizes the idea that it takes time to adopt or to abandon an asset. I study how asset acceptability evolves over time, and how asset prices and output evolve accordingly, converging to a stable monetary steady state. When there are multiple steady states, asset acceptability exhibits hysteresis.

This hysteresis in asset acceptability provides a novel explanation for the hysteresis in dollarization. In a dual-currency economy, the acceptability of the foreign currency may increase following an increase in domestic inflation. However, once the economy is sufficiently dollarized, then it may not be able to de-dollarize, even if inflation returns to its original value. Moreover, if agents form pessimistic beliefs on the domestic currency, believing that it will crash eventually, then the economy can end up highly dollarized even if domestic inflation is low.

I also explore the role of asset acceptability on monetary policy transmissions, both in the long run and in the short run. I show that the reactions of asset acceptability mitigates the effects of monetary policies, especially in the long run. The framework can be easily applied to other new payment method, e.g. CBDC, cryptocurrencies, etc., which is left to future work.

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Appendix A Proofs of propositions

Proof of Proposition 1 (Part 1). A steady-state equilibrium is triple (m^*, e^*, χ^*) that solves

$$\begin{aligned} \rho + \gamma - \frac{dM}{m_t} &= \alpha \chi_t L(m_t), && (41) \\ (\rho + \delta + e_t)\, \varphi'(e_t) - \varphi(e_t) &= \alpha \Psi(m_t), && (42) \\ e_t(1 - \chi_t) &= \delta \chi_t, && (43) \end{aligned}$$

where $\gamma = 0$ if $d > 0$. Equations (41), (42), and (43) give increasing relationships between m^* and χ^*, e^* and m^*, and χ^* and e^*, respectively. In particular, the left hand side of equation (42) is increasing in e because $e\varphi'(e) - \varphi(e)$ is non-negative and strictly increasing for all $e \in \mathbb{R}_+$. Combining (41) and (42), we obtain a mapping from χ^* to e^* that is strictly increasing. Let $e^* = g_1(\chi^*)$ define the explicit form of the mapping. Let $e^* = g_2(\chi^*)$ define the explicit form of (43). A steady state is an intersection between $e^* = g_1(\chi^*)$ and $e^* = g_2(\chi^*)$. It can be checked that both g_1 and g_2 are non-decreasing functions of χ. In particular, when $m < dM/\rho$, g_1 is strictly increasing. Moreover,

$$g_1(0) > 0 = g_2(0) \tag{44}$$

$$g_1(1) < \infty = g_2(1) \tag{45}$$

(44) and (44) imply that an interior (monetary) steady state must exist, and there can be an odd number of interior steady states.

■

Proof of Proposition 1 (Part 2). We start by considering the case where $dM/\rho < p\,(y^*)$, i.e., when liquidity is scarce. We focus on cases where there is a unique steady state, (m^*, e^*, χ^*). The Jacobian matrix of the system evaluated at the steady state is

$$J = \begin{bmatrix} \frac{\partial \dot{m}}{\partial m} & 0 & \frac{\partial \dot{m}}{\partial \chi} \\ \frac{\partial \dot{e}}{\partial m} & \frac{\partial \dot{e}}{\partial e} & 0 \\ 0 & \frac{\partial \dot{\chi}}{\partial e} & \frac{\partial \dot{\chi}}{\partial \chi} \end{bmatrix}_{(m^*, e^*, \chi^*)} = \begin{bmatrix} j_{11} & 0 & j_{13} \\ j_{21} & j_{22} & 0 \\ 0 & j_{32} & j_{33} \end{bmatrix} = \begin{bmatrix} \rho - \alpha\chi^* L(m^*) - \alpha\chi m^* L'(m^*) & 0 & -\alpha m^* L(m^*) \\ -\frac{\alpha}{\varphi''(e^*)}\left(1 - \frac{1}{p'(y(m^*))}\right) & \rho + \delta + e^* & 0 \\ 0 & 1 - \chi^* & -(e^* + \delta) \end{bmatrix}.$$

Let λ_1 λ_2 and λ_3 denote the three eigenvalues of matrix J. Then,

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 = tr(J) &= \rho - \alpha\chi^* L(m^*) - \alpha\chi^* m^* L'(m^*) + \rho + \delta + e^* - (e^* + \delta), \\ &= \rho + \frac{dM}{m^*} - \alpha^* \chi^* m^* L'(m^*) > 0. \end{aligned}$$

Moreover,

$$\lambda_1 \lambda_2 \lambda_3 = det(J) = \frac{\partial \dot{m}}{\partial m} \cdot \frac{\partial \dot{e}}{\partial e} \cdot \frac{\partial \dot{\chi}}{\partial \chi}\bigg|_{(m^*, e^*, \chi^*)} + \frac{\partial \dot{e}}{\partial m} \cdot \frac{\partial \dot{\chi}}{\partial e} \cdot \frac{\partial \dot{m}}{\partial \chi}\bigg|_{(m^*, e^*, \chi^*)}.$$

To continue discussing the sign of $\lambda_1\lambda_2\lambda_3$, note that $g_1(\chi)$ crosses $g_2(\chi)$ from above, and thus,

$$
\begin{aligned}
&\frac{\partial e}{\partial m}|_{\dot{e}=0,(m^*,e^*,\chi^*)} \cdot \frac{\partial m}{\partial \chi}|_{\dot{m}=0,(m^*,e^*,\chi^*)} < \frac{\partial e}{\partial \chi}|_{\dot{\chi}=0,(m^*,e^*,\chi^*)} \\
\Longleftrightarrow\ & \frac{\alpha}{(\rho+\delta+e^*)\varphi''(e^*)}\Big[1-\frac{1}{p'(y(m^*))}\Big] \cdot \frac{\alpha m^* L(m^*)}{\rho-\alpha\chi L(m^*)+\alpha\chi m^* L'(m^*)} < \frac{\delta}{(1-\chi^*)^2} \\
\Longleftrightarrow\ & \frac{(-j_{21})(-j_{13})}{j_{11}j_{22}} < \frac{\delta}{j_{32}^2} \\
\Longrightarrow\ & j_{21}j_{13}j_{32} < \frac{j_{12}j_{22}\delta}{j_{32}} < j_{11}j_{22}(-j_{33}) \\
\Longleftrightarrow\ & \lambda_1\lambda_2\lambda_3 = \det(J) = j_{11}j_{22}j_{33} + j_{21}j_{13}j_{32} < 0
\end{aligned}
$$

