The Irrelevance of Market Incompleteness for the Price of Aggregate Risk*

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Abstract

In models with a large number of agents who have constant relative risk aversion (CRRA) preferences, the absence of insurance markets for idiosyncratic labor income risk has no effect on the premium for aggregate risk if the distribution of idiosyncratic risk is independent of aggregate shocks. In spite of the missing markets, a representative agent who consumes aggregate consumption prices the excess returns on stocks correctly. This result holds regardless of the persistence of the idiosyncratic shocks, as long as they are not permanent, even when households face binding, and potentially very tight borrowing constraints. Consequently, in this class of models there is no link between the extent of self-insurance against idiosyncratic income risk and aggregate risk premia.

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1 Introduction

This paper examines whether closing down insurance markets for idiosyncratic risk increases the risk premium that stocks command over bonds, and we provide general conditions under which it does not.

We study a standard incomplete markets model populated by a continuum of agents who have CRRA preferences and who can only trade a risk-free bond and a stock. The presence of uninsurable idiosyncratic risk is shown to lower the equilibrium risk-free rate, but it has no effect on the price of aggregate risk in equilibrium if the distribution of idiosyncratic shocks is statistically independent of aggregate shocks. Consequently, in this class of models, the representative agent Consumption-CAPM (CCAPM) developed by Breeden (1979) and Lucas (1978) prices the excess returns on the stock correctly. Therefore, as long as idiosyncratic shocks are distributed independently of aggregate shocks, the extent to which households manage to insure against idiosyncratic income risk is irrelevant for risk premia. These results deepen the equity premium puzzle, because we show that Mehra and Prescott's (1985) statement of the puzzle applies to a much larger class of incomplete market models.[1]

Our result holds regardless of the persistence of the idiosyncratic shocks, as long as these shocks are not completely permanent, and it is robust to the introduction of various forms of borrowing and solvency constraints, regardless of the tightness of these constraints. Our result also survives when agents have non-time-additive preferences if the consumption aggregator is a homogeneous function.

In addition, we show that adding markets does not always lead to more risk sharing. In particular, if the logarithm of aggregate consumption follows a random walk, allowing agents to trade claims on payoffs that are contingent on *aggregate shocks,* in addition to the risk-free bond and the stock, does not help them to smooth their consumption. Agents only trade the stock to smooth their consumption, and introducing these additional state contingent claims leaves interest rates and asset prices unaltered. However, if there is predictability in aggregate consumption growth, agents want to hedge their portfolio against interest rate shocks, creating a role for trade in a richer menu of assets. The risk premium irrelevance

[1] Weil's (1989) statement of the risk-free rate puzzle, on the contrary, does not.

result, however, still applies. Finally, we also show that idiosyncratic uninsurable income risk does not contribute any variation in the conditional market price of risk, beyond what is built into aggregate consumption growth.

Most related to our paper is Constantinides and Duffie (henceforth CD) (1996), who consider an environment in which agents face only permanent idiosyncratic income shocks and can trade stocks and bonds. Their equilibrium is characterized by the absence of trade in financial markets. By choosing the right stochastic household income process, CD show they can deliver autarchic equilibrium asset prices with all desired properties. CD's results also imply that if the cross-sectional variance of consumption growth is orthogonal to returns, then the equilibrium risk premium is equal to the one in the representative agent model. We show that this characterization of the household consumption process is indeed the correct one in equilibrium in a large class of incomplete market models.[2] Relative to CD, our paper adds potentially binding solvency or borrowing constraints and the equilibrium does feature trade in financial markets, but we do not make any assumptions about the distribution of the underlying shocks other than mean reversion. The consumption growth distribution across household is the endogenous, equilibrium outcome of these trades, but we can still fully characterize equilibrium asset prices.[3]

In the quest towards the resolution of the equity premium puzzle identified by Hansen and Singleton (1983) and Mehra and Prescott (1985), uninsurable idiosyncratic income risk has been introduced into standard dynamic general equilibrium models.[4] Incomplete insurance of household consumption against idiosyncratic income risk was believed to increase the aggregate risk premium. Heaton and Lucas (1996) attribute the failure of incomplete market

[2]Krebs (2005) derives the same result in a production economy with human capital accumulation. In his world, as in ours, the wealth distribution is not required as a state variable to characterize equilibria, in spite of the presence of aggregate shocks. As in CD (1996), however, the equilibrium is autarkic so that households cannot diversify any of their risk in equilibrium. Finally, Lettau (2006) also shows that *if* household consumption (in logs) consists of an aggregate and an idiosyncratic part, the latter does not affect risk premia.

[3]In a separate class of continuous-time diffusion models, Grossman and Shiller (1982) have demonstrated that heterogeneity has no effect on risk premia, simply because the cross-sectional variance of consumption growth is locally deterministic. Our irrelevance result is obtained in a different class of discrete-time incomplete market models in which heterogeneity does potentially matter, as shown by Mankiw (1986), CD and others, but we find the model itself cannot endogenously activate the Mankiw-CD mechanism.

[4]For examples, see the work of Ayiagari and Gertler (1991), Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Krusell and Smith (1998), and Marcet and Singleton (1999).

models (or their partial success) in matching moments of asset prices to the household's success in smoothing consumption in this class of models. In contrast, our paper shows *analytically* that there is essentially no link between the extent to which idiosyncratic risk can be traded away by households and the size of the risk premium. The main contribution of our paper is to argue that Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Marcet and Singleton (1999) and others in this literature have reached the right conclusions -namely that adding uninsurable idiosyncratic income risk to standard models does not alter the asset pricing implications of the model - but not for the right reasons. As long as the distribution of idiosyncratic shocks is independent of aggregate shocks, the extent of self-insurance does not matter. We can make our solvency constraints arbitrarily tight or make the income process highly persistent, and our theoretical result still goes through. In equilibrium, all households bear the same amount of aggregate risk, and accordingly, they are only compensated in equilibrium for the aggregate consumption growth risk they take on by investing in stocks.

Most of the work on incomplete markets and risk premia documents the moments of model-generated data for particular calibrations, but there are few analytical results. Levine and Zame (2002) show that in economies populated by agents with infinite horizons, the equilibrium allocations in the limit, as their discount factors go to one, converge to the complete markets allocations. Consequently the pricing implications of the incomplete markets model converge to that of the representative agent model as households become perfectly patient. We provide a qualitatively similar equivalence result that applies only to the risk premium. Our result, however, does not depend on the time discount factor of households. For households with CARA utility, closed form solutions of the individual decision problem in incomplete markets models with idiosyncratic risk are sometimes available, as Willen (1999) shows.[5] In contrast to Willen, we employ CRRA preferences, and we obtain an unambiguous (and negative) result for the impact of uninsurable income risk for the equity premium in case the distribution of individual income shocks is independent of aggregate shocks.

The key ingredients underlying our irrelevance result are (i) a continuum of agents, (ii) CRRA utility, (iii) idiosyncratic labor income risk that is independent of aggregate risk

[5] CARA utility eliminates wealth effects which crucially simplifies the analysis.

and (iv) solvency constraints or borrowing constraints on total financial wealth that are proportional to aggregate income. We now discuss each of these assumptions in detail to highlight and explain the differences with existing papers in this literature.

First, we need to have a large number of agents in the economy. As forcefully pointed out by Denhaan (2001), in an economy with a finite number of agents, each idiosyncratic shock is by construction an aggregate shock because it changes the wealth distribution and, through these wealth effects, asset prices.

Second, our results rely on the homotheticity of CRRA utility, but not crucially on the time separability of the lifetime utility function; they can easily be extended to Epstein-Zin utility.

Third, in our model labor income grows with the aggregate endowment, as is standard in this literature.[6] In addition, for our results to go through, idiosyncratic income shocks must be distributed independently of aggregate shocks. This explicitly rules out that the variance of idiosyncratic shocks is higher in recessions (henceforth we call such correlation countercyclical cross-sectional variance of labor income shocks, or CCV).

Finally, we can allow households to face either constraints on total net wealth today or state-by-state solvency constraints on the value of their portfolio in each state tomorrow, and these constraints have to be proportional to aggregate income.[7] Our irrelevance result only survives short-sale constraints on *individual* securities if aggregate consumption growth is uncorrelated over time, as assumed in the benchmark case. In that case households only trade the stock to smooth their consumption. Therefore, in the benchmark case, our result is also robust to transaction costs in the bond market (but not to such costs in the stock market).

As a crucial step in demonstrating our main irrelevance result we prove a theorem that may be of independent theoretical interest. We show that equilibrium allocations and prices in a class of models with idiosyncratic and aggregate risk and incomplete markets can be

[6]By contrast, Weil (1992) derives a positive effect of background risk on the risk premium in a two-period model in which, however, labor income does not have an aggregate endowment component. This non-homotheticity invalidates our mapping from the growing to the stationary economy.

[7]It is more common in the literature to impose short-sale constraints on stocks and bonds separately instead of total financial wealth, but this is done mostly for computational reasons, to bound the state space. In fact, if the solvency constraints are meant to prevent default, they should be directly on total financial wealth (see e.g. Zhang (1996) and Alvarez and Jermann (2000)).

easily obtained from allocations and interest rates of a stationary equilibrium in a model with only idiosyncratic risk (as in Bewley (1986), Huggett (1993) or Aiyagari (1994)). One first computes a stationary equilibrium for the Bewley model with an appropriate probability measure for idiosyncratic shocks, including a stationary wealth distribution. We then show that scaling up the allocations of the Bewley equilibrium by the aggregate endowment delivers an equilibrium for the model with aggregate uncertainty. The distribution of wealth in that model, normalized by the aggregate endowment, coincides with the stationary wealth distribution of the Bewley equilibrium and thus is (up to the aggregate endowment) stationary as well. Since stationary equilibria in Bewley models are relatively straightforward to compute, our result implies an algorithm for computing equilibria in this class of models which appears to be simpler than the auctioneer algorithm devised by Lucas (1994) and its extension to economies with a continuum of agents. There is also no need for computing a law of motion for the aggregate wealth distribution, or approximating it by a finite number of moments, as in Krusell and Smith (1997, 1998).[8]

This result of our paper makes contact with the literature on aggregation. Constantinides (1982), building on work by Negishi (1960) and Wilson (1968), derives an aggregation result for heterogenous agents in complete market models, implying that assets can be priced off the intertemporal marginal rate of substitution of an agent who consumes the aggregate endowment. Our findings extend his result to a large class of incomplete market models with idiosyncratic income shocks.

The paper is structured as follows. In section 2, we lay out the physical environment of our model. This section also demonstrates how to transform an economy where aggregate income grows stochastically into a stationary economy with a constant aggregate endowment. In section 3 we study this stationary economy, called the Bewley model henceforth. The next section, section 4, introduces the Arrow model, a model with aggregate uncertainty and a full set of Arrow securities whose payoffs are contingent on the realization of the aggregate shock. We show that a stationary equilibrium of the Bewley model can be mapped into an

[8]This result also implies the existence of a recursive competitive equilibrium with only asset holdings in the state space, albeit under a transformed probability measure. Kubler and Schmedders (2002) establish the existence of such an equilibrium in more general models, but only under very strong conditions. Miao (2004) relaxes these conditions, but he includes continuation utilities in the state space.

equilibrium of the Arrow economy just by scaling up allocations by the aggregate endowment. In section 5 we derive the same result for a model where only a one-period risk-free bond can be traded. We call this the *THL* model (for *T*elmer-*H*eaton-*L*ucas model). After briefly discussing the classic *L*ucas-*B*reeden representative agent model (henceforth *LB* model), section 6 shows that risk premia in the representative agent model and the Arrow model (and by implication, in the *THL* model) coincide. Section 7 investigates the robustness of our results with respect to the assumptions about the underlying stochastic income process, and shows in particular that most of our results can be extended to the case where the aggregate shocks are correlated over time and where preferences are not time-separable, but rather follow an Epstein-Zin specification. Finally, section 8 concludes; all proofs are contained in the appendix.

2 Environment

Our exchange economy is populated by a continuum of individuals of measure 1. There is a single nonstorable consumption good. The aggregate endowment of this good is stochastic. Each individual's endowment depends, in addition to the aggregate shock, also on the realization of an idiosyncratic shock. Thus, the economy we study is identical to the one described by Lucas (1994), except that ours is populated by a continuum of agents (as in Bewley (1986), Aiyagari and Gertler (1991), Huggett (1993) and Aiyagari (1994)), instead of just two agents.

2.1 Representation of Uncertainty

We denote the current aggregate shock by $z_t \in Z$ and the current idiosyncratic shock by $y_t \in Y$. For simplicity, both Z and Y are assumed to be finite. Furthermore, let $z^t = (z_0, \ldots, z_t)$ and $y^t = (y_0, \ldots, y_t)$ denote the history of aggregate and idiosyncratic shocks. As shorthand notation, we use $s_t = (y_t, z_t)$ and $s^t = (y^t, z^t)$. We let the economy start at an initial aggregate node z_0. Conditional on an idiosyncratic shock y_0 and thus $s_0 = (y_0, z_0)$, the probability of a history s^t is given by $\pi_t(s^t|s_0)$. We assume that shocks follow a first order Markov process with transition probabilities given by $\pi(s'|s)$.

