Solid Coalitional Games

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Abstract

This paper introduces a game theoretic framework for analyzing the way in which groups interact with other groups called solid coalitional games. It is partly based on the literature on coalitional structures pioneered by Aumann and Dreze (1974). It is a dynamic analysis that focuses on whether or not the grand coalition can form and whether or not the grand coalition is stable. I refer to these concepts as formability and the core of cores[1] respectively. The paper presents general results on solid coalitional games and analyzes two examples, multiple markets for an indivisible good and infrastructure sharing.

† I would like to thank Professor. Donald G. Saari for his valuable advice.

[1] The term "core of cores" comes from Professor Saari.

1 Introduction

Traditional game theory models analyze strategic behavior in individual interaction. Coalitional models analyze the strategic behavior of individuals in forming groups. Neither of these frameworks are sufficient for studying the way in which groups interact with other groups. This study invents such a framework and calls it a solid coalitional game (or SCG).

Today's world is filled with examples of large coalitions (i.e. NATO) that are comprised of smaller coalitions (i.e. the United States, the United Kingdom, Canada, France, Germany, etc.). The small coalitions join together in order to achieve something they cannot achieve on their own. For example, the European economies have joined to form the European Union with a strong, stable currency and more mobile factors of production. Corporations merge to consolidate market power and cut costs. Municipalities join together to share the burden of public projects. In Iraq, the Sunnis, Shiites and Kurds are merging to form a coalitional government and create a balance of power and peace. In each of these examples, the individual coalitions are well defined by ethnic, legal, ideological or other boundaries. Those boundaries define the coalitions, so that when the coalitions interact, they do so as coalitions rather than individuals.

When looking at the above situations, several questions are pertinent. First, what are the terms? That is, we should know how the gains from joining a coalition of coalitions will be distributed among each of the coalitions and between the members of each of the coalitions. For example, when the United States negotiated the North American Free Trade Agreement with Mexico and Canada it not only considered the aggregate gains from trade, but it also decided on the distribution of those gains. Second, can the coalition of coalitions form? More specifically, we should know what political process or decision rule will allow a coalition of coalitions to form. For example, the terms on which two companies agree to merge will differ if the decisions are made by an individual, a board of directors, or an open vote among all shareholders. Third, is the coalition of coalitions stable? Just like simple coalitional games, it is important to know whether this group will dissolve into smaller groups or remain unified. Given the terms under which the Iraqi coalitional government forms, we must be concerned with whether those terms also make the government stable. Can the Sunnis and Shiites decide to exclude the Kurds? Answering these questions yields new insight into institutional design and dynamics.

This paper will do three things. First, it will introduce the framework of SCG's, including related terms and concepts. As with any new concept, it will be accompanied by an appropriate vocabulary and details of relevant properties. Second, it will develop the theoretical foundation of SCG's, including existence theorems related to the properties and definitions given. Third, this paper will present two concrete examples to illuminate the concepts of stability and formability.

2 Background and Literature

In coalitional game theory the core solution concept has received special attention. This is in part because core allocations can be viewed as fixed points in a bargaining dynamic wherein proposed members of the coalition threaten to break off into subcoalitions. Under a core allocation, these threats are empty and the bargaining process is stable. Much like a Nash equilibrium, there is no incentive to deviate. Formally,

Definition 1 *The payoff vector* $x \in V(N)$ *is in the* ***core*** *of the coalitional game* $\langle N, X, V, \succeq_i \rangle$ *if there is no* $S \subset N$ *and payoff vector* $y \in V(S)$ *such that* $y_i \succ_i x_i \; \forall \; i \in S$.

Here N is the set of players, X is the set of consequences and V assigns to every coalition S a set $V(S) \subseteq X$. Also, $\succeq_i$ is the players' preference relations over X. Therefore, the core is the set of feasible allocations such that no subset of players can do better by forming a subcoalition.

In the case of a **transferable payoff** game $\langle N, v \rangle$, v is the **value function** which assigns to each $S \subseteq N$ a non-negative real number which is the worth of S. In a transferable payoff game, $v(S)$ might be the amount of money that S makes. A game is **balanced** if no feasible allocation of time spent in each subcoalition can yield an aggregate payoff more than $v(N)$, the value of the coalition of all N players. The Bondareva-Shapley theorem says that in a transferable payoff game the core exists if and only if the game is balanced. (Similar conditions apply to the general game without transferable payoff $\langle N, X, V, \succeq_i \rangle$).