And thus, if all eigenvalues are real, then two of them must be positive while one of them is negative. The next step is to show that the eigenvalues are real. To see this, note that an eigenvalue λ solves the following equation:

$$(j_{11}-\lambda)(j_{22}-\lambda)(j_{33}-\lambda) + j_{13}j_{21}j_{22} = 0,$$

or

$$(\lambda-j_{11})(\lambda-j_{22})(\lambda-j_{33}) = j_{13}j_{21}j_{22}. \tag{46}$$

Define $h(\lambda) \equiv (\lambda-j_{11})(\lambda-j_{22})(\lambda-j_{33})$. We first observe that

$$h(0) = -j_{11}j_{22}j_{33} > j_{21}j_{13}j_{32}. \tag{47}$$

Next, we observe that as $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} h(\lambda) = \infty > j_{21}j_{13}j_{32}. \tag{48}$$

And finally, we observe that

$$h(j_{11}) = h(j_{22}) = h(j_{33}) = 0 < j_{21}j_{13}j_{32}. \tag{49}$$

Given (47), (48), (49), continuity implies that there exist a couple (λ_1, λ_2), with $\lambda_1 \in (j_{33}, 0)$ and $\lambda_1 \in (\max\{j_{11}, j_{22}\}, \infty)$, such that both λ_1 and λ_2 are solutions to equation (46). Now that we already find two real solutions to (46), the third solution cannot be complex.

Therefore, the Jacobian matrix of the linearized system at the unique steady state (a^*, e^*, χ^*) has two positive eigenvalues and one negative eigenvalue, and thus there should be a one-dimensional stable manifold around the steady state.

Now we show that this stable manifold exists for all $\chi \in [0,1] \setminus \{\chi^*\}$. The goal is to show that there does not exist cases where the equilibrium path requires that e or m fall outside of their domain for some χ. By assumption, e is defined from 0 to ∞. Now suppose that at a time point $\tilde{t}$, $m \geq p(y^*)$. Then (6) becomes

$$\rho - \frac{dM + \dot{m}_{\tilde{t}}}{m_{\tilde{t}}} = 0,$$

and thus

$$\dot{m}_{\tilde{t}} = \rho m_{\tilde{t}} - dM,$$

which violates the transversality condition. Therefore, along the equilibrium path m_t is always less than $p(y^*)$. Therefore, e or m will not fall outside of their domain for some $\chi \in [0,1] \setminus \{\chi^*\}$. ■

Proof of Lemma 1. We focus on equilibria where m^h and m^f are not both zero, and prove by contradiction.

1. First, we show that when $\chi \in [0, \underline{\chi}]$, $m^h > 0 = m^f$. Suppose instead that $m^h = 0$, then $m^f > 0$. Equations (19) and (20) become

$$\rho - r^h \geq \alpha\chi L\left(m^f\right) + \alpha(1-\chi)L\left(0\right), \tag{50}$$

$$\rho - r^f = \alpha\chi L\left(m^f\right). \tag{51}$$

   Subtracting (51) from (50), we obtain

$$r^f - r^h \geq \alpha(1-\chi)L(0) \implies \chi \geq 1 - \frac{r^f - r^h}{\alpha L(0)} = \overline{\chi},$$

   which is a contradiction. Therefore, when $\chi \in [0, \underline{\chi}]$, m^h must be strictly positive. Now we show that m^h and m^f cannot both be strictly positive. To see this, suppose instead that $m^f > 0$, then both (19) and (20) hold at equality. Subtracting (20) from (19), we obtain

$$r^f - r^h = \alpha(1-\chi)L(m^h).$$

   Rearranging, we obtain

$$\chi = 1 - \frac{r^f - r^h}{\alpha L(m^h)} > 1 - \frac{r^f - r^h}{\alpha L(m^h + m^f)} = 1 - \frac{r^f - r^h}{\rho - r^f} \cdot \chi$$
$$\implies \left(1 + \frac{r^f - r^h}{\rho - r^f}\right)\chi > 1 \implies \chi > \frac{\rho - r^f}{\rho - r^h} = \underline{\chi},$$

   which is a contradiction. Therefore, when $\chi \in [0, \underline{\chi}]$, it must be that $m^h > 0 = m^f$.

2. Next, we show that when $\chi \in \left(\underline{\chi}, \overline{\chi}\right)$, both m^h and m^f are strictly positive. From above, we know that if $m^h = 0$, then $\chi \geq \overline{\chi}$, which is a contradiction. In the following, we show that if $m^f = 0$, then $\chi \leq \underline{\chi}$, which is also a contradiction. Suppose instead that $m^f = 0$, then $m^h > 0$. Equations (19) and (20) become

$$\rho - r^h = \alpha\chi L\left(m^h\right) + \alpha(1-\chi)L\left(m^h\right) = \alpha L(m^h), \tag{52}$$

$$\rho - r^f \geq \alpha\chi L\left(m^h\right). \tag{53}$$

   Substituting (52) into (53), we obtain

$$\rho - r^f \geq \chi\left(\rho - r^h\right) \implies \chi \leq \frac{\rho - r^f}{\rho - r^h} = \underline{\chi},$$

   which is a contradiction. Therefore, when $\chi \in \left(\underline{\chi}, \overline{\chi}\right)$, it must be that both m^h and m^f are strictly positive.

3. Finally, we show that when $\chi \in [\overline{\chi}, 1]$, an inflation-targeting monetary policy is unsustainable. From above, we know that if $m^f = 0$, then $\chi \leq \underline{\chi}$. Therefore, it must be that $m^f > 0$. Now we show that m^h and m^f cannot both be strictly positive. To see this, suppose instead that $m^h > 0$, then both (19) and (20) hold at equality. Subtracting (20) from (19), we obtain

$$r^f - r^h = \alpha(1-\chi)L(m^h).$$

Rearranging, we obtain

$$\chi = 1 - \frac{r^f - r^h}{\alpha L(m^h)} < 1 - \frac{r^f - r^h}{\alpha L(0)} = \overline{\chi},$$

which is a contradiction. Therefore, when $\chi \in [\overline{\chi}, 1]$, it must be that $m^f > 0 = m^h$. However, by definition,

$$r^h = \frac{\dot{m}^h}{m^h} > 0,$$

which is not possible when m^h is fixed at 0. Therefore, an inflation targeting monetary policy is not sustainable when $\chi \geq \bar{\chi}$.