2.2 Preferences and Endowments

Consumers rank stochastic consumption streams $\{c_t(s^t)\}$ according to the following homothetic utility function:

$$U(c)(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \geq s^0} \beta^t \pi(s^t|s_0) \frac{c_t(s^t)}{1-\gamma}^{1-\gamma} \tag{1}$$

where $\gamma > 0$ is the coefficient of relative risk aversion and $\beta \in (0,1)$ is the constant time discount factor. We define $U(c)(s^t)$ to be the continuation expected lifetime utility from a consumption allocation $c = \{c_t(s^t)\}$ in node s^t. This utility can be constructed recursively as follows:

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1})$$

where we made use of the Markov property the underlying stochastic processes. The economy's aggregate endowment process $\{e_t\}$ depends only on the aggregate event history; we let $e_t(z^t)$ denote the aggregate endowment at node z^t. Each agent draws a 'labor income' share $\eta(y_t, z_t)$, as a fraction of the aggregate endowment in each period. Her labor income share only depends on the current individual and aggregate event. We denote the resulting individual labor income process by $\{\eta_t\}$, with

$$\eta_t(s^t) = \eta(y_t, z_t) e_t(z^t) \tag{2}$$

where $s^t = (s^{t-1}, y_t, z_t)$. We assume that $\eta(y_t, z_t) > 0$ in all states of the world. The stochastic growth rate of the endowment of the economy is denoted by $\lambda(z^{t+1}) = e_{t+1}(z^{t+1})/e_t(z^t)$. We assume that aggregate endowment growth only depends on the current aggregate state.

Condition 2.1. *Aggregate endowment growth is a function of the current aggregate shock only:*

$$\lambda(z^{t+1}) = \lambda(z_{t+1})$$

Furthermore, we assume that a Law of Large Numbers holds[9], so that $\pi(s^t|s_0)$ is not only

[9] See e.g. Hammond and Sun (2003) for conditions under which a LLN holds with a continuum of random variables.

a household's individual probability of receiving income $\eta_t(s^t)$, but also the fraction of the population receiving that income.

In addition to labor income, there is a Lucas tree that yields a constant share α of the total aggregate endowment as capital income, so that the total dividends of the tree are given by $\alpha e_t(z^t)$ in each period. The remaining fraction of the total endowment accrues to individuals as labor income, so that $1-\alpha$ denotes the labor income share. Therefore, by construction, the labor share of the aggregate endowment equals the sum over all individual labor income shares:

$$\sum_{y_t \in Y} \Pi_{z_t}(y_t)\eta(y_t, z_t) = (1-\alpha), \tag{3}$$

for all z_t, where $\Pi_{z_t}(y_t)$ represents the cross-sectional distribution of idiosyncratic shocks y_t, conditional on the aggregate shock z_t. By the law of large numbers, the fraction of agents who draw y in state z only depends on z. An increase in the capital income share α translates into proportionally lower individual labor income shares $\eta(y,z)$ for all (y,z).[10]

At time 0, the agents are endowed with initial wealth θ_0. This wealth represents the value of an agent's share of the Lucas tree producing the dividend flow in units of time 0 consumption, as well as the value of her labor endowment at date 0. We use Θ_0 to denote the initial joint distribution of wealth and idiosyncratic shocks (θ_0, y_0).

Most of our results are derived in a de-trended version of our economy. This de-trended economy features a constant aggregate endowment and a growth-adjusted transition probability matrix. The agents in this de-trended economy, discussed now, have stochastic time discount factors.

2.3 Transformation of Growth Economy into a Stationary Economy

We transform our growing economy into a stationary economy with a stochastic time discount rate and a growth-adjusted probability matrix, following Alvarez and Jermann (2001). First,

[10] Our setup nests the baseline model of Heaton and Lucas (1996), except for the fact that they allow for the capital share α to depend on z.

we define growth deflated consumption allocations (or consumption shares) as

$$\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)} \text{ for all } s^t. \tag{4}$$

Next, we define *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}} \text{ and } \hat{\beta}(s_t) = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}.$$

Note that $\hat{\pi}$ is a well-defined Markov matrix in that $\sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) = 1$ for all s_t and that $\hat{\beta}(s_t)$ is stochastic as long as the original Markov process is not *iid* over time. For future reference, we also define

$$\hat{\beta}(s^t) = \hat{\beta}(s_0) \cdot \ldots \cdot \hat{\beta}(s_t)$$

and note that $\frac{\hat{\beta}(s^t)}{\hat{\beta}(s^{t-1})} = \hat{\beta}(s_t)$. Finally, let $\hat{U}(\hat{c})(s^t)$ denote the lifetime expected continuation utility in node s^t, under the new transition probabilities and discount factor, defined over consumption shares $\{\hat{c}_t(s^t)\}$

$$\hat{U}(\hat{c})(s^t) = u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t)\hat{U}(\hat{c})(s^t, s_{t+1}) \tag{5}$$

In the appendix we prove that this transformation does not alter the agents' ranking of different consumption streams.

Proposition 2.1. *Households rank consumption share allocations in the de-trended economy in exactly the same way as they rank the corresponding consumption allocations in the original growing economy: for any* s^t *and any two consumption allocations* c, c'

$$U(c)(s^t) \geq U(c')(s^t) \Longleftrightarrow \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

where the transformation of consumption into consumption shares is given by (4).

This result is crucial for demonstrating that equilibrium allocations c for the stochastically growing economy can be found by solving for equilibrium allocations $\hat{c}$ in the transformed economy.

2.4 Independence of Idiosyncratic Shocks from Aggregate Conditions

Next, we assume that idiosyncratic shocks are independent of the aggregate shocks. This assumption is crucial for most of the results in this paper.

Condition 2.2. *Individual endowment shares* $\eta(y_t, z_t)$ *are functions of the current idiosyncratic state* y_t *only, that is* $\eta(y_t, z_t) = \eta(y_t)$*. Also, transition probabilities of the shocks can be decomposed as*

$$\pi(z_{t+1}, y_{t+1}|z_t, y_t) = \varphi(y_{t+1}|y_t)\phi(z_{t+1}|z_t).$$

That is, individual endowment *shares* and the transition probabilities of the idiosyncratic shocks are independent of the aggregate state of the economy z. In this case, the growth-adjusted probability matrix $\hat{\pi}$ and the re-scaled discount factor is obtained by adjusting only the transition probabilities for the *aggregate* shock, ϕ, but not the transition probabilities for the idiosyncratic shocks:

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}|z_t), \text{ and } \hat{\phi}(z_{t+1}|z_t) = \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}.$$

Furthermore, the growth-adjusted discount factor only depends on the aggregate state z_t:

$$\hat{\beta}(z_t) = \beta \sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma} \tag{6}$$

The first part of our analysis, until section 6 inclusively, assumes that the aggregate shocks are independent over time:

Condition 2.3. *Aggregate endowment growth is i.i.d.:*

$$\phi(z_{t+1}|z_t) = \phi(z_{t+1}).$$

In this case the growth rate of aggregate endowment is uncorrelated over time, so that the logarithm of the aggregate endowment follows a random walk with drift.[11] As a result,

[11] In section 7 we show that most of our results survive the introduction of persistence in the growth rates if a complete set of contingent claims on aggregate shocks is traded.

the growth-adjusted discount factor is a constant: $\hat{\beta}(z_t) = \hat{\beta}$, since $\hat{\phi}(z_{t+1}|z_t) = \hat{\phi}(z_{t+1})$. There are two competing effects on the growth-adjusted discount rate: consumption growth itself makes agents more impatient, while the consumption risk makes them more patient.[12]

2.5 A Quartet of Economies

In order to derive our results, we study four models, whose main characteristics are summarized in table 1. The first three models are endowment economies with aggregate shocks. The models differ along two dimensions, namely whether agents can trade a full set of Arrow securities against aggregate shocks, and whether agents face idiosyncratic risk, in addition to aggregate risk. Idiosyncratic risk, if there is any, is never directly insurable.

Table 1: Summary of Four Economies

Model	*Aggregate Shocks*	*Idiosyncr. Shocks*	*Arrow Securities*
THL	Yes	Yes	No
Arrow	Yes	Yes	Yes
BL	Yes	No	Yes
Bewley	No	Yes	N/A

Our primary goal is to understand asset prices in the first model in the table, the *THL* model. This model has idiosyncratic and aggregate risk, as well as incomplete markets. Agents can only insure against idiosyncratic and aggregate shocks by trading a single bond and a stock.

The standard representative agent complete markets Breeden (1979)-Lucas (1978) (*BL*) model lies on the other end of the spectrum; there is no idiosyncratic risk and there is a full menu of Arrow securities for the representative agent to hedge against aggregate risk. Through our analysis we will demonstrate that in the *THL* model the standard representative

[12]The growth-adjusted measure $\hat{\phi}$ is obviously connected to the risk-neutral measure commonly used in asset pricing (see e.g. Harrison and Kreps, 1979). Under our hatted measure, agents can evaluate utils from consumption streams while abstracting from aggregate risk; under a risk-neutral measure, agents can price payoffs by simply discounting at the risk-free rate.

agent Euler equation for excess returns is satisfied:

$$E_t\left[\beta\left(\lambda_{t+1}\right)^{-\gamma}\left(R^s_{t+1}-R_t\right)\right]=0, \tag{7}$$

where R^s_{t+1} is the return on the stock, R_t is the return on the bond and λ_{t+1} is the growth rate of the aggregate endowment. Hence, the aggregate risk premium is identical in the *THL* and the *BL* model. Constantinides (1982) had already shown that, in the case of complete markets, even if agents are heterogeneous in wealth, there exists a representative agent who satisfies the Euler equation for excess returns (7) and also the Euler equation for bonds:

$$E_t\left[\beta\left(\lambda_{t+1}\right)^{-\gamma}R_t\right]=1. \tag{8}$$

The key to Constantinides' result is that markets are *complete*. We show that the first Euler equation in (7) survives market incompleteness and potentially binding solvency constraints. The second one does not. To demonstrate this, we employ a third model, the Arrow model (second row in the table). Here, households trade a full set of Arrow securities against aggregate risk, but not against idiosyncratic risk. The fundamental result underlying our asset pricing findings is that equilibria in both the *THL* and the Arrow model can be found by first determining equilibria in a model with *only* idiosyncratic risk (the Bewley model, fourth row in the table) and then by simply scaling consumption allocations in that model by the stochastic aggregate endowment.

We therefore start in section 3 by characterizing equilibria for the Bewley model, a stationary economy with a constant aggregate endowment in which agents trade a single discount bond and a stock.[13] This model merely serves as a computational device. Then we turn to the stochastically growing economy (with different market structures), the one whose asset pricing implications we are interested in, and we show that equilibrium consumption allocations from the Bewley model can be implemented as equilibrium allocations in the stochastically growing *THL* and Arrow model.

[13]One of the two assets will be redundant for the households, so that this model is a standard Bewley model, as studied by Bewley (1986), Huggett (1993) or Aiyagari (1994). The presence of both assets will make it easier to demonstrate our equivalence results with respect to the *THL* and Arrow model later on.

3 The Bewley Model

In this model the aggregate endowment is constant at 1. Households face idiosyncratic shocks y that follow a Markov process with transition probabilities $\varphi(y'|y)$. The household's preferences over consumption shares $\{\hat{c}(y^t)\}$ are as defined in equation (5), with the time discount factor $\hat{\beta}$ as defined in equation (6). The adjusted discount factor is $\hat{\beta}$ constant, because the aggregate shocks are i.i.d. (see Condition (2.3)).

3.1 Market Structure

Agents trade only a riskless discount bond and shares in a Lucas tree that yields safe dividends of α in every period. The price of the Lucas tree at time t is denoted by v_t.[14] The riskless bond is in zero net supply. Each household is indexed by an initial condition (θ_0, y_0), where θ_0 denotes its wealth (including period 0 labor income) at time 0.

The household chooses consumption $\{\hat{c}_t(\theta_0, y^t)\}$, bond positions $\{\hat{a}_t(\theta_0, y^t)\}$ and share holdings $\{\hat{\sigma}_t(\theta_0, y^t)\}$ to maximize its normalized expected lifetime utility $\hat{U}(\hat{c})(s^0)$, subject to a standard budget constraint:[15]

$$\hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t = \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})(\hat{v}_t + \alpha).$$

Finally, each household faces one of two types of borrowing constraints. The first one restricts household wealth at the end of the current period. The second one restricts household wealth at the beginning of the next period: [16]

$$\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t) \text{ for all } y^t. \tag{9}$$

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) \geq \hat{M}_t(y^t) \text{ for all } y^t. \tag{10}$$

[14] The price of the tree is nonstochastic due to the absence of aggregate risk.

[15] We suppress dependence on θ_0 for simplicity whenever there is no room for confusion.

[16] This distinction is redundant in the Bewley model, but it will become meaningful in our models with aggregate risk.

3.2 Equilibrium in the Bewley Model

The definition of equilibrium in this model is standard.

Definition 3.1. *For an initial distribution* Θ_0 *over* (θ_0, y_0) *, a competitive equilibrium for the Bewley model consists of trading strategies* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ *, and prices* $\{\hat{R}_t, \hat{v}_t\}$ *such that*

1. *Given prices, trading strategies solve the household maximization problem*

2. *The goods markets and asset markets clear in all periods t*

$$\begin{aligned}
\int \sum_{y^t} \varphi(y^t|y_0)\hat{c}_t(\theta_0, y^t) d\Theta_0 &= 1. \\
\int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, y^t) d\Theta_0 &= 0. \\
\int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, y^t) d\Theta_0 &= 1.
\end{aligned}$$

In the absence of aggregate risk, the bond and the stock are perfect substitutes for households, and no-arbitrage implies the following restriction on equilibrium stock prices and interest rates:

$$\hat{R}_t = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}.$$

In addition, with these equilibrium prices household portfolios are indeterminate. Without loss of generality one can therefore focus on trading strategies in which households only trade the stock, but not the bond: $\hat{a}_t(\theta_0, y^t) \equiv 0$.[17]

A *stationary equilibrium* in the Bewley model consists of a constant interest rate $\hat{R}$, a share price $\hat{v}$, optimal household allocations and a time-invariant measure Φ over income shocks and financial wealth.[18] In the stationary equilibrium households move within the invariant wealth distribution, but the wealth distribution itself is constant over time.