Aumann and Dreze (1974) extend many cooperative game theory solution concepts to games with coalitional structure. While they derive important generalizations of the core, the nucleolus, the kernel, the value (i.e. Shapley Value), and the Von-Neumann Morgenstern solution, their formulation differs from the framework discussed here in two major ways. First, their coalitional structure $\mathcal{B}$ is organic in the sense that it could arise spontaneously from a game with N players. Therefore, Aumann and Dreze's definition of the core of a game with partition $\mathcal{B}$ considers the outside opportunities of any player within a particular coalition $\mathbf{B_k} \in \mathcal{B}$. In other words, if players $i \in \mathbf{B_k}$ and $j \in \mathbf{B_l}$ could form a coalition $S' = \{i, j\}$ and do strictly better than before, they are allowed. In my framework, there are ethnic, legal, national or other boundaries that the players cannot violate. The second way in which Aumann-Dreze's paper differs from the analysis here, is that they do not consider the way in which individual coalitions join a larger coalition. This is a key point because as the decision rule for each of the individual coalitions changes, so will the terms under which the coalition of coalitions is stable and/or formable.

Aumann and Dreze's research has given rise to a string of theoretical and applied models. Banerjee *et al.* (2001) extend Aumann and Dreze's work by showing that neither anonimity nor additive separability guarantee the existence of the core in games with partitional structure. They continue to propose two properties that do guarantee the core's existence. There are also applications of Aumann and Dreze's work to political economy problems, such as Greenberg and Weber (1993) and Haimonko, Le Breton, and Weber (2003). They look at the stability of nations with respect to the cost of government and the preferences of the people. Finally, the coalitional structures framework has been used to analyze public goods problems. Haeringer (2000), Demange (1994), Greenberg and Weber (1986) and Guesnerie and Oddou (1981) have all looked for stability in the allocation of public goods using this framework. Another interesting paper in coalitional structures is Derks and Gilles (1995). This paper examines the case in which some players are superiors and others are subordinates. They show that there is a limited collection of formable coalitions and that this collection is a lattice, which allows for infinite exploitation of subordinates.

3 Definitions and Framework

This section will lay out some important definitions and concepts related to SCG's.

First, let $\mathcal{S} = \{S_1, S_2, ..., S_I\}$ stand for the set of coalitions. $\mathcal{S}$ can also be referred to as the **grand coalition**. Any subset T of $\mathcal{S}$ is a **sub-grand coalition**. Also, let $N = \sum_{i=1}^{I} |S_i|$ be the total number of players in $\mathcal{S}$ and $N_i = |S_i|$ stand for the number of players in coalition i.

Definition 2 *A* ***solid coalitional structure*** *is a set of coalitions* $\mathcal{S} =$

$\{S_1, S_2, ..., S_I\}$ whereby no $R \subset S_i$ can form a coalition with any subset of coalitions, $T \subseteq \mathcal{S} \setminus S_i$, or any subset of players from other coalitions.

Therefore a solid coalitional structure is one with boundaries between the individual coalitions. Any coalition an agent joins he joins together with his own coalition or not at all.

Let X be a set of consequences and $V(S) \subseteq X$ is the set of consequences available to coalition S. The vector $x \in V(S)$ assigns to each member in S an outcome. The agents in S have preferences over these outcomes. For a coalition S_i the set of preferences are denoted $\succeq_i$. A **solid coalitional game** (SCG) is given by a five-tuple $\langle N, V, X, \mathcal{S}, \succeq_i \rangle$. In addition, each S_i will have a decision rule, $\mathcal{R}_i$. The collection of decision rules is $\mathcal{R} = \{\mathcal{R}_1, ..., \mathcal{R}_I\}$. The decision rules could be unanimous rule, majority rule, or even designate a dictator. While much of the analysis will treat the decision rules as given, they should ultimately be endogenous to the formation of the coalitions S_i. That is, when S_i forms, it designates an allocation (the core) and a decision rule. In addition, R_i determines the way S_i interacts with the other coalitions by defining a winning coalition.

Definition 3 *A **winning coalition** $w(S_i)$ for T under the decision rule $\mathcal{R}_i$ is a set of players in S_i that can force S_i to join a sub-grand coalition T (provided that the other coalitions in T also have a winning coalition for T).*

A winning coalition is then a group of players in S_i that can dictate which sub-grand coalition S_i is willing to join. Next, I will give a definition of the core of cores solution concept.