■

Proof of Proposition 2.

1. We start by checking that $(e, \chi) = (0, 0)$ is always a steady state. When $\chi = 0$, equation (24) gives $e = 0$; Lemma 1 implies that $m^h > 0 = m^f$, which implies that the right hand side of equation (23) is zero, and thus the left hand side of (23) is zero, meaning that $e = 0$. As a result, $(e, \chi) = (0, 0)$ satisfies both (23) and (24), as well as (19) and (20). When $(e, \chi) = (0, 0)$, then conditional on $m^{h*} \neq 0$, m^{h*} solves

$$\rho + \pi^h = \alpha L(m^{h*}). \tag{54}$$

Under gradual bargaining and the Inada conditions, there must be an interior solution to equation (54). Now check the possibility for other steady states. A steady state is an intersection between the χ isocline and the e isocline. Let $e = \tilde{g}_1(\chi)$ denote the explicit form of the e isocline and $e = \tilde{g}_2(\chi)$ denote the explicit form of the χ isocline. Then, $\tilde{g}_1(\chi)$ and $\tilde{g}_2(\chi)$ do not intersect within the interval $(0, \underline{\chi})$, as $\tilde{g}_1(\chi) = 0 < \tilde{g}_2(\chi)$ within the region. When $\chi = \underline{\chi}$, $\tilde{g}_1(\underline{\chi}) = 0 < \tilde{g}_2(\underline{\chi})$. When $\chi = 1$, $\tilde{g}_1(\chi) = \tilde{g}_1(1) < \infty = \tilde{g}_2(1)$. $\tilde{g}_1(1)$ is finite since the seller's gain from trade is finite. Therefore, there can be $2k (k \in \mathbb{N})$ steady states within the region $\chi \in [\underline{\chi}, 1]$, and $2k + 1 (k \in \mathbb{N})$ steady states in total.

2. Now we prove the second part of the lemma. We start by showing that when π^h increases, the e isocline shift up. To see this, consider an increase in π^h from π_1 to π_2. Then $\underline{\chi}$ decreases from $\underline{\chi}_1 = (\rho - r^f)/(\rho + \pi_1)$ to $\underline{\chi}_2 = (\rho - r^f)/(\rho + \pi_2)$. Therefore, the e isocline shifts up for $\chi \in [\underline{\chi}_2, \underline{\chi}_1]$. Now consider $\chi \in (\underline{\chi}_1, 1]$, in which case both m^f and m^h are positive according to Lemma **??**. Therefore,

both (19) and (20) hold at equality, and can be rewritten as

$$\rho - r^f = \alpha\chi L(m^h + m^f), \tag{55}$$

$$\pi^h + r^f = \alpha(1-\chi)L(m^h). \tag{56}$$

Fixing χ, an increase in π^h implies that m^h decreases, while $m^h + m^f$ stays constant. Since $\Psi(m)$ is an increasing function of m, this implies that $\Psi(m^h + m^f) - \Psi(m^h)$ increases. From (23), e also increases, i.e., the e isocline shifts up.

Next, we show that the two isoclines intersect multiple times when $\pi^h \to \infty$. When $\pi^h \to \infty$, $\underline{\chi} \to 0$. Let χ^* be an arbitrary value between $\underline{\chi}$ and 1. From (24), $\tilde{g}_2(\chi^*) = \delta\chi/(1-\chi)$. From (55) and (56), m^h and m^f, and thus $\Psi(m^h + m^f)$ and $\Psi(m^h)$ are pinned down uniquely, with $\Psi(m^h + m^f) > \Psi(m^h)$, and therefore the right hand side of (23) is strictly positive. Define $f(e) = (\rho + e)\varphi'(e) - \varphi(e)$. One can show that $f(e)$ is an increasing function of e and ranges from 0 to ∞ and e goes from 0 to ∞. Then, when $\delta \to 0$, $\tilde{g}_2(\chi^*) \to f^{-1}\left[\alpha\left(\Psi(m^h + m^f) - \Psi(m^h)\right)\right] > 0$. On the other hand, as $\delta \to 0$, $\tilde{g}_1(\chi^*) \to 0$. Therefore $\tilde{g}_2(\chi^*) > \tilde{g}_1(\chi^*)$ and $\tilde{g}_2(1) < \tilde{g}_1(1) = \infty$, and thus there must be an intersection between $\tilde{g}_1$ and $\tilde{g}_2$ at a point where $\chi \in (\chi^*, 1)$. By continuity, such an intersection exists when δ is sufficiently small and when π^h is sufficiently large.

Now, when $\pi^h \to -r^f$, $\underline{\chi} \to 1$, in which case the e isocline becomes a straight line $e = 0$ for $\chi \in [0, 1]$, and the two isoclines do not intersect except at $(0, 0)$. By continuity, there exists an $\varepsilon_2 \in \mathbb{R}_{++}$ such that for all $\pi^h \in (-r^f - \varepsilon_2, -r^f]$, the two isoclines intersect once and only once at $(0, 0)$.

Define $\tilde{\pi}$ the largest π^h under which the two isoclines intersect more than once. We show that for all $\pi > \tilde{\pi}$, the two isoclines intersect more than once. To see this, suppose that π^h increases from $\tilde{\pi}$ to $\tilde{\pi}' > \tilde{\pi}$. Define (χ_1, e_1) one dollarization steady state when $\pi^h = \tilde{\pi}$. Then when $\pi^h = \tilde{\pi}'$, define $\tilde{g_1}'(\chi)$ the e isocline at $\pi^h = \tilde{\pi}'$, it must be than

$$\tilde{g_1}'(\chi_1) = e_1' > e_1 = \tilde{g_2}(\chi_1).$$

From part 1, $\tilde{g_1}'(\infty) < \tilde{g_2}(\infty) = \infty$. Therefore, $\tilde{g_1}'$ and $\tilde{g_2}$ must intersect at least once between $(\chi_1, 1)$. Moreover, there is a non-dollarization steady state. Therefore, For all $\pi^h > \tilde{\pi}$, there exists more than one steady state.