[17] Alternatively, we could have agents simply trade in the bond and adjust the net supply of bonds to account for the positive capital income α in the aggregate. We only introduce both assets into the Bewley economy to make the mapping to allocations in the Arrow and *THL* models simpler.

[18] See Chapter 17 of Ljungqvist and Sargent (2004) for the standard formal definition and the straightforward algorithm to compute such a stationary equilibrium.

4 The Arrow Model

We now turn to our main object of interest, the economy with aggregate risk. We first consider the Arrow market structure in which households can trade shares of the stock and a complete menu of contingent claims on aggregate shocks. Idiosyncratic shocks are still uninsurable. We demonstrate in this section that the allocations and prices of a stationary Bewley equilibrium can be made into equilibrium allocations and prices in the Arrow model with aggregate risk.

4.1 Trading

Let $a_t(s^t, z_{t+1})$ denote the quantity purchased of a security that pays off one unit of the consumption good if aggregate shock in the next period is z_{t+1}, irrespective of the idiosyncratic shock y_{t+1}. Its price today is given by $q_t(z^t, z_{t+1})$. In addition, households trade shares in the Lucas tree. We use $\sigma_t(s^t)$ to denote the number of shares a household with history $s^t = (y^t, z^t)$ purchases today and we let $v_t(z^t)$ denote the price of one share.

An agent starting period t with initial wealth $\theta_t(s^t)$ buys consumption commodities in the spot market and trades securities subject to the usual budget constraint:

$$c_t(s^t) + \sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \leq \theta_t(s^t). \tag{11}$$

If next period's state is $s^{t+1} = (s^t, y_{t+1}, z_{t+1})$, her wealth is given by her labor income, the payoff from the contingent claim purchased in the previous period as well as the value of her position on the stock, including dividends:

$$\theta_{t+1}(s^{t+1}) = \eta(y_{t+1}, z_{t+1}) e_{t+1}(z_{t+1}) + a_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1}). \right] \tag{12}$$

In addition to the budget constraints, the households' trading strategies are subject to solvency constraints of one of two types. The first type of constraint imposes a lower bound on the value of the asset portfolio at the end of the period today,

$$\sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \geq K_t(s^t), \tag{13}$$

while the second type imposes state-by-state lower bounds on net wealth tomorrow,

$$a_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1}) \right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \tag{14}$$

We assume these solvency constraints are at least tight enough to prevent Ponzi schemes. In addition, we impose restrictions on the solvency constraints that make them proportional to the aggregate endowment in the economy:

Condition 4.1. *We assume that the borrowing constraints only depend on the aggregate history through the level of the aggregate endowment. That is, we assume*

$$K_t(y^t, z^t) = \hat{K}_t(y^t) e_t(z^t),$$

and

$$M_t(y^t, z^t, z_{t+1}) = \hat{M}_t(y^t) e_{t+1}(z^{t+1}).$$

If the constraints did not have this feature in a stochastically growing economy, the constraints would become more or less binding as the economy grows, clearly not a desirable feature[19]. The definition of an equilibrium is completely standard (see section A.1 of the Appendix).

Instead of working with the model with aggregate risk, we transform the Arrow model into a stationary model. As we are about to show, the equilibrium allocations and prices in the de-trended model are the same as the allocations and prices in a stationary Bewley equilibrium.

[19] In the incomplete markets literature the borrowing constraints usually have this feature (see e.g. Heaton and Lucas (1996)). It is easy to show that solvency constraints that are not too tight in the sense of Alvarez and Jermann (2000) satisfy this condition.

4.2 Equilibrium in the De-trended Arrow Model

We use hatted variables to denote the variables in the stationary model. Households rank consumption shares $\{\hat{c}_t\}$ in exactly the same way as original consumption streams $\{c_t\}$. Dividing the budget constraint (11) by $e_t(z^t)$ and using equation (12) yields the deflated budget constraint:

$$\begin{aligned} & \hat{c}_t(s^t) + \sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) \\ \leq \quad & \eta(y_t) + \hat{a}_{t-1}(s^{t-1}, z_t) + \sigma_{t-1}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha \right], \end{aligned} \tag{15}$$

where we have defined the deflated Arrow positions as $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})}$ and prices as $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1}) \lambda(z_{t+1})$. The deflated stock prices are given by $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$. Similarly, by deflating the solvency constraints (13) and (14), using condition (4.1), yields:

$$\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) \quad \geq \quad \hat{K}_t(y^t). \tag{16}$$

$$\hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[\hat{v}_{t+1}(z^{t+1}) + \alpha \right] \quad \geq \quad \hat{M}_t(y^t) \text{ for all } z_{t+1}. \tag{17}$$

Finally, the goods market clearing condition is given by[20]

$$\int \sum_{y^t} \pi(y^t | y_0) \hat{c}_t(\theta_0, s^t) d\Theta_0 = 1. \tag{18}$$

The asset market clearing conditions are exactly the same as before. In the stationary economy, the household maximizes $\hat{U}(\hat{c})(s_0)$ by choosing consumption, Arrow securities and shares of the Lucas tree, subject to the budget constraint (44) and the solvency constraint (16) or (17) in each node s^t. The definition of a competitive equilibrium in the de-trended Arrow economy is straightforward.

Definition 4.1. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the de-trended Arrow model consists of trading strategies $\{\hat{a}_t(\theta_0, s^t, z_{t+1})\}$, $\{\hat{\sigma}_t(\theta_0, s^t)\}$, $\{\hat{c}_t(\theta_0, s^t)\}$ and prices $\{\hat{q}_t(z^t, z_{t+1})\}$, $\{\hat{v}_t(z^t)\}$ such that*

[20] The conditional probabilities simplify due to condition (2.2).

1. *Given prices, trading strategies solve the household maximization problem*

2. *The goods market clears, that is, equation* (18) *holds for all* z^t.

3. *The asset markets clear*

$$\begin{aligned}
\int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, s^t)d\Theta_0 &= 1 \\
\int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, s^t, z_{t+1})d\Theta_0 &= 0 \textit{ for all } z_{t+1} \in Z
\end{aligned}$$

The first order conditions and complementary slackness conditions, together with the appropriate transversality condition, are listed in the appendix in section (A.1.1). These are necessary and sufficient conditions for optimality on the household side. Now we are ready to establish the equivalence between equilibria in the Bewley model and in the Arrow model.

4.3 Equivalence Results

We now show that equilibria in the Bewely model can be mapped into equilibria of the stochastically growing Arrow model.

Theorem 4.1. *An equilibrium of the Bewley model* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t),\ \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the Arrow model with growth,* $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ *and* $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t)e_t(z^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1}) \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} = \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}} \qquad (19)
\end{aligned}$$

The proof is given in the appendix, but here we provide its main intuition. Conjecture that equilibrium prices of Arrow securities in the de-trended Arrow model are given by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t}. \tag{20}$$

The Euler equation for an unconstrained household with respect to Arrow securities reads as (see section (A.1.1) in the appendix)

$$1 = \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1}|s^t, z_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))}.$$

But under the maintained assumptions 2.2 and 2.3 and under the conjecture that consumption allocations in the de-trended Arrow model only depend on idiosyncratic shock histories y^t and not on $s^t = (y^t, z^t)$ this Euler equation reduces to

$$1 = \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{21}$$

$$= \hat{\beta}\hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}, \tag{22}$$

where we used the conjectured form of prices in (20). But this is exactly the Euler equation with respect to bonds in the Bewley model. Since Bewley equilibrium consumption allocations satisfy this condition, they therefore satisfy the Euler equation in the de-trended Arrow model, if prices are of the form (20). The proof in the appendix shows that a similar argument applies for the Euler equation with respect to the stock (under the conjectured stock prices), and also shows that for agents whose solvency constraints binds the Lagrange multipliers on the constraints in the Bewley equilibrium are also valid Lagrange multipliers for the constraints in the de-trended Arrow model. This implies, in particular, that our results go through independent of how tight the solvency constraints are. Once one has established that allocations and prices of a Bewley equilibrium are an equilibrium in the de-trended Arrow model, one simply needs to scale up allocation and prices by the appropriate growth factors to obtain the equilibrium prices and allocations in the stochastically growing Arrow model, as stated in the theorem.

It is straightforward to compute risk-free interest rates for the Arrow model. By summing over aggregate states tomorrow on both sides of equation (20), we obtain that the risk-free rate in the de-trended Arrow model coincides with that of the Bewley model:

$$\hat{R}_t^A = \frac{1}{\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})} = \hat{R}_t. \tag{23}$$

Once we have determined risk free interest rates for the de-trended economy, $\hat{R}_t^A = \hat{R}_t$, we can back out the implied interest rate for the original growing Arrow economy, using (19) in the previous theorem.[21]

Corollary 4.1. *If equilibrium risk-free interest rates in the de-trended Arrow model are given by* (23)*, equilibrium risk-free interest rates in the Arrow model with aggregate risk are given by*

$$R_t^A = \frac{1}{\sum_{z_{t+1}} q_t(z^t, z_{t+1})} = \hat{R}_t \frac{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}. \tag{24}$$

This result implies that, in the absence of aggregate risk (λ is only a function of time, but not of z_{t+1}) the risk-free rates in the original and deflated economy are related by the familiar relation $R_t^A = \hat{R}_t^A \lambda_{t+1}$where λ_{t+1} is the gross growth rate of endowment between period t and $t+1$.

The theorem implies that we can solve for an equilibrium in the Bewley model of section 3 (and in, in particular, a stationary equilibrium), including risk free interest rates $\hat{R}_t$, and we can deduce the equilibrium allocations and prices for the Arrow economy from those in the Bewley economy, using the mapping described in theorem 4.1. As shown, the key to this result is that households in the Bewley model face exactly the same Euler equations as the households in the de-trended version of the Arrow model.

This theorem has several important implications. First, we will use this equivalence result to show below that asset prices in the Arrow economy are identical to those in the representative agent economy, except for a lower risk-free interest rate (and a higher price/dividend ratio for stocks).[22] Second, the existence proofs in the literature for stationary equilibria

[21] The dependence of $\hat{R}_t^A$ on time t is not surprising since, for an arbitrary initial distribution of assets Θ_0, we cannot expect the equilibrium to be stationary. In the same way we expect that $\hat{v}_t(z^t)$ is only a function of t as well, but not of z^t.

[22] The fact that the risk-free interest rate is lower comes directly from the fact that interest rates in the

in the Bewley model directly carry over to the stochastically growing economy[23]. Third, the moments of the wealth distribution vary over time but proportionally to the aggregate endowment. If the initial wealth distribution in the de-trended model corresponds to an invariant distribution in the Bewley model, then for example the ratio of the mean to the standard deviation of the wealth distribution is constant in the Arrow model with aggregate risk as well. Finally, an important result of the previous theorem is that, in the Arrow equilibrium, the trade of Arrow securities is simply proportional to the aggregate endowment: $a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1})$, or, equivalently, in the de-trended Arrow model households choose not to make their contingent claims purchases contingent on next period's aggregate shock: $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$. Furthermore, since in the Bewley model without loss of generality $\hat{a}_t(\theta_0, y^t) = 0$, we can focus on the situation where Arrow securities are not traded at all: $a_t(\theta_0, s^t, z_{t+1}) = 0$. This no-trade result for contingent claims suggests that the equivalence will carry over to economies with more limited asset structures. That is what we show in the next section.

5 The *THL* Model

We now turn our attention to the main model of interest, namely the model with a stock and a single uncontingent bond. This section establishes the equivalence of equilibria in the *THL* model and the Bewley model by showing that the optimality conditions in the de-trended Arrow and the de-trended *THL* model are identical. In addition, we show that in the benchmark case with i.i.d. aggregate endowment growth shocks agents do not even trade bonds in equilibrium.

5.1 Market Structure

In the *THL* economy, agents only trade a one-period discount bond and a stock. An agent who starts period t with initial wealth composed on bond and stock payout and labor income

Bewley model are lower than in the corresponding representative agent model without aggregate risk.

[23]See e.g. Huggett (1993), Aiyagari (1994) or Miao (2002) for (elements of) existence proofs. Uniqueness of a stationary equilibrium is much harder to establish. Our equivalence result shows that for any stationary Bewley equilibrium there exists a corresponding Arrow equilibrium in the model with aggregate risk. Furthermore note that our result does not rule out other Arrow equilibria either.

buys consumption commodities in the spot market and trades a one-period bond and the stock, subject to the budget constraint:

$$c_t(s^t) + \frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)v_t(z^t) \leq \eta(y_t)e_t(z_t) + b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1})\left[v_t(z^t) + \alpha e_t(z_t)\right] \quad (25)$$

Here, $b_t(s^t)$ denotes the amount of bonds purchased and $R_t(z^t)$ is the gross interest rate from period t to $t+1$. As was the case in the Arrow model, short-sales of the bond and the stock are constrained by a lower bound on the value of the portfolio today,

$$\frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t), \quad (26)$$

or a state-by-state constraint on the value of the portfolio tomorrow,

$$b_t(s^t) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \quad (27)$$

Since $b_t(s^t)$ and $\sigma_t(s^t)$ are chosen before z_{t+1} is realized, at most one of the constraints (27) will be binding at a given time. The definition of an equilibrium for the *THL* model follows directly. (see section (A.2) in the appendix). We now show that the equilibria in the Arrow and the *THL* model coincide. As a corollary, it follows that the asset pricing implications of both models are identical. In order to do so, we first transform the model with growth into a stationary, de-trended model.

5.2 Equilibrium in the De-trended *THL* Model

Dividing the budget constraint (25) by $e_t(z^t)$ we obtain

$$\hat{c}_t(s^t) + \frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \leq \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t(z^t) + \alpha\right],$$

where we define the deflated bond position as $\hat{b}_t(s^t) = \frac{b_t(s^t)}{e_t(z^t)}$. Using condition (4.1), the solvency constraints in the de-trended economy are simply:

$$\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \geq \hat{K}_t(y^t), \text{ or} \tag{28}$$

$$\frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] \geq \hat{M}_t(y^t) \text{ for all } z_{t+1}. \tag{29}$$

The definition of equilibrium in the de-trended *THL* model is straightforward and hence omitted.[24] We now show that equilibrium consumption allocations in the de-trended *THL* model coincide with those of the Arrow model. For simplicity, we first abstract from binding borrowing constraints and then extend our results to that case later on.

5.3 Equivalence Results

As for the Arrow economy, we can show that the Bewley equilibrium allocations and prices constitute, after appropriate scaling by endowment (growth) factors, an equilibrium of the *THL* model with growth.

Theorem 5.1. *An equilibrium of a stationary Bewley economy, given by trading strategies* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ *and prices* $\{\hat{R}_t, \hat{v}_t\}$*, can be made into an equilibrium for the* THL *model with growth,* $\{b_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$, $\left\{\sigma_t^{THL}(\theta_0, s^t)\right\}$ *and* $\{R_t(z^t)\}$ *and* $\{v_t(z^t)\}$

[24] We list the first order conditions for household optimality and the transversality conditions in section (A.2) of the appendix.

where

$$
\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\
\sigma_t^{THL}(\theta_0, s^t) &= \frac{\hat{a}_t(\theta_0, y^t)}{[\hat{v}_{t+1} + \alpha]} + \hat{\sigma}_t(\theta_0, y^t) \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
R_t(z^t) &= \hat{R}_t * \frac{\sum_{z_{t+1}} \phi(z_{t+1}) \lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}) \lambda(z_{t+1})^{-\gamma}}
\end{aligned}
$$

and bond holdings given by $b_t(\theta_0, s^t) = 0$.

The crucial step of the proof, given in the appendix, shows that Bewley allocations, given the prices proposed in the theorem, satisfy the necessary and sufficient conditions for household optimality and all market clearing conditions in the de-trended *THL* model.

This theorem again has several important consequences. First, equilibrium risk-free rates in the Arrow and in the *THL* model coincide, despite the fact that the set of assets agents can trade to insure consumption risk differs in the two models. Second, in equilibrium of the *THL* model, the bond market is inoperative: $b_{t-1}(s^{t-1}) = \hat{b}_{t-1}(s^{t-1}) = 0$ for all s^{t-1}. Therefore all consumption smoothing is done by trading stocks, and agents keep their net wealth proportional to the level of the aggregate endowment.[25]

In summary, our results show that one can solve for equilibria in a standard Bewley model and then map this equilibrium into an equilibrium for both the Arrow model and the *THL* model *with* aggregate risk. The risk-free interest rate and the price of the Lucas tree coincide in the stochastic Arrow and *THL* economies. Finally, without loss of generality, we can restrict attention to equilibria in which bonds are not traded; consequently transaction costs in the bond market would not change our results. Transaction costs in the stock market of course would (see section (7)).

In both the Arrow and the *THL* model, households do not have a motive for trading bonds, unless there are short-sale constraints on stocks. We do not deal with this case. In

[25] There is a subtle difference between this result and the corresponding result for the Arrow model. In the Arrow model we demonstrated that contingent claims positions were in fact uncontingent: $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$ and equal to the Bond position in the Bewley equilibrium, but not necessarily equal to zero. In the *THL* model bond positions *have to be* zero. But since bonds in the Bewley equilibrium are a redundant asset, one can restrict attention to the situation where $\hat{a}_t(\theta_0, y^t) = 0$, although this is not necessary for our results.

addition, the no-trade result depends critically on the i.i.d assumption for aggregate shocks, as we will show in section (7). If the aggregate shocks are not i.i.d, agents want to hedge against the implied shocks to interest rates. We will show in section (7) that these interest rate shocks look like taste shocks in the de-trended model.

But first we compare the asset pricing implications of the equilibria just described in the Arrow and the *THL* models to those emerging from the *BL* (standard representative agent) model.

6 Asset Pricing Implications

This section shows that the multiplicative risk premium on a claim to aggregate consumption in the *THL* model -and the Arrow model- equals the risk premium in the representative agent model. Uninsurable idiosyncratic income risk only lowers the risk-free rate.

6.1 Consumption-CAPM

The benchmark model of consumption-based asset pricing is the representative agent *BL* model. The representative agent owns a claim to the aggregate 'labor' income stream $\{(1-\alpha)e_t(z^t)\}$ and she can trade a stock (a claim to the dividends $\alpha e_t(z^t)$ of the Lucas tree), a bond and a complete set of Arrow securities.[26]

First, we show that the *B*reeden-*L*ucas Consumption-CAPM also prices excess returns on the stock in the *THL* model and the Arrow model. R^s denotes the return on a claim to aggregate consumption.

Lemma 6.1. *The Consumption-CAPM prices excess returns in the Arrow economy and the* THL *Economy:*

$$E_t\left[\left(R^s_{t+1}-R_t\right)\beta\left(\lambda_{t+1}\right)^{-\gamma}\right]=0$$

This result follows directly from the Euler equation in (22). It has important implications for empirical work in asset pricing. First, in spite of the market incompleteness and binding solvency constraints, an econometrician can estimate the coefficient of risk aversion directly

[26]see section (A.3) in the Appendix for a complete description.

from aggregate consumption data and the excess return on stocks, as in Hansen and Singleton (1982). Second, it provides a strong justification for explaining the cross-section of excess returns using the CCAPM, without trying to match the risk-free rate.

6.2 Risk Premia

Not surprisingly, the equilibrium risk premium is identical to the one in the representative agent economy.[27] Since the risk-free rate is higher than in the Arrow and *THL* model, the price of the stock is correspondingly lower. However, the multiplicative risk premium is the same in all three models and it is constant.

The stochastic discount factors that prices stochastic payoffs in the representative agent economy and the Arrow economy only differ by a non-random multiplicative term, equal to the ratio of (growth-deflated) risk-free interest rates in the two models. We use the superscript RE to denote the Representative Agent economy.

Proposition 6.1. *In the Arrow economy, there is a unique SDF given by*

$$m^A_{t+1} = m^{RE}_{t+1}\kappa_t$$

with a non-random multiplicative term given by:

$$\kappa_t = \frac{\hat{R}_t^{RE}}{\hat{R}^A_t} \geq 1$$

Note that the term κ_t is straightforward to compute since $\hat{R}^A_t$ equals the interest rate in the Bewley model discussed in section (3). What about the *THL* economy? We have shown, using the equivalence result between the Arrow and the *THL* model, that the Arrow economy's stochastic discount factor m^A_{t+1} is also a valid stochastic discount factor in the *THL* economy. So, our results below carry over to the *THL* economy as well.

The proof that risk premia are unchanged between the representative agent model and the Arrow model now follows directly from the previous decomposition of the SDF.[28] Let

[27] This does not immediately follow from Lemma 6.1.

[28] The proof strategy follows Alvarez and Jermann (2001) who derive a similar result in the context of a *complete markets* model populated by two agents that face endogenous solvency constraints.

$R_{t,j}[\{d_{t+k}\}]$ denote the j-period holding return on claim to $\{d_{t+k}\}$. Consequently $R_{t,1}[1]$ is the gross risk-free rate and $R_{t,1}[\alpha e_{t+k}]$ is the one-period holding return on a k-period strip of the aggregate endowment (a claim to α times the aggregate endowment k periods from now). With this notation in place, we can state our main result.

Theorem 6.1. *The multiplicative risk premium in the Arrow economy equals that in the representative agent economy*

$$1+\nu_t^A = 1+\nu_t^{RE} = \frac{E_t R_{t,1}[\{e_{t+k}\}]}{R_{t,1}[1]}$$

Thus, the extent to which households smooth idiosyncratic income shocks (in the Arrow model or in the *THL* model) amount has absolutely no effect on the size of the risk premia; it merely lowers the risk-free rate. Exactly the same result applies to the *THL* model as well. Luttmer (1991) and Cochrane and Hansen (1992) had already established a similar aggregation result for the case in which households face market wealth constraints, but in a complete markets environment. We show this result survives even if households trade only a stock and a bond. The market incompleteness does not generate any dynamics in the conditional risk premia either: the conditional risk premium is constant.

7 Robustness and Extensions of the Main Results

In this section, we investigate how robust our results are, and we demonstrate that our assumption that the aggregate shocks are *i.i.d* over time, which means the growth rates of the aggregate endowment are *i.i.d* over time, is not crucial for our results.

7.1 Non-iid Aggregate Shocks

When the aggregate shocks z follow an arbitrary, finite state, Markov chain, the growth-adjusted time discount factor $\hat{\beta}(z)$ depends on the current aggregate state, and, as a result, the aggregate endowment shock acts as an aggregate taste shock in the deflated economy. This shock renders all households more or less impatient. Of course, households are not able to insure against this shock at all, since it affects all households in the same way. As a consequence, non-iid aggregate endowment shocks, acting as taste shocks in the deflated economy, only affect the price/dividend ratio and the interest rate, but not the risk premium. However, in this case, there is trade in the Arrow securities market.

Stationary Bewley economy In order to establish these results, as before we use a stationary Bewley economy as a computational device. Denote by $\hat{\Pi}(z')$ the invariant distribution over the aggregate shock under the hatted measure. Agents in the stationary economy discount future utility flows using the deterministic time discount factor process $\left\{\widetilde{\beta}_t\right\}$, where $\widetilde{\beta}_t$ denotes the average subjective discount factor:

$$\widetilde{\beta}_t = \sum_{z_0} \hat{\Pi}(z_0) \sum_{z^{t-1}} \widehat{\phi}(z^{t-1}|z_0)\hat{\beta}_{0,t-1}(z^{t-1}|z_0), t \geq 1, \tag{30}$$

and where we have used $\hat{\beta}_{0,\tau}(z^\tau|z_0)$ to denote the product of time discount factors:

$$\hat{\beta}_{0,\tau}(z^\tau|z_0) = \hat{\beta}(z_0)\hat{\beta}(z_1)\ldots\hat{\beta}(z_\tau).$$

Given this non-random sequence of subjective time discount factors, we can define an equilibrium for the Bewley economy in the standard way. In the simplest case of i.i.d shocks we obtain $\widetilde{\beta}_t = \hat{\beta}^t$, as in the case we discussed in the previous section.

We choose this particular sequence of subjective time discount factors because it ensures that the time zero budget constraint in the deflated Arrow economy is satisfied given the same initial wealth distribution Θ_0 as in the stationary Bewley economy. In order to construct equilibrium allocations, we first determine equilibrium allocations and interest rates in the Bewley economy with time discount factors $\left\{\widetilde{\beta}_t\right\}$, analogous to the analysis in section (3). In a second step, we make these allocations and interest rates into an equilibrium of the *actual* Arrow economy with time-varying discount factors by adjusting the risk-free interest rate in proportion to the taste shock $\hat{\beta}(z)$. We do not change the allocations, and we back out the implied Arrow securities positions. To understand the effect of these aggregate taste shocks on the time discount rate in the Bewley economy, we consider a simple example.

Example 7.1. *If the $\hat{\beta}(z)$ are lognormal and i.i.d with variance σ^2 , then the effective average* *time discount rate between time 0 and time t is given by:*

$$\frac{\widetilde{\rho}_t}{t} = \widehat{\rho} - \frac{1}{2}\sigma^2 \textit{ for any } t \geq 1$$

where $\widetilde{\beta}_t = e^{-\widetilde{\rho}_t}$ *and* $E\hat{\beta}(z) = e^{-\widehat{\rho}}$. *As a result, the time discount rate in the Bewley economy is lower than the actual discount rate, because of the risk associated with the taste shocks.*

We use $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and prices $\left\{\hat{R}_t, \hat{v}_t\right\}$ to denote the Bewley equilibrium for given $\left\{\widetilde{\beta}_t\right\}$. Only the total net wealth positions in the Bewley economy are pinned down (since the bond and the stock, by a no-arbitrage condition, have exactly the same return, and are both risk-free assets):

$$\hat{b}_t(\theta_0, y^t) = \hat{a}_t(\theta_0, y^t) + \hat{\sigma}_t(\theta_0, y^t)(\hat{v}_t + \alpha).$$

Without loss of generality, we focus on the case where $\hat{a}_t(\theta_0, y^t) = 0$ for all y^t. Our goal is to show that the consumption and share allocation $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ can be made into an Arrow equilibrium, and then to argue why we need to choose the specific discount factor sequence in (30) for the Bewley model. To fix notation, let

$$\tilde{Q}_{t,\tau} = \prod_{j=0}^{\tau-t-1} \hat{R}_{t+j}^{-1} = \frac{1}{\widehat{R}_{\tau,t}}$$

denote the price (in the Bewley equilibrium) for one unit of consumption to be delivered at time τ, in terms of consumption at time t. Also, let us use the convention $\tilde{Q}_\tau = \tilde{Q}_{t=0,\tau}$ and $\tilde{Q}_{t=\tau} = 1$.

Arrow economy Next, we follow the same recipe as before, by implementing the Bewley allocations as part of an equilibrium for the deflated Arrow economy, but in this case, this will inolve trade in the contingent claims market. To start off, we conjecture that the Arrow-Debreu prices in the deflated Arrow economy are given by:

$$\hat{Q}(z^t|z_0) = \frac{\hat{\beta}_{0,t-1}(z^{t-1})\hat{\phi}(z^t|z_0)}{\widetilde{\beta}_t \hat{R}_{0,t}},$$

from which we can easily recover the Arrow prices from the usual relationship:

$$\frac{\hat{Q}(z^{t+1}|z_0)}{\hat{Q}(z^t|z_0)} = q(z_{t+1}, z_t) = \hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t)\frac{1}{\hat{R}_t}\frac{\widetilde{\beta}_t}{\widetilde{\beta}_{t+1}} \tag{31}$$

where $\frac{1}{\hat{R}_t} = \frac{\hat{R}_{0,t}}{\hat{R}_{0,t+1}}$. These Arrow prices are Markovian, since $\hat{R}_t$ and $\widetilde{\beta}_t$ are deterministic.