■

Proof of Lemma 3. Taking total derivatives with respect to χ on both sides of (23), we obtain

$$[\varphi'(e) + (\rho + \delta + e)\varphi''(e) - \varphi'(e)]\frac{de}{d\chi} = g(\chi),$$

where

$$g(\chi) = \frac{\partial}{\partial\chi}\alpha\Big\{\Big[p[y(m^h + m^f)] - y(m^h + m^f)\Big] - \Big[p[y(m^h)] - y(m^h)\Big]\Big\}.$$

And thus, the slope of the e isocline is

$$\frac{de}{d\chi}|_{\dot{e}=0} = \frac{g(\chi)}{(\rho+\delta+e)\varphi''(e)}.$$

The slope of the χ isocline is

$$\frac{de}{d\chi}|_{\dot{\chi}=0} = \frac{\delta}{(1-\chi)^2}.$$

The Jacobian matrix at the steady state(s) is

$$J = \begin{bmatrix} \frac{\partial \dot{e}}{\partial e} & \frac{\partial \dot{e}}{\partial \chi} \\ \frac{\partial \dot{\chi}}{\partial e} & \frac{\partial \dot{\chi}}{\partial \chi} \end{bmatrix}_{(\chi^*, e^*)} = \begin{bmatrix} \rho+\delta+e^* & -\frac{g(\chi^*)}{\varphi''(e^*)} \\ 1-\chi^* & -(e^*+\delta) \end{bmatrix}$$

When the e isocline crosses the χ isocline from above,

$$\frac{de}{d\chi}|_{\dot{e}=0} < \frac{de}{d\chi}|_{\dot{\chi}=0},$$

and thus

$$g(\chi^*) < \frac{\delta(\rho+\delta+e^*)\varphi''(e^*)}{(1-\chi^*)^2},$$

which implies that

$$\begin{aligned} \det(J)|_{(\chi^*,e^*)} &= -(\rho+\delta+e^*)(e^*+\nu) + \frac{(1-\chi^*)g(\chi^*)}{\varphi''(e^*)} \\ &< -(\rho+\delta+e^*)(e^*+\delta) + \frac{\delta(\rho+\delta+e^*)}{1-\chi^*} \\ &= (\rho+\delta+e^*)\Big(-e^* + \frac{\delta\chi^*}{1-\chi^*}\Big) \\ &= 0, \end{aligned}$$

and thus, the steady state is a *saddle*.

Similarly, we can show that when

$$\frac{de}{d\chi}|_{\dot{e}=0} > \frac{de}{d\chi}|_{\dot{\chi}=0},$$

it must be that

$$\det(J)|_{(\chi^*,e^*)} > 0.$$

This result, combined with the arrows of motions in the left panel of Figure 5, implies that when the e isocline crosses the χ isocline from below, the steady state is an *unstable spiral.* ■

Appendix B Additional lemmas

Lemma 4 *There exists a unique monetary steady state if the following conditions hold: for all* $m \in [dM/\rho, p(y^*))$,

$$(a)\ \varphi'''(e) \geq 0; \quad (b)\ \left|\frac{mL''(m)}{L'(m)}\right| \geq 2; \quad (c)\ \left|\frac{mL'(m)}{L(m)}\right| \geq \frac{1}{2}\left|\frac{mL''(m)}{L'(m)}\right|.$$

Lemma 4 provides a sufficient condition for uniqueness in the one-asset case. Condition (a) states that the cost function $\varphi(e)$ becomes more convex as e increases. Condition (b) states that the elasticity of the "marginal liquidity premium", $L'(m)$, is sufficiently large (greater than 2). Condition(c) states that the elasticity of the liquidity premium, $L(m)$, is also sufficiently large (at lease one half of that of $L'(m)$).

Proof of Lemma 4.

1. We start by studying the shape of g_1. From equation (41), m is defined over $\left[dM/\rho, \widehat{m^*}\right]$, where $\widehat{m^*}$ is the solution to

$$\rho - \frac{dM}{\widehat{m^*}} = \alpha L\left(\widehat{m^*}\right).$$

When $m = dM/\rho$, equation (41) implies that $\chi = 0$, and equation (42) implies that e is positive. When $m \to \widehat{m^*}$, (41) implies that $\chi \to 1$, and equation (42) implies that e is finite.

In order to study the concavity of function g_1, we study separately the equation (41) and (42). From (41),

$$\rho - \frac{dM}{m} = \alpha\chi L(m). \tag{57}$$

Totally differentiating both sides of (57) with respect to χ, we obtain

$$\frac{dM}{m^2}\frac{dm}{d\chi} = \alpha L(m) + \alpha\chi L'(m)\frac{dm}{d\chi}. \tag{58}$$

Rearranging (58), we obtain

$$\frac{dm}{d\chi} = \frac{\alpha L(m)}{\frac{dM}{m^2} - \alpha\chi L'(m)}, \tag{59}$$

and therefore,

$$\begin{aligned} \frac{d^2m}{d\chi^2} &= \frac{\alpha L'(m)\frac{dm}{d\chi}\left[\frac{dM}{m^2} - \alpha\chi L'(m)\right] - \alpha L(m)\left[-\frac{2dM}{m^3}\frac{dm}{d\chi} - \alpha L'(m) - \alpha\chi L''(m)\frac{dm}{d\chi}\right]}{\left[\frac{dM}{m^2} - \alpha\chi L'(m)\right]^2}, \\ &= \frac{\alpha^2 L'(m)L(m) - \alpha L(m)\left[-\frac{2dM}{m^3}\frac{dm}{d\chi} - \alpha L'(m) - \alpha\chi L''(m)\frac{dm}{d\chi}\right]}{\left[\frac{dM}{m^2} - \alpha\chi L'(m)\right]^2}. \end{aligned} \tag{60}$$

2. Now, we study the function f_2. From equation (42), m is defined over $[dM/\rho, \infty)$.

Totally differentiate both sides of equation (42) with respect to m, and we obtain

$$(\rho + \delta + e)\varphi''(e)\frac{de}{dm} = \alpha\left[1 - \frac{1}{p'(y)}\right], \tag{61}$$

and thus

$$\frac{de}{dm} = \frac{\alpha\left[1 - \frac{1}{p'(y)}\right]}{(\rho + \delta + e)\varphi''(e)}, \tag{62}$$

Totally differentiate both sides of equation (61) with respect to m, and we obtain

$$[\varphi''(e) + (\rho + \delta + e)\varphi'''(e)]\left(\frac{de}{dm}\right)^2 + (\rho + \delta + e)\varphi''(e)\frac{d^2e}{dm^2} = \alpha\frac{p''(y)}{[p'(y)]^3}. \tag{63}$$

Rearranging (61), we obtain

$$\frac{d^2e}{dm^2} = \frac{\alpha\frac{p''(y)}{[p'(y)]^3} - [\varphi''(e) + (\rho + \delta + e)\varphi'''(e)]\left(\frac{de}{dm}\right)^2}{(\rho + \delta + e)\varphi''(e)}. \tag{64}$$

By assumption, $p'(y) > 0$, $p''(y) < 0$, and $\varphi''(e) > 0$. Therefore, if $p''(y) < 0$ and $\varphi'''(e) \geq 0$, then $d^2e/dm^2 < 0$.