Lemma 7.1. *The household Euler equations are satisfied in the Arrow model at the Bewley allocations $\{\hat{c}_{t+1}(y^t, y_{t+1})\}$ and Arrow prices $\{q(z_{t+1}, z_t)\}$ given by (31).*

Naturally, this implies that the state-contingent interest rates in the Arrow model are given by

$$\frac{1}{\hat{R}_t^A(z_t)} = \hat{\beta}(z_t)\frac{\widetilde{\beta}_t}{\hat{R}_t\widetilde{\beta}_{t+1}},$$

which can easily be verified from equation (31).

Trading Trading starts before the initial aggregate shock z_0 is observed. In the Arrow economy, there is trade in the contingent claims markets. First, we need to check that these contingent claims positions implied by the Bewley allocations clear the securities markets.

Proposition 7.1. *The contingent claims positions implied by the Bewley allocations:*

$$\begin{aligned}\hat{a}_{t-1}(y^{t-1}, z^t, \theta_0) &= \hat{c}_t(y^t, \theta_0) - \eta(y_t) + \sum_{\tau=t+1}^{\infty}\sum_{z^\tau, y^\tau}\hat{Q}_t(z^\tau|z_t)\left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\ &\quad -\sigma_{t-1}(y^{t-1})\left[\hat{v}_t(z_t) + \alpha\right] = \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)\end{aligned} \tag{32}$$

clear the bond market

The contingent claims positions are used to hedge against interest rate shocks, as is clear from this restatement of the bond position:

$$\begin{aligned}\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) &= \sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau\right)\sum_{\eta^\tau}\left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\ &\quad -\sigma_{t-1}(y^{t-1})\alpha\sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau\right)\end{aligned}$$

The difference between $\hat{Q}_\tau(z^\tau|z_t)$ and $\widetilde{Q}$ is governed by the interest rate shocks. If the aggregate consumption growth shocks are i.i.d, this gap is zero. Second, to close our argument, we need to make sure that no wealth transfers are required to implement the Bewley equilibrium allocations in the deflated Arrow economy. In other words, we need to make sure that the state contingent bond portfolio at time zero, before the realization of z_0, is worth exactly zero. For this we proceed in two steps. First, we show that at time zero the state prices in the Arrow and Bewley economy coincide.

Lemma 7.2. *As a result of our choice of* $\left\{\widetilde{\beta}_t\right\}$ *in the Bewley economy, on average these AD prices coincide with the Bewley prices at time 0, before the realization of* z_0*:*

$$\sum_{z_0} \hat{\Pi}(z_0) \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_0) = \tilde{Q}_{0,\tau}.$$

Second, using the result about prices, we show that the cost of the bond portfolio is zero.

Lemma 7.3. *The cost at time 0 of the time 0 state-contingent claims portfolio is zero:*

$$\sum_{z^t} \hat{\Pi}(z_0) \left[\hat{a}_{-1}(y^{-1}, z_0, \theta_0)\right] = 0$$

Since we know the contingent bond positions are zero cost, the time 0 budget constraint is satisfied in the deflated Arrow economy for the exact same initial wealth distribution Θ_0 as in the Bewley economy.

Theorem 7.2. *An equilibrium of a stationary Bewley economy, populated by households with constant discount factor* $\widetilde{\beta}$, $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t) = 0, \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the Arrow economy with growth,* $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$

and $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)e_t(z^t) \text{ defined above} \\
v_t(z_t) &= \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1}|z_t)}{\lambda(z_{t+1})} \left[\frac{v_{t+1}(z_{t+1}) + \alpha e_{t+1}(z_{t+1})}{\hat{R}_t^A(z_t)} \right] \\
\hat{R}_t^A(z_t) &= \frac{\hat{R}_t \widetilde{\beta}_{t+1}}{\hat{\beta}(z_t)\widetilde{\beta}_t} \\
q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t^A(z_t)} * \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}
\end{aligned}$$

Evidently, as before one can back out the Arrow prices for the deflated economy from the prices in the deflated Bewley economy. The contingent claims positions required to implement the Bewley allocations where determined above.

Risk Premia Of course, this implies that our baseline irrelevance result for risk premia survives the introduction of non-i.i.d. aggregate shocks, provided that a complete menu of aggregate-state-contingent securities is traded. These aggregate taste shocks only affect interest rates and price/dividend ratios, not risk premia. When agents in the transformed economy become more impatient, the interest rises and the price/dividend ratio decreases, but the conditional expected excess return is unchanged.

Solvency Constraints So far we have abstracted from solvency constraints. Remember that we originally assumed that the solvency constraints satisfy $K_t(s^t) = \hat{K}_t(y^t)e_t(z^t)$ and $M_{t+1}(s^{t+1}) = \hat{M}_t(y^t)e_t(z^{t+1})$. The allocations computed in the stationary economy using $\hat{K}_t(y^t)$ and $\hat{M}_t(y^t)$ as solvency constraints, satisfy a modified version of the solvency constraints $K_t(s^t)$ and $M_{t+1}(s^{t+1})$.

Lemma 7.4. *The allocations from the ergodic economy satisfy the modified solvency con-*

straints:

$$
\begin{aligned}
K_t^*(s^t) &= K_t(s^t) - \sum_{z_{t+1}} q_t(z^t, z_{t+1}) a_t(s^t, z_{t+1}) \\
M_{t+1}^*(s^{t+1}) &= M_{t+1}(s^{t+1}) - a_t(s^t, z_{t+1})
\end{aligned}
$$

If the allocations satisfy the constraints in the stationary Bewley economy, they satisfy the modified solvency constraints in the actual Arrow economy, but not the ones we originally specified, because of the nonzero state-contingent claims positions.[29] Nevertheless, in principle, one could reverse-engineer a sequence of auxiliary solvency constraints such that in the actual equilibrium the modified version of the auxiliary constraints coincides with the actual constraints we want to impose, $K_t(s^t)$. As a consequence, the irrelevance result goes through, but of course, it would be hard to actually compute this equilibrium.

***THL* Economy** For the *THL* model, the same equivalence result obviously no longer holds, because the market for state contingent Arrow securities is now operative in the Arrow economy. When there is predictability in aggregate consumption growth, households actually find it optimal to trade the state-contingent Arrow securities, but the market structure in the *THL* economy prevents them from doing so.

7.2 Preferences

What role do preferences play in our results? Well, it is key to have homogeneous preferences, but time separability is not critical, at least not in the benchmark case of i.i.d. aggregate shocks. In section (A.1) of the appendix, we study the case of Epstein-Zin preferences, and we show that, in the case of i.i.d. aggregate consumption growth, the irrelevance result survives. However, in the more general case of non-i.i.d. aggregate shocks, the irrelevance result seems to break down.

[29] However, it is easy to verify using the result in Corollary (7.3) in the appendix that these modified solvency constraints coincide with the actual ones on average (averaged across z shocks). These deviations are completely due to the impact of interest rate changes on the value of the portfolio. The risk premia are still constant over the business cycle.

8 Related Literature and Conclusion

Recently Krusell and Smith (1997) and Storesletten, Telmer and Yaron (2006) have argued that models with idiosyncratic income shocks and incomplete markets can generate an equity premium that is substantially larger than the CCAPM if there is counter-cyclical cross-sectional variance (CCV) in labor income shocks. Storesletten, Telmer and Yaron (2004) argue that this condition is satisfied in the data, although it is not clear the CCV in the data is strong enough to explain equity risk premia at reasonable levels of risk aversion. Our paper demonstrates analytically that CCV is not only sufficient, but *necessary* to make uninsurable idiosyncratic income shocks potentially useful for explaining the equity premium.

Mankiw (1986) already showed in his analysis of the standard incomplete markets Euler equation that if the marginal utility of consumption is convex (a property that CRRA utility and also CARA utility satisfies), then *if* the cross-sectional variance of equilibrium household consumption growth increases in recessions, larger risk premia obtain. The CCV mechanism is one way to induce time variation in the equilibrium consumption growth distribution across households. Our work shows that solvency constraints and transaction costs in incomplete market models alone, without the CCV mechanism, cannot produce this time variation as an equilibrium outcome.

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A Additional Definitions

A.1 Arrow Model

The definition of an equilibrium in the Arrow model is standard. Each household is assigned a label that consists of its initial financial wealth θ_0 and its initial state $s_0 = (y_0, z_0)$. A household of type (θ_0, s_0) then chooses consumption allocations $\left\{c_t(\theta_0, s^t)\right\}$, trading strategies for Arrow securities $\left\{a_t(\theta_0, s^t, z_{t+1})\right\}$ and shares $\{\sigma_t(\theta_0, s^t)\}$ to maximize her expected utility (1), subject to the budget constraints (11) and subject to solvency constraints (13) or (14).

Definition A.1. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the Arrow model consists of household allocations $\left\{a_t(\theta_0, s^t, z_{t+1})\right\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\left\{c_t(\theta_0, s^t)\right\}$ and prices $\left\{q_t(z^t, z_{t+1})\right\}$, $\{v_t(z^t)\}$ such that*

1. *Given prices, household allocations solve the household maximization problem*
2. *The goods market clears for all z^t,*

$$\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} c_t(\theta_0, s^t) d\Theta_0 = e_t(z^t)$$

3. *The asset markets clear for all z^t*

$$\begin{aligned} \int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 &= 1 \\ \int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} a_t(\theta_0, s^t, z_{t+1}) d\Theta_0 &= 0 \text{ for all } z_{t+1} \in Z \end{aligned}$$

A.1.1 Optimality Conditions for De-trended Arrow Model

Define Lagrange multiplier

$$\hat{\beta}(s^t)\hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) \hat{\mu}(s^t) \geq 0$$

for the constraint in (16) and

$$\hat{\beta}(s^t)\hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) \hat{\kappa}_t(s^t, z_{t+1}) \geq 0$$

for the constraint in (17). The Euler equations of the de-trended Arrow model are given by:

$$\begin{aligned} 1 &= \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1}|s^t, z_{t+1}} \hat{\pi}(s_{t+1} | s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\ &\quad + \hat{\mu}_t(s^t) + \frac{\hat{\kappa}_t(s^t, z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \; \forall z_{t+1}. \end{aligned} \tag{33}$$

$$\begin{aligned} 1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1} | s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\ &\quad + \hat{\mu}_t(s^t) + \sum_{z_{t+1}} \hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right]. \end{aligned} \tag{34}$$

Only one of the two Lagrange multipliers enters the equations, depending on which version of the solvency constraint we consider. The complementary slackness conditions for the Lagrange multipliers are given by

$$\begin{aligned} \hat{\mu}_t(s^t)\left[\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \hat{\sigma}_t(s^t)\hat{v}_t(z^t) - \hat{K}_t(y^t)\right] &= 0 \\ \hat{\kappa}_t(s^t, z_{t+1})\left[\hat{a}_t(s^t, z_{t+1}) + \hat{\sigma}_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] - \hat{M}_t(y^t)\right] &= 0. \end{aligned}$$

The appropriate transversality conditions read as

$$\begin{aligned} \lim_{t\to\infty}\sum_{s^t} \hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))[\hat{a}_{t-1}(s^{t-1}, z_t) - \hat{M}_{t-1}(y^{t-1})] &= 0 \\ \lim_{t\to\infty}\sum_{s^t} \hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))[\hat{\sigma}_{t-1}(s^{t-1})(\hat{v}_t(z^t) + \alpha) - \hat{M}_{t-1}(y^{t-1})] &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{t\to\infty}\sum_{s^t} \hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\left[\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) - \hat{K}_t(y^t)\right] &= 0 \\ \lim_{t\to\infty}\sum_{s^t} \hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\left[\hat{\sigma}_t(s^t)\hat{v}_t(z^t) - \hat{K}_t(y^t)\right] &= 0. \end{aligned}$$

Since the household optimization has a concave objective function and a convex constraint set the first order conditions and complementary slackness conditions, together with the transversality condition, are necessary and sufficient conditions for optimality of household allocation choices.