3. Now we can study the concavity of g_1. By definition,

$$g_1'(\chi) = \frac{de}{d\chi} = \frac{de}{dm}\frac{dm}{d\chi}, \tag{65}$$

where $\frac{de}{dm}$ and $\frac{dm}{d\chi}$ are defined by (62) and (59), both of which are positive, and thus $g_1(\chi)$ is increasing in χ. Now, (65) implies that

$$g_1''(\chi) = \frac{d^2e}{d\chi^2} = \frac{d^2e}{dm^2}\left(\frac{dm}{d\chi}\right)^2 + \frac{de}{dm}\frac{d^2m}{d\chi^2}. \tag{66}$$

From (64), $d^2e/dm^2 < 0$ if $\varphi'''(e) \geq 0$. Therefore, if $\frac{d^2m}{d\chi^2} \leq 0$, then $g_1''(\chi) < 0$. Now we find conditions under which $\frac{d^2m}{d\chi^2} \leq 0$.

$$\begin{aligned}
\alpha^2 L'(m)L(m) - \alpha L(m)\left[-\frac{2dM}{m^3}\frac{dm}{d\chi} - \alpha L'(m) - \alpha\chi L''(m)\frac{dm}{d\chi}\right] &\leq 0, \\
\iff \frac{dm}{d\chi}\left[\frac{2dM}{m^3} + \alpha\chi L''(m)\right] &\leq -2\alpha L'(m). \qquad (67)
\end{aligned}$$

From (57),

$$\alpha\chi = \frac{\rho - \frac{dM}{m}}{L(m)},$$

and thus equation (67) can be written as

$$\begin{aligned}
\frac{\alpha L(m)}{\frac{dM}{m^2} - \frac{L'(m)}{L(m)}\left(\rho - \frac{dM}{m}\right)}\left[\frac{2dM}{m^3} + \frac{L''(m)}{L(m)}\left(\rho - \frac{dM}{m}\right)\right] &\leq -2\alpha L'(m), \\
\iff \frac{\frac{2dM}{m^3} + \frac{L''(m)}{L(m)}\left(\rho - \frac{dM}{m}\right)}{\frac{dM}{m^2} - \frac{L'(m)}{L(m)}\left(\rho - \frac{dM}{m}\right)} &\leq -2\frac{L'(m)}{L(m)}, \\
\iff -\frac{L''(m)}{L'(m)} + \frac{\frac{dM}{m^3}\left[2 + \frac{mL''(m)}{L'(m)}\right]}{\frac{dM}{m^2} - \frac{L'(m)}{L(m)}\left(\rho - \frac{dM}{m}\right)} &\leq -2\frac{L'(m)}{L(m)}. \qquad (68)
\end{aligned}$$

By assumption, $L'(m) < 0$. Equation (68) holds if the following two conditions are satisfied:

$$\begin{aligned}
\frac{L(m)L''(m)}{[L'(m)]^2} &\leq 2, \qquad (69) \\
\frac{mL''(m)}{L'(m)} &\leq -2. \qquad (70)
\end{aligned}$$

If condition (69) is satisfied, then the first term on the right hand side of (68) is no greater than the right hand side. Under condition (70), the second term on the left hand side of (68) is negative. Therefore, if (69) and (70) are satisfied, then (68) holds. One can rewrite (70) as Condition **??** in Lemma **??**, and rewrite (69) as Condition **??** in Lemma **??**.

4. So far, we have shown that if Conditions **??**–**??** in Lemma **??** are satisfied, then $g_1(\chi)$ is concave. Now we show that if $g_1(\chi)$ is concave, then $g_1(\chi)$ and $g_2(\chi)$ intersect once and only once. From (43), we know that
$$g_2(\chi) = \frac{e}{e+\delta}.$$
Therefore,
$$g_2''(\chi) = \frac{-2\delta}{(e+\delta)^2} < 0.$$
From Proposition **??**, there must exist at least one steady state. Denote $S_1 = (\chi_1, e_1)$ the steady state that is closest to the origin. Because $g_1(0) > 0 = g_2(0)$, at χ_1, $g_1(\chi)$ must intersect $g_2(\chi)$ from above, i.e., $g_1'(\chi_1) < g_2'(\chi_1)$. Therefore, suppose that there exists another steady state, $S_2 = (\chi_2, e_2)$, that is to the right of S_1, i.e., $\chi_2 > \chi_1$, then $g_1(\chi)$ must intersect $g_2(\chi)$ from below, i.e., $g_1'(\chi_1) > g_2'(\chi_1)$. However, since $g_1''(\chi) < 0$ and $g_2''(\chi) > 0$, it must be that for all $\chi_2 > \chi_1$,
$$g_1'(\chi_2) < g_1'(\chi_1) < g_2'(\chi_1) < g_2'(\chi_2),$$
which is a contradiction. Therefore, if Conditions (a)-(c) hold, then there exists one and only one steady state.

■

Appendix C Derivation of the Hamilton-Jacobi-Bellman equations

In this section, we derive the Hamilton-Jacobi-Bellman equations for the buyers and sellers in an economy where there are a finite number J types of assets.

C.1 HJB for the buyers

We focus on equilibria where buyers adjust their asset holdings only at the beginning of time, and immediately after the pairwise meetings. Otherwise, they consume or produce in flow. At time 0, the buyer's value function solves
$$V_0^b(\boldsymbol{a}_0) = \max_{\boldsymbol{a}_t, c_t, \Delta C_0} \Big\{ \Delta C_0 + \mathbb{E}\int_0^T e^{-\rho t} c_t dt + e^{-\rho T} W_T^b(\boldsymbol{a}_T) \Big\}, \tag{71}$$
$$\text{s.t. } (\mathbf{1} \cdot \dot{\boldsymbol{a}}_t) = \boldsymbol{r_t} \cdot \boldsymbol{a_t} - c_t + \tau_t, \tag{72}$$
$$\Delta C_0 = \mathbf{1} \cdot (\boldsymbol{a}_0 - \boldsymbol{a}_0^+), \tag{73}$$
$$\boldsymbol{a}_0 \text{ is given.} \tag{74}$$
where T is the time the next pairwise meeting occurs, and $W_T^b(\boldsymbol{a}_T)$ is the expected continuation value at the moment the buyer enters the pairwise meeting. T follows an exponential distribution with parameter $1/\alpha$. And finally, we assume the following transversality condition:
$$\lim_{t\to\infty} \mathbb{E}_0[e^{-\rho t}(\mathbf{1} \cdot \boldsymbol{a}_t)] = 0. \tag{75}$$

We can rewrite (71) and obtain the following equation:

$$V_0^b(\boldsymbol{a}_0) = \mathbf{1}\cdot\boldsymbol{a}_0 + \max_{\boldsymbol{a}_t, c_t}\Big\{ -\mathbf{1}\cdot\boldsymbol{a}_0^+ + \int_0^\infty e^{-(\rho+\alpha)t}[c_t + \alpha W_t^b(\boldsymbol{a}_t)]dt\Big\}. \tag{76}$$

From (72), we can rewrite

$$\int_0^\infty e^{-(\rho+\alpha)t}c_t dt = \int_0^\infty e^{-(\rho+\alpha)t}(\boldsymbol{r}_t\cdot\boldsymbol{a}_t + \tau_t)dt - \int_0^\infty e^{-(\rho+\alpha)t}(\mathbf{1}\dot{\cdot}\boldsymbol{a}_t)dt.$$

Using integrating by part and the transversality condition,

$$\int_0^\infty e^{-(\rho+\alpha)t}(\mathbf{1}\dot{\cdot}\boldsymbol{a}_t)dt = -\mathbf{1}\cdot\boldsymbol{a}_0^+ + \int_0^\infty e^{-(\rho+\alpha)t}(\rho+\alpha)\mathbf{1}\cdot\boldsymbol{a}dt.$$

And thus, (76) can be rewritten as

$$\begin{aligned} V_0^b &= \max_{\boldsymbol{a}_t}\int_0^\infty e^{-(\rho+\alpha)t}[\boldsymbol{r}_t\cdot\boldsymbol{a}_t + \tau_t - (\rho+\alpha)\mathbf{1}\cdot\boldsymbol{a}_t + \alpha W_t^b(\boldsymbol{a}_t)]dt, \\ &= \max_{\boldsymbol{a}_t}\int_0^\infty e^{-(\rho+\alpha)t}\Big\{ -(\rho\mathbf{1} - \boldsymbol{r}_t)\cdot\boldsymbol{a}_t + \tau_t + \alpha\big[W_t^b(\boldsymbol{a}_t) - \mathbf{1}\cdot\boldsymbol{a}_t\big]\Big\}dt. \end{aligned} \tag{77}$$

where $V_0^b = V_0^b(\boldsymbol{a}_0) - \mathbf{1}\cdot\boldsymbol{a}_0$.

Let $\mathcal{P}$ be the power set of $\{1, 2, \ldots, J\}$. $\mathcal{P}$ has 2^J elements, each corresponding to a type of sellers. For example, $\{1, 2\}$ corresponds to a seller who recognizes asset 1 and asset 2. It follows that there are 2^J types of meetings. Let Pr_i be the probability of being in the type i meeting. Let $\mathbb{1}_i$ denote an indicator vector that indicates the set of assets that can be recognized in meeting i. For example, if in the type i meeting, only asset 1 and asset 2 can be recognized, then $\mathbb{1}_i = (1, 1, 0, \ldots, 0)^T$. Therefore,

$$W^b(\boldsymbol{a}_t) = \sum_{i=1}^{2^J}\text{Pr}_{i,t}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}_t}\{u(y_i) + V_t^b(\boldsymbol{a}'_{i,t})\} \text{ s.t. } \mathbf{1}\cdot\boldsymbol{a}'_{i,t} = \mathbf{1}\cdot\boldsymbol{a}_t - p(y_i) \tag{78}$$

By the linearity of $V^b(\boldsymbol{a})$, (78) can be rewritten as

$$\begin{aligned} W_t^b(\boldsymbol{a}_t) &= \sum_{i=1}^{2^J}\text{Pr}_{i,t}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}_t}\Big\{[u(y_i) - p(y_i)] + V_t^b(\boldsymbol{a}_t)\Big\} \\ &= \sum_{i=1}^{2^J}\text{Pr}_{i,t}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}_t}[u(y_i) - p(y_i)] + V_t^b(\boldsymbol{a}_t) \end{aligned} \tag{79}$$

Substituting (79) into (77), we obtain

$$V_0^b = \max_{\boldsymbol{a}_t}\int_0^\infty e^{-(\rho+\alpha)t}\Big\{ -(\rho\mathbf{1} - \boldsymbol{r}_t)\cdot\boldsymbol{a}_t + \tau_t + \alpha\big\{\sum_{i=1}^{2^J}\text{Pr}_{i,t}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}_t}[u(y_i) - p(y_i)] + V_t^b\big\}\Big\}dt. \tag{80}$$

We can renormalize time and rewrite (80). Along the optimal path,

$$V_t^b = \int_0^\infty e^{-(\rho+\alpha)x}\Big\{ -(\rho\mathbf{1} - \boldsymbol{r}_{t+x})\cdot\boldsymbol{a}^*_{t+x} + \tau_{t+x} + \alpha\big\{\sum_{i=1}^{2^J}\text{Pr}_{i,t+x}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_{t+x}}[u(y_i) - p(y_i)] + V_{t+x}^b\big\}\Big\}dx. \tag{81}$$