A.2 *THL* Model

Agents only trade a single bond a single stock. Wealth tomorrow in state $s^{t+1} = (s^t, y_{t+1}, z_{t+1})$ is given by

$$\theta_{t+1}(s^{t+1}) = \eta(y_{t+1})e_{t+1}(z_{t+1}) + b_t(s^t) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})\right].$$

Definition A.2. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the* THL *economy consists of household allocations $\{b_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$, $\{\sigma_t(\theta_0, s^t)\}$ and interest rates $\{R_t(z^t)\}$ and share prices $\{v_t(z^t)\}$ such that*

1. *Given prices, allocations solve the household maximization problem*
2. *The goods market clears for all z^t*

$$\int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} c_t(\theta_0, s^t)d\Theta_0 = e_t(z^t)$$

3. The asset markets clear for all z^t

$$\begin{aligned} \int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 &= 1 \\ \int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} b_t(\theta_0, s^t) d\Theta_0 &= 0. \end{aligned}$$

A.2.1 Optimality Conditions for *THL* Economy

Define the Lagrange multiplier

$$\hat{\beta}(s^t)\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\hat{\mu}(s^t) \geq 0$$

for the constraint in (28) and

$$\hat{\beta}(s^t)\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\hat{\kappa}_t(s^t, z_{t+1}) \geq 0$$

for the constraint in (29). In the detrended *THL* economy the Euler equations read as

$$\begin{aligned} 1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\ &\quad + \hat{\mu}_t(s^t) + \sum_{z_{t+1}} \hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \qquad (35) \\ 1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\ &\quad + \hat{\mu}_t(s^t) + \sum_{z_{t+1}} \hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right], \qquad (36) \end{aligned}$$

with complementary slackness conditions given by:

$$\begin{aligned} \hat{\mu}_t(s^t) \left[\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \hat{\sigma}_t(s^t)\hat{v}_t(z^t) - \hat{K}_t(y^t)\right] &= 0 \\ \hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \hat{\sigma}_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] - \hat{M}_t(y^t)\right] &= 0 \end{aligned}$$

The transversality conditions read as

$$\lim_{t\to\infty} \sum_{s^t} \hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t)) \left[\frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} - \hat{M}_{t-1}(y^{t-1})\right] = 0$$

$$\lim_{t\to\infty} \sum_{s^t} \hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))[\hat{\sigma}_{t-1}(s^{t-1})(\hat{v}_t(z^t) + \alpha) - \hat{M}_{t-1}(y^{t-1})] = 0$$

and

$$\begin{aligned}
\lim_{t\to\infty}\sum_{s^t}\hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\left[\frac{\hat{b}_t(s^t)}{R_t(z^t)}-\hat{K}_t(y^t)\right] &= 0 \\
\lim_{t\to\infty}\sum_{s^t}\hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\left[\hat{\sigma}_t(s^t)\hat{v}_t(z^t)-\hat{K}_t(y^t)\right] &= 0.
\end{aligned}$$

A.3 Representative Agent Model

The budget constraint of the representative agent who consumes aggregate consumption $c_t(z^t)$ reads as

$$\begin{aligned}
& c_t(z^t)+\sum_{z_{t+1}}a_t(z^t,z_{t+1})q_t(z^t,z_{t+1})+\sigma_t(z^t)v_t(z^t) \\
\leq\ & e_t(z^t)+a_{t-1}(z^{t-1},z_t)+\sigma_{t-1}(z^{t-1})\left[v_t(z^t)+\alpha e_t(z_t)\right]
\end{aligned}$$

After deflating by the aggregate endowment $e_t(z^t)$, the budget constraint reads as

$$\begin{aligned}
& \hat{c}_t(z^t)+\sum_{z_{t+1}}\hat{a}_t(z^t,z_{t+1})\hat{q}_t(z^t,z_{t+1})+\sigma_t(z^t)\hat{v}_t(z^t) \\
\leq\ & 1+\hat{a}_{t-1}(z^{t-1},z_t)+\sigma_{t-1}(z^{t-1})\left[\hat{v}_t(z^t)+\alpha\right],
\end{aligned}$$

where $\hat{a}_t(z^t,z_{t+1})=\frac{a_t(z^t,z_{t+1})}{e_{t+1}(z^{t+1})}$ and $\hat{q}_t(z^t,z_{t+1})=q_t(z^t,z_{t+1})\lambda(z_{t+1})$ as well as $\hat{v}_t(z^t)=\frac{v_t(z^t)}{e_t(z^t)}$, precisely as in the Arrow model. Obviously, in an equilibrium of this model the representative agent consumes the aggregate endowment.

Lemma A.1. *Equilibrium asset prices are given by*

$$\begin{aligned}
\hat{q}_t(z^t,z_{t+1}) &= \hat{\beta}\hat{\phi}(z_{t+1})=\hat{q}(z_{t+1}) \text{ for all } z_{t+1} \\
\hat{v}_t(z^t) &= \hat{\beta}\sum_{z_{t+1}}\hat{\phi}(z_{t+1})\left[\hat{v}_{t+1}(z^{t+1})+\alpha\right]
\end{aligned}$$

A.4 Recursive Utility

We consider the class of preferences due to Epstein and Zin (1989) . Let $V(c^i)$ denote the utility derived from consuming c^i :

$$V(c^i)=\left[(1-\beta)c_t^{1-\rho}+\beta(\mathcal{R}_tV_1)^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where the risk-adjusted expectation operator is defined as:

$$\mathcal{R}_tV_{t+1}=\left(E_tV_{t+1}^{1-\alpha}\right)^{1/1-\alpha}.$$

α governs risk aversion and ρ governs the willingness to substitute consumption intertemporally. These preferences impute a concern for the timing of the resolution of uncertainty to agents. In the special case where $\rho=\frac{1}{\alpha}$, these preferences collapse to standard power utility preferences with CRRA coefficient α. As before, we can define *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\begin{aligned}
\hat{\pi}(s_{t+1}|s_t) &= \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}} \\
\text{and } \hat{\beta}(s_t) &= \beta \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha} \right)^{\frac{1-\rho}{1-\alpha}}
\end{aligned}$$

As before, $\hat{\beta}(s_t)$ is stochastic as long as the original Markov process is not *iid* over time. Note that the adjustment of the discount rate is affected by both ρ and α. If $\rho = \frac{1}{\alpha}$, this transformation reduces to the case we discussed in section (2).

Finally, let $\hat{V}_t(\hat{c})(s^t)$ denote the lifetime expected continuation utility in node s^t, under the new transition probabilities and discount factor, defined over consumption shares $\left\{\hat{c}_t(s^t)\right\}$:

$$\hat{V}_t(\hat{c})(s^t) = \left[(1-\beta)\hat{c}_t^{1-\rho} + \hat{\beta}(s_t)(\hat{\mathcal{R}}_t \hat{V}_{t+1}(s^{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where $\mathcal{R}$ denotes the following operator:

$$\hat{\mathcal{R}}_t V_{t+1} = \left(\hat{E}_t \hat{V}_{t+1}^{1-\alpha}\right)^{1/1-\alpha}.$$

and $\hat{E}$ denotes the expectation operator under the hatted measure $\hat{\pi}$.

Proposition A.1. *Households rank consumption share allocations in the de-trended economy in exactly the same way as they rank the corresponding consumption allocations in the original growing economy: for any s^t and any two consumption allocations c, c'*

$$V(c)(s^t) \geq V(c')(s^t) \Longleftrightarrow \hat{V}(\hat{c})(s^t) \geq \hat{V}(\hat{c}')(s^t)$$

where the transformation of consumption into consumption shares is given by (4).

Detrended Arrow Economy We proceed as before, by conjecturing that the equilibrium consumption shares only depend on y^t. Our first result states that if the consumption shares in the de-trended economy do not depend on the aggregate history z^t, then it follows that the interest rates in this economy are deterministic.

Proposition A.2. *In the de-trended Arrow economy, if there exists a competitive equilibrium with equilibrium consumption allocations $\left\{\hat{c}_t(\theta_0, y^t)\right\}$, then there is a deterministic interest rate process $\left\{\hat{R}_t^A\right\}$ and equilibrium prices $\left\{\hat{q}_t(z^t, z_{t+1})\right\}$, that satisfy:*

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A} \tag{37}$$

All the results basically go through. We can map an equilibrium of the Bewley economy into an equilibrium of the detrended Arrow economy.

Theorem A.1. *An equilibrium of the Bewley model $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and $\{\hat{R}_t, \hat{v}_t\}$ can be made into an equilibrium for the Arrow economy with growth, $\left\{a_t(\theta_0, s^t, z_{t+1})\right\}$, $\{\sigma_t(\theta_0, s^t)\}$,*

$\{c_t(\theta_0, s^t)\}$ *and* $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1}) \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\alpha}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\alpha}}
\end{aligned}$$

As a result, even for an economy with agents who have these Epstein-Zin preferences, the risk premium is not affected.

B Proofs

- Proof of Proposition 2.1:

Proof. Denote $U(c)(s^t)$ as continuation utility of an agent from consumption stream c, starting at history s^t. This continuation utility follows the simple recursion

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1}),$$

where it is understood that $(s^t, s_{t+1}) = (z^t, z_{t+1}, y^t, y_{t+1})$. Divide both sides by $e_t(s^t)^{1-\gamma}$ to obtain

$$\frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}} = u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \frac{e_{t+1}(z^{t+1})^{1-\gamma}}{e_t(z^t)^{1-\gamma}} \frac{U(c)(s^t, s_{t+1})}{e_{t+1}(z^{t+1})^{1-\gamma}}.$$

Define a new continuation utility index $\hat{U}(\cdot)$ as follows:

$$\hat{U}(\hat{c})(s^t) = \frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}}.$$

It follows that

$$\begin{aligned}
\hat{U}(\hat{c})(s^t) &= u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \lambda(z_{t+1})^{1-\gamma} \hat{U}(\hat{c})(s^t, s_{t+1}) \\
&= u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \hat{U}(\hat{c})(s^t, s_{t+1})
\end{aligned}$$

Thus it follows, for two consumption streams c and c', that

$$U(c)(s^t) \geq U(c')(s^t) \text{ if and only if } \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

i.e. the household orders original and growth-deflated consumption streams in exactly the same way. □

- Proof of Theorem 4.1:

Proof. The proof consists of two parts. In a first step, we argue that Bewley equilibrium allocations and prices can be made into an equilibrium for the de-trended Arrow model, and in a second step, we argue that by scaling the allocations and prices by the appropriate endowment (growth) factors results in an equilibrium of the stochastically growing Arrow model.

Step 1: Take allocations and prices from a Bewley equilibrium, $\left\{\hat{c}_t(y^t), \hat{a}_t(y^t), \hat{\sigma}_t(y^t)\right\}, \{\hat{R}_t, \hat{v}_t\}$ and let the associated Lagrange multipliers on the solvency constraints be given by

$$\hat{\beta}^t \varphi(y^t|y_0) u'(\hat{c}_t(y^t)) \hat{\mu}(y^t) \geq 0$$

for the constraint in (9) and

$$\hat{\beta}^t \varphi(y^t|y_0) u'(\hat{c}_t(y^t)) \hat{\kappa}_t(y^t) \geq 0$$

for the constraint in (10). The first order conditions (which are necessary and sufficient for household optimal choices together with the complementary slackness and transversality conditions) in the Bewley model, once combined to the Euler equations, read as

$$1 = \hat{R}_t \hat{\beta} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{R}_t \hat{\kappa}_t(y^t) \tag{38}$$

$$= \hat{\beta} \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}\right] \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}\right] \hat{\kappa}_t(y^t). \tag{39}$$

The corresponding Euler equations for the de-trended Arrow model, evaluated at the Bewley equilibrium allocations and Lagrange multipliers $\hat{\mu}(y^t)$ and $\hat{\kappa}_t(y^t)\hat{\phi}(z_{t+1})$, read as (see (33) and (34)):

$$\begin{aligned} 1 &= \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1}|s^t, z_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &\quad + \hat{\mu}_t(y^t) + \frac{\hat{\kappa}_t(y^t)\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \; \forall z_{t+1}. \\ 1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &\quad + \hat{\mu}_t(y^t) + \hat{\kappa}_t(y^t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right]. \end{aligned}$$

Evaluated at the conjectured prices

$$\hat{v}_t(z^t) = \hat{v}_t \tag{40}$$

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t}, \tag{41}$$

and using the independence and i.i.d. assumptions, which imply (i)

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1})$$

and (ii) $\hat{\beta}(s_t) = \hat{\beta}$ these Euler equations can be restated as follows:

$$
\begin{aligned}
1 &= \frac{\hat{\beta}\hat{R}_t}{\hat{\phi}(z_{t+1})} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{R}_t \hat{\kappa}_t(y^t) \\
1 &= \hat{\beta} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t}\right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\
&+ \hat{\mu}_t(y^t) + \hat{\kappa}_t(y^t) \left[\frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t}\right] \sum_{z_{t+1}} \hat{\phi}(z_{t+1}),
\end{aligned}
$$

which are, given that $\sum_{z_{t+1}} \hat{\phi}(z_{t+1}) = 1$, exactly the Euler conditions (38) and (39) of the Bewley model and hence satisfied by the Bewley equilibrium allocations. A similar argument applies to the complementary slackness conditions, which for the Bewley model read as

$$
\hat{\mu}_t(y^t) \left[\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t - \hat{K}_t(y^t)\right] = 0 \tag{42}
$$

$$
\hat{\kappa}_t(y^t) \left[\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) - \hat{M}_t(y^t)\right] = 0 \tag{43}
$$

and for the de-trended Arrow model, evaluated at Bewley equilibrium allocations and conjectured prices, read as

$$
\begin{aligned}
\hat{\mu}_t(y^t) \left[\frac{\hat{a}_t(y^t)}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) + \hat{\sigma}_t(y^t) - \hat{K}_t(y^t)\right] &= 0 \\
\hat{\kappa}_t(y^t) \left[\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1} + \alpha\right] - \hat{M}_t(y^t)\right] &= 0/\hat{\phi}(z_{t+1})
\end{aligned}
$$

Again, the Bewley equilibrium allocations satisfy the complementary slackness conditions in the de-trended Arrow model. The argument is exactly identical for the transversality conditions. Finally, we have to check whether the Bewley equilibrium allocation satisfies the de-trended Arrow budget constraints. Plugging in the allocations yields

$$
\hat{c}_t(s^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) + \hat{\sigma}_t(y^t)\hat{v}_t \leq \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \sigma_{t-1}(y^{t-1})\left[\hat{v}_t + \alpha\right], \tag{44}
$$

which is exactly the budget constraint in the Bewley model. Thus, given the conjectured prices Bewley equilibrium allocations are optimal household choices also in the de-trended Arrow model.

Since the market clearing conditions for assets and consumption goods coincide in the two models, Bewley allocations satisfy the market clearing conditions in the de-trended Arrow model. Thus we conclude that Bewley equilibrium allocations, together with prices (40) and (41) are an equilibrium in the de-trended Arrow model.