Differentiating both sides by t, we obtain

$$\begin{aligned}
\dot{V}_t^b &= \int_0^\infty e^{-(\rho+\alpha)x}\frac{d}{dt}\Big\{-(\rho\mathbf{1}-\boldsymbol{r}_{t+x})\cdot\boldsymbol{a}^*_{t+x}+\tau_{t+x}+\alpha\big\{\sum_{i=1}^{2^J}\mathrm{Pr}_{i,t+x}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_{t+x}}[u(y_i)-p(y_i)]+V^b_{t+x}\big\}\Big\}dx \\
&= \int_0^\infty e^{-(\rho+\alpha)x}\frac{d}{d(t+x)}\Big\{-(\rho\mathbf{1}-\boldsymbol{r}_{t+x})\cdot\boldsymbol{a}^*_{t+x}+\tau_{t+x}+\alpha\big\{\sum_{i=1}^{2^J}\mathrm{Pr}_{i,t+x}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_{t+x}}[u(y_i)-p(y_i)]+V^b_{t+x}\big\}\Big\}dx \\
&= \int_0^\infty e^{-(\rho+\alpha)x}d\Big\{-(\rho\mathbf{1}-\boldsymbol{r}_{t+x})\cdot\boldsymbol{a}^*_{t+x}+\tau_{t+x}+\alpha\big\{\sum_{i=1}^{2^J}\mathrm{Pr}_{i,t+x}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_{t+x}}[u(y_i)-p(y_i)]+V^b_{t+x}\big\}\Big\} \qquad (82)
\end{aligned}$$

Using integration by part, we rewrite (82) as

$$\begin{aligned}
\dot{V}_t^b &= \lim_{x\to\infty} e^{-(\rho+\alpha)x}\Big\{-(\rho\mathbf{1}-\boldsymbol{r}_{t+x})\cdot\boldsymbol{a}^*_{t+x}+\tau_{t+x}+\alpha\big\{\sum_{i=1}^{2^J}\mathrm{Pr}_{i,t+x}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_{t+x}}[u(y_i)-p(y_i)]+V^b_{t+x}\big\} \\
&\quad +(\rho\mathbf{1}-\boldsymbol{r}_t)\cdot\boldsymbol{a}^*_t-\tau_t-\alpha\big\{\sum_{i=1}^{2^J}\mathrm{Pr}_{i,t}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_t}[u(y_i)-p(y_i)]-V^b_t\big\} \\
&\quad +(\rho+\alpha)\int_0^\infty e^{-(\rho+\alpha)x}d\Big\{-(\rho\mathbf{1}-\boldsymbol{r}_{t+x})\cdot\boldsymbol{a}^*_{t+x}+\tau_{t+x}+\alpha\big\{\sum_{i=1}^{2^J}\mathrm{Pr}_{i,t+x}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_{t+x}}[u(y_i)-p(y_i)]+V^b_{t+x}\big\}\Big\} \\
&= (\rho\mathbf{1}-\boldsymbol{r}_t)\cdot\boldsymbol{a}^*_t-\tau_t-\alpha\big\{\sum_{i=1}^{2^J}\mathrm{Pr}_{i,t}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_t}[u(y_i)-p(y_i)]-V^b_t\big\}+(\rho+\alpha)V^b_t \qquad (83)
\end{aligned}$$

Rearranging equation (83), we obtain the Hamilton-Jacobi-Bellman equation for the buyers:

$$\rho V_t^b = -(\rho\mathbf{1}-\boldsymbol{r}_t)\cdot\boldsymbol{a}^*_t+\tau_t+\alpha\big\{\sum_{i=1}^{2^J}\mathrm{Pr}_{i,t}\max_{p(y_i)\le\mathbb{1}_i\cdot\boldsymbol{a}^*_t}[u(y_i)-p(y_i)]+\dot{V}^b_t \qquad (84)$$

When $J=1$, (84) coincides with (4). (36) and (18) are special cases of (84) with $J=2$.

Appendix D A one-asset economy with abundant liquidity

So far, we have only considered cases where liquidity is scarce, i.e., $dM/\rho < p(y^*)$. When $dM/\rho \ge p(y^*)$, i.e., liquidity is abundant, liquidity premium $L(m)$ becomes zero. Therefore, the only possible path for m that satisfies the transversality condition (12) is $m_t = dM/\rho$ for all t, and thus (10) becomes

$$\varphi''(e)\dot{e} = (\rho+\delta+e)\varphi'(e)-\alpha\Psi\left(\frac{dM}{\rho}\right)-\varphi(e). \qquad (85)$$

Figure 19 plots the phase diagram of this economy. When liquidity is abundant, the e isocline is a horizontal line. Therefore, for any initial state χ_0, there is a unique solution to the ODE. The equilibrium trajectory coincides with the e isocline.

Appendix E More on dollarization

E.1 The seigniorage rule

In this section, I consider a different monetary policy regime—the seigniorage rule.[14] Consider an economy where the government is committed to a fixed real consumption stream g. The government consumption is

[14]The way the seigniorage rule is modeled in this section follows from Rocheteau (2023).

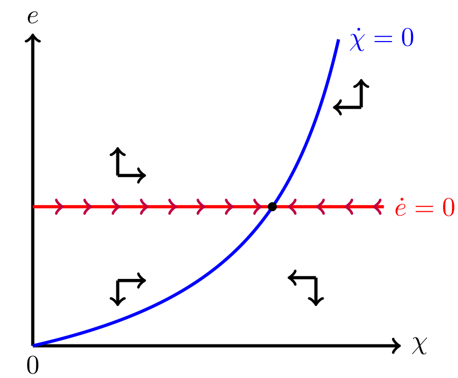

Figure 19: Phase diagram when liquidity is abundant, i.e., $dM/\rho \geq p\left(y^*\right)$.

funded solely by issuing money. I show through a numerical example that there can be limit cycles in this case. The seigniorage income requirement pins down the speed of money creation:

$$\dot{M}^h \phi^h = g, \tag{86}$$

where ϕ^h is the price of the domestic currency in terms of the numeraire. Equation (86) equates the real value of the created money with government consumption. Under market clearing, we can rewrite (86) as

$$\frac{\dot{M}^h}{M^h} = \frac{g}{m^h}, \tag{87}$$

and thus the rate of return of the domestic currency is determined endogenously by

$$r^h = \frac{\dot{\phi}^h}{\phi^h} = \frac{\dot{m}^h}{m^h} - \frac{\dot{M}^h}{M^h} = \frac{\dot{m}^h}{m^h} - \frac{g}{m^h}. \tag{88}$$