Step 2: Now we need to show that an equilibrium of the de-trended Arrow model is, after appropriate scaling, an equilibrium in the stochastically growing economy. But this was established in section 4.2 where we showed that by with the transformations $\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)}$, $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})}$, $\hat{\sigma}_t(s^t) = \sigma_t(s^t)$, $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1})\lambda(z_{t+1})$, $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$

household problems and market clearing conditions in the de-trended and the stochastically growing Arrow model coincide. □

- Proof of Theorem 5.1:

Proof. As in the Arrow model the crucial part of the proof is to argue that Bewley equilibrium allocations and prices can be made into an equilibrium for the de-trended *THL* model. The Euler equations of the Bewley model where given in (38) and (39).

The corresponding Euler equations for the de-trended *THL* model, evaluated at the Bewley equilibrium allocations and Lagrange multipliers $\hat{\mu}(y^t)$ and $\hat{\kappa}_t(y^t)\hat{\phi}(z_{t+1})$, read as (see (35) and (36))

$$\begin{aligned} 1 &= \hat{\beta}(s_t)\sum_{s^{t+1}|s^t}\hat{\pi}(s_{t+1}|s_t)\left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right]\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &+\hat{\mu}(y^t)+\hat{\kappa}_t(y^t)\sum_{z_{t+1}}\hat{\phi}(z_{t+1})\left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \\ 1 &= \hat{\beta}(s_t)\sum_{s^{t+1}|s^t}\hat{\pi}(s_{t+1}|s_t)\left[\frac{\hat{v}_{t+1}(z^{t+1})+\alpha}{\hat{v}_t(z^t)}\right]\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &+\hat{\mu}(y^t)+\hat{\kappa}_t(y^t)\sum_{z_{t+1}}\hat{\phi}(z_{t+1})\left[\frac{\hat{v}_{t+1}(z^{t+1})+\alpha}{\hat{v}_t(z^t)}\right]. \end{aligned}$$

Evaluated at the conjectured prices, we obtain:

$$\begin{aligned} \hat{v}_t(z^t) &= \hat{v}_t \\ R_t(z^t) &= \hat{R}_t * \frac{\sum_{z_{t+1}}\phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}}\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}} \end{aligned}$$

and using the independence and i.i.d. assumptions, we obtain (i) $\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1})$ and (ii) $\hat{\beta}(s_t) = \hat{\beta}$ and, by definition of $\hat{\phi}(z_{t+1})$,

$$R_t(z^t)\sum_{z_{t+1}}\frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} = R_t(z^t)\frac{\sum_{z_{t+1}}\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}}\phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}} = \hat{R}_t$$

these equations can be restated as:

$$\begin{aligned} 1 &= \hat{\beta}\hat{R}_t\sum_{y^{t+1}|y^t}\varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))}+\hat{\mu}(y^t)+\hat{\kappa}_t(y^t)\hat{R}_t \\ 1 &= \hat{\beta}\sum_{y^{t+1}|y^t}\varphi(y_{t+1}|y_t)\left[\frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t}\right]\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))}\sum_{z_{t+1}}\hat{\phi}(z_{t+1}) \\ &+\hat{\mu}(y^t)+\hat{\kappa}_t(y^t)\left[\frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t}\right]\sum_{z_{t+1}}\hat{\phi}(z_{t+1}), \end{aligned}$$

which again are, given that $\sum_{z_{t+1}}\hat{\phi}(z_{t+1}) = 1$, exactly the Euler conditions (38) and (39) of the Bewley model and hence satisfied by the Bewley equilibrium allocations. For the Bewley model, the complementary slackness conditions were given in (42) and (43), and for the de-trended *THL* model, evaluated at the proposed allocations in the theorem (which had bond holdings equal to zero) these equations are given by:

$$\begin{aligned}\hat{\mu}_t(y^t)\left[\hat{\sigma}_t^{THL}(y^t)\hat{v}_t - \hat{K}_t(y^t)\right] &= \hat{\mu}_t(y^t)\left[\left(\frac{\hat{a}_t(y^t)}{[\hat{v}_{t+1}+\alpha]} + \hat{\sigma}_t(y^t)\right)\hat{v}_t - \hat{K}_t(y^t)\right] \\ &= \hat{\mu}_t(y^t)\left[\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t - \hat{K}_t(y^t)\right] = 0\end{aligned}$$

and

$$\begin{aligned}\hat{\kappa}_t(y^t)\left[\hat{\sigma}_t^{THL}(y^t)\left[\hat{v}_{t+1}+\alpha\right] - \hat{M}_t(y^t)\right] &= \hat{\kappa}_t(y^t)\left[\left(\frac{\hat{a}_t(y^t)}{[\hat{v}_{t+1}+\alpha]} + \hat{\sigma}_t(y^t)\right)\left[\hat{v}_{t+1}+\alpha\right] - \hat{M}_t(y^t)\right] \\ &= \hat{\kappa}_t(y^t)\left[\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1}+\alpha\right] - \hat{M}_t(y^t)\right] = 0/\hat{\phi}(z_{t+1})\end{aligned}$$

where we use the fact that the Bewley equilibrium prices and interest rates satisfy

$$\hat{R}_t = \frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t}$$

These complementary slackness conditions are satisfied since the Bewley equilibrium allocations satisfy the complementary slackness conditions in the Bewley model. The argument is exactly identical for the transversality conditions. Finally we have to check whether the allocations proposed in the theorem satisfy the de-trended *THL* model budget constraints. Plugging into the de-trended *THL* model budget constraint yields

$$\hat{c}_t(y^t) + \frac{\hat{b}_t(y^t)}{R_t} + \hat{\sigma}_t^{THL}(y^t)\hat{v}_t \leq \eta(y_t) + \frac{\hat{b}_{t-1}(y^{t-1})}{\lambda(z_t)} + \hat{\sigma}_{t-1}^{THL}(y^{t-1})\left[\hat{v}_t+\alpha\right] \tag{45}$$

$$\hat{c}_t(y^t) + \left[\frac{\hat{a}_t(y^t)}{[\hat{v}_{t+1}+\alpha]} + \hat{\sigma}_t(y^t)\right]\hat{v}_t \leq \eta(y_t) + \left[\frac{\hat{a}_{t-1}(y^{t-1})}{[\hat{v}_t+\alpha]} + \hat{\sigma}_{t-1}(y^{t-1})\right]\left[\hat{v}_t+\alpha\right]$$

$$\hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \leq \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})\left[\hat{v}_t+\alpha\right]$$

which is exactly the budget constraint in the Bewley model. Thus, given the conjectured prices the allocations proposed in the theorem are optimal household choices in the de-trended *THL* model. Note that equation (45) shows why, in contrast to the Arrow model, in the *THL* model bond positions have to be zero. Nothing in this equation depends in the aggregate shock z_t but the term $\frac{\hat{b}_{t-1}(y^{t-1})}{\lambda(z_t)}$. Therefore the budget constraint can only be satisfied if $\hat{b}_{t-1}(y^{t-1}) = 0$.[30]

The market clearing conditions for bonds in the de-trended *THL* model is trivially satisfied because bond positions are identically equal to zero. The goods market clearing condition is identical to that of the Bewley model and thus satisfied by the Bewley equilibrium consump-

[30]Intuitively, in the model with growth households want to keep wealth at the beginning of the period proportional to the aggregate endowment in the economy. But since bond positions are chosen yesterday and thus cannot depend on the realization of the aggregate shock today bond positions have to be zero to achieve proportionally of wealth and the aggregate endowment.

tion allocations. It remains to be shown that the stock market clears. But

$$\begin{aligned}
& \int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t^{THL}(\theta_0, y^t) d\Theta_0 \\
= & \int \sum_{y^t} \varphi(y^t|y_0) \left[\frac{\hat{a}_t(\theta_0, y^t)}{[\hat{v}_{t+1}+\alpha]} + \hat{\sigma}_t(\theta_0, y^t) \right] d\Theta_0 \\
= & \frac{1}{[\hat{v}_{t+1}+\alpha]} \int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, y^t) d\Theta_0 + \int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, y^t) d\Theta_0 \\
= & \; 0 + 1
\end{aligned}$$

where the last line follows from the fact that the bond and stock market clears in the Bewley equilibrium. Thus we conclude that the allocations and prices proposed in the theorem form indeed an equilibrium in the de-trended *THL* model, and, after appropriate scaling, in the original *THL* model. □

Proof of Lemma 6.1:

Proof. First, consider a household whose borrowing constraints do not bind. The Euler equation of the Arrow/*THL* economy for the stock respectively reads as follows:

$$1 = \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}.$$

The Euler equation in the Arrow/*THL* economy for the bond reads as follows:

$$1 = \hat{\beta}\hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}$$

This implies that

$$\hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} - \hat{R}_t \right] = 0$$

which then also holds if we replace $\hat{\beta}$ by β. This immediately implies that

$$E_t[\left(R^s_{t+1} - R_t\right) \beta \left(\frac{e_{t+1}}{e_t}\right)^{-\gamma}] = 0.$$

In the Arrow economy, the Consumption-CAPM prices any excess return $R^i_{t+1} - R_t$ as long as the returns only depend on the aggregate state z_{t+1}. In the *THL* model, it only prices a claim to aggregate consumption.

□

- Proof of Proposition 6.1:

Proof. Let us define

$$1/\hat{R}^{RE}_\tau = 1/\hat{R}^{RE} = \sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1}) = \sum_{z_{t+1}} \hat{\beta}\hat{\phi}(z_{t+1}) = \hat{\beta}.$$

We know that in the Arrow economy, the equilibrium Arrow prices are given by:

$$q_t^A(z^t, z_{t+1}) = \frac{\hat{q}_t^A(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})\hat{R}_t^A(z^t)}$$

whereas in the representative agent economy Arrow prices are given as

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \hat{\beta}\frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})}$$

This implies that the stochastic discount factor in the Arrow economy equals the SDF in the representative agent economy multiplied by a non-random number κ_t :

$$m_{t+1}^A = m_{t+1}^{RE}\kappa_t$$

with $\kappa_t = \frac{\hat{R}_t^E}{\hat{R}_t^A}$ and $\left(\hat{R}^E\right)^{-1} = \hat{\beta}$. □

- Proof of Theorem 6.1:

Proof. First, note that the multiplicative risk premium on a claim to aggregate consumption can be stated as a weighted sum of risk premia on strips (Alvarez and Jermann (2001)):

$$\begin{aligned} 1 + \nu_t &= E_t m_{t+1} E_t \left(\frac{\sum_{k=1}^{\infty} E_{t+1} m_{t+1,t+k} \alpha e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k} \alpha e_{t+k}} \right) \\ &= \sum_{k=1}^{\infty} \frac{\frac{E_{t+1} m_{t,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}}}{1/E_t m_{t+1}} \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k} e_{t+k}} \\ &= \sum_{k=1}^{\infty} \omega_k \frac{E_t R_{t,1}\left[e_{t+k}\right]}{R_{t,1}\left[1\right]}, \end{aligned}$$

where the weights are

$$\omega_k = \frac{E_t m_{t,t+k} e_{t+k}(z^{t+k})}{\sum_{l=1}^{\infty} E_t m_{t,t+k} e_{t+k}(z^{t+k})} \tag{46}$$

The multiplicative risk premium on a one-period strip of aggregate consumption (a claim to the Lucas tree's dividend next period only, not the entire stream) is the same in the Arrow economy as in the representative agent economy.

First, we show that the one-period ahead conditional strip risk premia are identical:

$$\frac{E_t \frac{\alpha e_{t+1}(z^{t+1})}{E_t[m_{t+1}^A \alpha e_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m_{t+1}^A]}} = \frac{E_t \frac{\lambda_{t+1}(z^{t+1})}{E_t[m_{t+1}^A \lambda_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m_{t+1}^A]}} = \frac{E_t \frac{\lambda_{t+1}(z^{t+1})}{E_t[m_{t+1}^{RE} \lambda_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m_{t+1}^{RE}]}}$$

since $m_{t+1}^A = m_{t+1}^{RE}\kappa$. Next, we follow Alvarez and Jermann's proof strategy (in a different setting), and we show that the risk premia on k-period strips are identical. Here is the risk premium on a $k-$period strip:

$$\frac{E_t R_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R_{t,t+1}\left[1\right]} = \frac{E_t \frac{E_{t+1}[m^A_{t+1,t+k}\alpha e_{t+k}(z^{t+k})]}{E_t[m^A_{t,t+k}\alpha e_{t+k}(z^{t+k})]}}{\frac{1}{E_t[m^A_{t+1}]}}$$

Now, since the aggregate shocks are i.i.d., the term structure of risk premia on strips in the representative agent economy is flat (i.e. the risk premia does not depend on k) :

$$\frac{E_t R^{RE}_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R^{RE}_{t,t+1}\left[1\right]} = \frac{E_t \frac{[E\lambda(z)^{1-\gamma}]^{k-1}}{[E\lambda(z)^{1-\gamma}]^k}}{\frac{1}{E_t[m^{RE}_{t+1}]}} = \frac{\frac{1}{E[\lambda(z)^{1-\gamma}]}}{\frac{1}{E[\lambda(z)^{-\gamma}]}} = \frac{E[\lambda(z)^{-\gamma}]}{E[\lambda(z)^{1-\gamma}]}$$

(same proof by Alvarez and Jermann on page 37.) Now, to keep things simple, assume the $R^A_t = R^{RE}$ for all $t, and\ \kappa_t$ is constant as a result, then this implies that then the term structure of risk premia in the Arrow economy is flat as well.

$$\frac{E_t R^{A}_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R^{A}_{t,t+1}\left[1\right]} = \frac{E_t \frac{[E\lambda(z)^{1-\gamma}\kappa]^{k-1}}{[E\lambda(z)^{1-\gamma}\kappa]^k}}{\frac{1}{E_t[m^{RE}_{t+1}]}} = \frac{\frac{1}{E[\lambda(z)^{1-\gamma}\kappa]}}{\frac{1}{E[\lambda(z)^{-\gamma}\kappa]}} = \frac{E[\lambda(z)^{-\gamma}]}{E[\lambda(z)^{1-\gamma}]}$$

But this implies that the multiplicative risk premia is unchanged since the risk premia on the consumption strips are invariant in k. This means the multiplicative risk premium on a claim to aggregate consumption is the same in the Arrow economy as in the RE economy. The proof goes through for time-varying κ_t, but the algebra is a little messier. □

- Proof of Lemma 7.1:

Proof. By construction, the Euler equations of the Arrow model with these prices and the consumption allocations from the Bewley model are satisfied. We start by assuming the borrowing constraint does not bind. First, note that the Euler equation in the Bewley economy is given by:

$$1 = \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t}\hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}.$$

The Euler equation in the detrended Arrow economy implies that:

$$\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1}) = \hat{\beta}(z_t)\sum_{y_{t+1}} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}\forall z_{t+1},$$

which in turn implies that:

$$\frac{1}{\hat{R}_t}\frac{\widetilde{\beta}_t}{\widetilde{\beta}_{t+1}} = \sum_{y_{t+1}} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}\forall z_{t+1}$$

This is exactly the same Euler equation as in the Bewley economy. □

The state-contingent interest rate in this economy is given by:

$$\frac{1}{\hat{R}_t^A(z_t)} = \hat{\beta}(z_t)\frac{\widetilde{\beta}_t}{\hat{R}_t\widetilde{\beta}_{t+1}}$$

which can easily be verified from equation (31).