Given (88), we can rewrite the buyers' optimal condition (19) as

$$\rho + \frac{g}{m^h} - \frac{\dot{m}^h}{m^h} = \alpha\chi L(m^h + m^f) + \alpha(1-\chi)L(m^h). \tag{89}$$

An equilibrium is a list of time paths $(m_t^h, m_t^f, e_t, \chi_t)$ that solves (89), (20), (21), (11), and the transversality condition (22). In the following, I show numerically that there can be limit cycles. The model is parameterized as follows: $u(y) = [(y+b)^{1-\sigma} - b^{1-\sigma}]/(1-\sigma)$ with $\sigma = 1/2$ and $b = 0.0001$, $p(y) = \theta y + (1-\theta)u(y)$, with $\theta = 0.5$. $\rho = 0.03$, $\delta = 0.02$, $\kappa = 5$, $r^f = -0.01$, and $g = 0.1$. In Figure 20, I plot the phase diagram of the system from two perspectives. The red, blue and green surfaces represent the m^h, the e and the χ isoclines, respectively. The m^h isocline,

$$\rho + \frac{g}{m^h} = \alpha\chi L(m^h + m^f) + \alpha(1-\chi)L(m^h),$$

describes a mapping from χ to m^h. Unlike the previous sections, the m^h isocline is hump-shaped, suggesting that one χ corresponds to two m^h-s. Intuitively, this is because there is a trade-off between the speed of money creation and the value of money in equilibrium. In order to collect g units of seigniorage income, the central bank may issue new money at a faster speed, in which case money also depreciates faster, or it may issue new money at a lower speed, but the money has high value. The three surfaces intersect trice, suggesting two steady states:

$$S_1 = (m_1^{h*}, m_1^{f*}, e_1^*, \chi_1^*) = (0.1375, 0.0007, 0.0011, 0.0529),$$

$$S_2 = (m_2^{h*}, m_2^{f*}, e_2^*, \chi_2^*) = (1.0533, 0.0492, 0.0109, 0.3519).$$

Steady states S_1 is a saddle, and S_2 is a sink. Moreover, S_1 is a steady state with little real balances

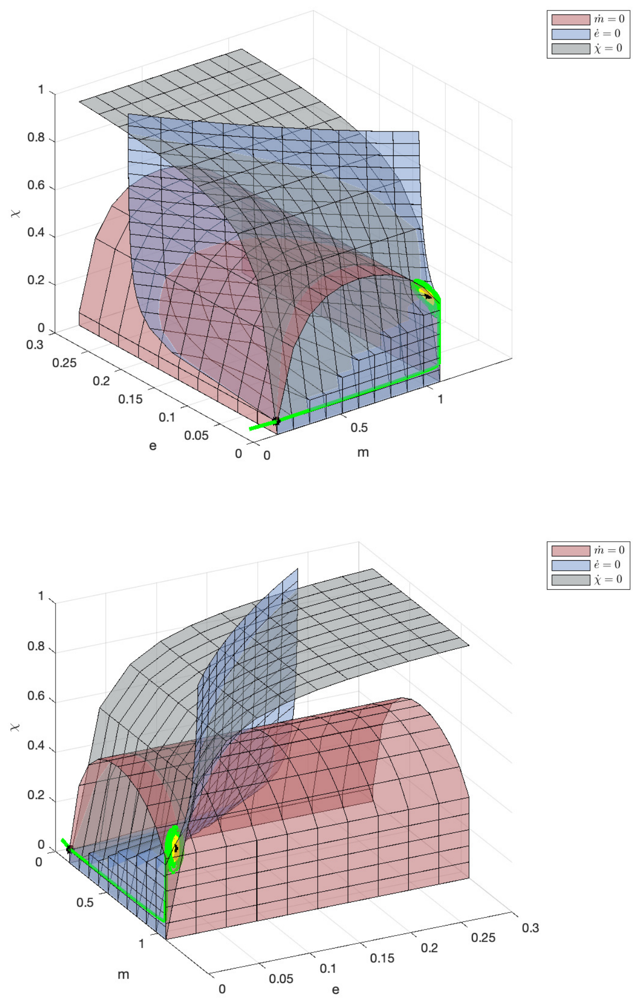

Figure 20: Numerical example: seigniorage income: two perspectives

of both currencies, and S_2 is a steady state where dollarization is high and the real balances of both currencies are high. The green curve represents the stable manifold of the lower steady state, and the red curve represents the stable manifold around the higher steady state. The two manifolds approach an unstable limit cycle asymptotically from both sides.[15] The model suggests that there exists a quadruple

[15]Related work that includes limit cycles include Boldrin et al. (1993) and Coles and Wright (1998).

$(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}) \in \mathbb{R}^4$, with $\varepsilon_{11} < \varepsilon_{21}$ and $\varepsilon_{21} < \varepsilon_{22}$, such that when the initial acceptability, χ_0, is such that $\chi_0 \in (\chi_2^* - \varepsilon_{11}, \chi_2^* + \varepsilon_{12})$, then there exists a continuum of perfect foresight equilibria, indexed by the initial e_0 and m_0, that spiral towards S_2, as well as a continuum of perfect foresight equilibria that spiral outwards and eventually approach S_1. When $\chi_0 \in (\chi_2^* - \varepsilon_{21}, \chi_2^* - \varepsilon_{11}) \cup (\chi_2^* + \varepsilon_{12}, \chi_2^* + \varepsilon_{22})$, then the only equilibria are a continuum of non-stationary perfect-foresight equilibria that spiral outwards. When When $\chi_0 \in (0, \chi_2^* - \varepsilon_{21})$, the equilibrium trajectory is monotone and approaches S_1. And finally, when $\chi(0) > \chi_2^* + \varepsilon_{22}$, there is no perfect foresight equilibrium.

The exercise suggests that, first, when the monetary authority follows the seigniorage rule, equilibrium may not exist for any initial χ_0. Moreover, if the initial χ_0 is within a certain range, the equilibrium may fall into a *dollarization trap*, where the equilibrium trajectory fluctuates around a limit cycle for a long period of time before it stabilizes. Moreover, the instability of the limit cycle suggests that small perturbations may have significant effects. For example, a small, exogenous change in χ, e.g., when a number of type 1 sellers enter or leave the economy, might switch the equilibrium trajectory from the yellow region to the green region, or vice versa, resulting in different long-run equilibrium outcomes.