- Proof of Proposition 7.1:

Proof. We need to check that these allocations satisfy the budget constraint in the economy with time-varying discount factors. Let us take $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t) = 0, \hat{\sigma}_t(\theta_0, y^t)\}$ from the Bewley model and let us back out the implied state-contingent Arrow securities positions recursively, for each y^t, z_t, from the budget constraint of the Arrow model

$$\begin{aligned} \hat{a}_{t-1}(y^{t-1}, z^t, \theta_0) \quad &= \quad \hat{c}_t(y^t, \theta_0) - \eta(y_t) + \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, y^\tau} \hat{Q}_t(z^\tau|z_t)\left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \quad (47) \\ &\quad -\sigma_{t-1}(y^{t-1})\left[\hat{v}_t(z_t) + \alpha\right] = \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) \end{aligned}$$

This ensures that $\left\{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0), \sigma_{t-1}(y^{t-1}, \theta_0)\right\}$ finance the consumption allocations $\{\hat{c}_t(\theta_0, y^t)\}$. Next, we need to check that the market for each Arrow security clears. That is, we have to check that

$$\begin{aligned} \int \sum_{y^t-1} \pi(y^{t-1}|y_0)\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)d\Theta_0 \quad &= \\ \sum_{y^t|y^{t-1}} \pi(y^t|y^{t-1}) \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)d\Theta_0 \quad &= \\ \int \sum_{y^t} \pi(y^t|y_0)\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)d\Theta_0 \quad &= \quad 0 \end{aligned}$$

for each z_t. We now multiply equation (47) by $\pi(y^t|y_0)$ and then sum over all y^t. For the first term we obtain, using the aggregate resource constraint,

$$\begin{aligned} &\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \sum_{y^t} \pi(y^t|y_{t-1})\left(\hat{c}_t(y^t, \theta_0) - \eta(y_t)\right) d\Theta_0 \\ = \quad &\int \sum_{y^t} \pi(y^t|y_0)\left(\hat{c}_t(y^t, \theta_0) - \eta(y_t)\right) d\Theta_0 = \alpha. \end{aligned} \quad (48)$$

By the same token we obtain

$$\begin{aligned} &\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t)\left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) d\Theta_0 \quad (49) \\ = \quad &\sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t)\alpha. \end{aligned}$$

Furthermore we know from the market clearing in the market for shares that

$$\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\sigma_{t-1}(y^{t-1},\theta_0)\left[\hat{v}_t(z_t)+\alpha\right]d\Theta_0 = \left[\hat{v}_t(z_t)+\alpha\right], \tag{50}$$

and finally we know that the share price can be written as follows:

$$\left[\hat{v}_t(z_t)+\alpha\right] = \sum_{\tau=t+1}^{\infty}\sum_{z^\tau}\hat{Q}_t(z^\tau|z_t)\alpha+\alpha, \tag{51}$$

which in turn implies, substituting (48), (49) and (50) into (47) that

$$\frac{1}{\lambda(z_t)}\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\hat{a}_{t-1}(y^{t-1},z_t)d\Theta_0 = 0$$

Thus, each of the Arrow securities markets clears for the new trading strategies. □

- Proof of Lemma 7.2:

Proof. From the definition of $\left\{\widetilde{\beta}_t\right\}$, it follows that the Arrow-Debreu prices at time 0 before the realization of z_0:

$$\begin{aligned}
\hat{Q}_{0,\tau} &= \sum_{z_0}\hat{\Pi}(z_0)\sum_{z^\tau}\hat{Q}_\tau(z^\tau|z_0) = \sum_{z_0}\hat{\Pi}(z_0)\sum_{z^{\tau-1}}\frac{\widehat{\phi}(z^\tau|z_0)\hat{\beta}_{0,\tau-1}(z^{\tau-1}|z_0)}{\widetilde{\beta}_\tau\widehat{R}_{0,\tau}} \\
&= \frac{1}{\widetilde{\beta}_\tau^\tau\widehat{R}_{0,\tau}}\sum_{z_0}\hat{\Pi}(z_0)\sum_{z^\tau}\widehat{\phi}(z^\tau|z_0)\hat{\beta}_{0,\tau-1}(z^{\tau-1}|z_0) \\
&= \frac{\widetilde{\beta}_\tau^\tau}{\widetilde{\beta}_\tau^\tau\widehat{R}_{0,\tau}} = \tilde{Q}_{0,\tau}
\end{aligned}$$

where we have used the result in equation (30). □

Proof of Corollary 7.3:

Proof.

$$\begin{aligned}
\sum_{z_0}\hat{\Pi}(z_0)\hat{a}_{-1}(z_0,\theta_0) &= \sum_{z_0}\hat{\Pi}(z_0)\sum_{\tau=1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_0)-\widetilde{Q}_\tau\right)\sum_{y^\tau}\left(\hat{c}_\tau(y^\tau,\theta_0)-\eta(y_\tau)\right) \\
&\quad -\sigma_{-1}\sum_{z_0}\hat{\Pi}(z_0)\left[\hat{v}_t(z_t)-\widetilde{\nu}_t(z^t)\right]. \\
&= \sum_{z_0}\hat{\Pi}(z_0)\sum_{\tau=1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_0)-\widetilde{Q}_\tau\right)\sum_{y^\tau}\left(\hat{c}_\tau(y^\tau,\theta_0)-\eta(y_\tau)\right) \\
&\quad -\sigma_{-1}\alpha\sum_{z_0}\hat{\Pi}(z_0)\sum_{\tau=1}^{\infty}\sum_{z^\tau}\left(\hat{Q}_\tau(z^\tau|z_0)-\widetilde{Q}_\tau\right) = 0,
\end{aligned}$$

because we know that:

$$\sum_{z_0} \hat{\Pi}(z_0) \sum_{z^t} \hat{Q}_t(z^t|z_0) \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau \right) = 0$$

□

- Proof of of Lemma 7.4:

Proof. Suppose the stationary Bewley allocation $\{\widetilde{\hat{a}}_t(y^t), \hat{\sigma}_t(y^t)\}$ satisfies the constraint

$$\frac{\widetilde{\hat{a}}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t)$$

which seems the natural constraint to impose on the stationary Bewley economy. We want to show that the allocation for the stochastic Arrow economy satisfies the borrowing constraint if the allocation for the stationary Bewley economy does. Multiply both sides by $e_t(z^t)\hat{\phi}(z_{t+1})$ to obtain

$$\frac{\widetilde{a}_t(s^t, z_{t+1})\hat{\phi}(z_{t+1})}{\hat{R}_t \lambda(z_{t+1})} + \sigma_t(s^t)v_t(z^t)\hat{\phi}(z_{t+1}) \geq K_t(s^t)\hat{\phi}(z_{t+1})$$

Using the fact that

$$q_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t \lambda(z_{t+1})}$$

and summing over all z_{t+1} yields

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1})\widetilde{a}_t(s^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t),$$

exactly the constraint of the stochastic Arrow economy, but the actual bond position differs from $\widetilde{a}_t(s^t, z_{t+1})$, because

$$\begin{aligned} \frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} - \frac{\widetilde{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} &= \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau \right) \sum_{\eta^\tau} (\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)) \\ &\quad - \sigma_{t-1}(y^{t-1})\alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau \right) \end{aligned}$$

So, we know that

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1})a_t(s^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) \begin{pmatrix} \widetilde{a}_t(s^t, z_{t+1}) \\ -a_t(s^t, z_{t+1}) \end{pmatrix},$$

So, we can simply re-define the solvency constraints as :

$$K_t^*(s^t) = K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) \begin{pmatrix} \widetilde{a}_t(s^t, z_{t+1}) \\ -a_t(s^t, z_{t+1}) \end{pmatrix}$$

Next consider the state-by state wealth constraint. In the stationary Bewley economy we may impose

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1} + \alpha\right] \geq \hat{M}_t(y^t)$$

Multiplying by $e_{t+1}(z^{t+1})$ yields

$$\widetilde{a}_t(s^t, z_{t+1}) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}$$

By the same token, we can redefine:

$$M^*_{t+1}(s^{t+1}) = M_{t+1}(s^{t+1}) + \widetilde{a}_t(s^t, z_{t+1}) - a_t(s^t, z_{t+1})$$

□

- Proof of Lemma A.1

Proof. The first order conditions for the representative agent are given by:

$$\begin{aligned} 1 &= \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}\frac{u'(\hat{c}_{t+1}(z^t, z_{t+1}))}{u'(\hat{c}_t(z^t))}\ \forall z_{t+1} \\ 1 &= \hat{\beta}\sum_{z_{t+1}}\hat{\phi}(z_{t+1})\left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right]\frac{u'(\hat{c}_{t+1}(z^t, z_{t+1}))}{u'(\hat{c}_t(z^t))} \end{aligned} \quad (52)$$

□

Proof of Proposition A.1:

Proof. First, we divided through by $e_t(z^t)$ on both sides in equation (A.4):

$$\begin{aligned} \frac{V_t(s^t)}{e_t(z^t)} &= \left[(1-\beta)\frac{c_t^{1-\rho}}{e_t^{1-\rho}} + \beta\frac{(\mathcal{R}_t V_{t+1})^{1-\rho}}{e_t^{1-\rho}}\right]^{\frac{1}{1-\rho}} \\ \hat{V}_t(s^t) &= \left[(1-\beta)\hat{c}_t^{1-\rho} + \beta(\frac{\mathcal{R}_t V_{t+1}}{e_t})^{1-\rho}\right]^{\frac{1}{1-\rho}}. \end{aligned} \quad (53)$$

Note that the risk-adjusted continuation utility can be stated as:

$$\begin{aligned} \frac{\mathcal{R}_t V_{t+1}}{e_t(z^t)} &= \left(E_t\left(\frac{e_{t+1}}{e_t}\right)^{1-\alpha}\frac{V_{t+1}^{1-\alpha}}{e_{t+1}^{1-\alpha}}\right)^{1/1-\alpha} \\ &= \left(\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\hat{V}_{t+1}^{1-\alpha}(s_{t+1})\right)^{1/1-\alpha} \end{aligned}$$

Next, we define growth-adjusted probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}{\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}} \text{ and } \hat{\beta}(s_t) = \beta\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}.$$

and note that:

$$\frac{\mathcal{R}_t V_{t+1}}{e_t(z^t)} = \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\hat{V}_{t+1}^{1-\alpha}(s_{t+1})\right)^{1/1-\alpha}$$
$$= \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\right)^{1/1-\alpha} \hat{\mathcal{R}}_t \hat{V}_{t+1}(s_{t+1})$$

Using the definition of $\hat{\beta}(s_t)$:

$$\hat{\beta}(s_t) = \beta \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}},$$

we finally obtain the desired result:

$$\hat{V}_t(s^t) = \left[(1-\beta)\hat{c}_t^{1-\rho} + \hat{\beta}(s_t)(\hat{\mathcal{R}}_t \hat{V}_{t+1}(s^{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

As before, if the z shocks are ii.d, then $\hat{\beta}$ is constant. □

Proof of Proposition A.2:

Proof. First, we suppose the borrowing constraints are not binding, which is the easiest case. Assume the equilibrium allocations only depend on y^t, not on z^t. Then conditions 2.2 and 2.3 imply that the Euler equations of the Arrow economy, for the contingent claim and the stock respectively, read as follows:

$$1 = \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \quad \forall z_{t+1}$$

$$1 = \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \tag{54}$$

$$* \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \textit{for all } z_{t+1}. \tag{55}$$

In the first Euler equation, the only part that depends on z_{t+1} is $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}$ which therefore implies that $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}$ cannot depend on z_{t+1} : $\hat{q}_t(z^t, z_{t+1})$ is proportional to $\hat{\phi}(z_{t+1})$. Thus define $\hat{R}_t^A(z^t)$ by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A(z^t)} \tag{56}$$

as the risk-free interest rate in the stationary Arrow economy. Using this condition, the Euler equation in (**??**) simplifies to the following expression:

$$1 = \hat{\beta}\hat{R}^A_t(z^t)\sum_{y_{t+1}}\varphi(y_{t+1}|y_t)\left(\frac{\hat{c}_{t+1}(y^t,y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \tag{57}$$

$$\left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \tag{58}$$

First, notice that apart from $\hat{R}^A_t(z^t)$ noting in this condition depends on z^t, so we can choose $\hat{R}^A_t(z^t) = \hat{R}^A_t$. □