Definition 4 *A vector of outcomes $x \in V(\mathcal{S})$ is in the* ***core of cores*** *of $\langle N, V, X, \mathcal{S}, \succeq_i \rangle$ if there is no $y \in V(T)$ for any sub-grand coalition $T \subset \mathcal{S}$ and winning coalitions $w(S_i)$ such that $y \succ_j x$ for all $j \in w(S_i)$ and all $i \in T$. Further, $w(S_i)$ must be stable under $\mathcal{R}_i$.*

So, the core of cores has a similar interpretation as the core itself. It is a payoff vector such that the grand coalition does not dissolve. The main difference is that the core of cores is imposed on the solid coalitional structure $\mathcal{S}$ and therefore implies a decision rule $\mathcal{R}$. In many games we are not only concerned with the stability of the grand coalition but also whether or not it can form from the solid coalitional structure $\mathcal{S}$.

Definition 5 *The grand coalition is* ***formable*** *from a partition of $\mathcal{S}$, denoted $\tilde{\mathcal{S}}$, if for each $y \in V(\tilde{\mathcal{S}})$ there exists an $x \in V(\mathcal{S})$ and winning coalitions $w(S_i)$ such that $x \succ_j y$ for all $j \in w(S_i)$ and $i \in \mathcal{S}$. Further, $w(S_i)$ must be stable under $\mathcal{R}_i$.*

While the core of cores prevents the grand coalition from dissolving once it has formed, formability allows the grand coalition to form in the first place. In the language of dynamical systems, the core of cores is a fixed point of the solid coalitional game and formability says whether or not that point is an attractor.

Definition 6 *Let $\mathcal{Z} = \{\tilde{\mathcal{S}}_1, \tilde{\mathcal{S}}_2, ..., \tilde{\mathcal{S}}_n\}$ be a sequence of partitions of $\mathcal{S}$. Then, $\tilde{\mathcal{S}}_n$ is **formable by $\mathcal{Z}$ from** $\tilde{\mathcal{S}}_1$ if $\tilde{\mathcal{S}}_{l+1}$ is formable from $\tilde{\mathcal{S}}_l$ for all l in the sequence.*

The above definition of an SCG assumes a general payoff structure. In a more specific case where payoffs are transferrable, the following classifications apply.

Definition 7 *An SCG is (a) **upper transferrable** if the value of the coalition $T \subseteq \mathcal{S}$ is transferrable between the coalitions in T, (b) **lower transferrable** if the allocation to S_i is transferrable between the members of S_i, and (c) **fully transferrable** if it is both upper and lower transferrable. It is **non-transferrable** if it is neither upper nor lower transferrable.*

As mentioned in the introduction, v is the **value function**. Here, $v(T)$ gives the value of a sub-grand coalition T. Also x is a **T-feasible payoff vector** if $x(T) = v(T)$ where $x(T) = \sum_i \sum_j x_{ij}$. The classifications of transferrable games are also the basis of several general results. The next set of definitions extend the balanced property of Bondareva and Shapley to fully transferrable SCG's.

Definition 8 *An upper transferrable SCG is **superadditive** if $v(T \cup H) \geq v(T) + v(H)$ for all sub-grand coalitions T and H (**strictly superadditive** for strict inequality).*

This property assures that the value of a sub-grand coalition does not decrease as the number of member coalitions increases. Strict superadditivity will be sufficient for the formation and stability of the grand coalition in some cases.

As per Osbourne and Rubinstein (1994), denote by $\mathcal{C}$ the collection of sub-grand coalitions, and by $1_T \in \mathbb{R}^I$ the characteristic vector of T given by

$$(1_T)_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Therefore, when S_i is in T the ith entry of the characteristic vector is one. We say that (λ_T) is a **balanced collection of weights** if for every coalition i the sum of λ_T over all the sub-grand coalitions that contain i is 1: $\sum_{T \in \mathcal{C}} \lambda_T 1_T = 1_I$. An SCG is **upper balanced** if for every balanced collection of weights $\sum_{T \in \mathcal{C}} \lambda_T v(T) \leq v(\mathcal{S})$. Note that this is a straightforward adaptation of the balanced property to the hierarchical coalitional structure. As the balanced property can be interpreted as a restriction on the individual's time allocation among coalitions, the upper balanced property can also be interpreted as a restriction on a coalition's time allocation among sub-grand coalitions. The next section synthesizes the above framework into several general results.

4 Results

This section will present some general results on SCG's, relating various decision rules and payoff structures to the core of cores and formability.

Proposition 1 *In a fully transferrable SCG where each coalition S_i employs unanimity as its decision rule, the grand coalition is formable from any partition of the solid coalitional structure if the SCG is strictly superadditive.*

Proof: The first requirement of formability is that for every feasible payoff vector y in the partition $\tilde{\mathcal{S}}$, there is some feasible payoff vector x of the grand coalition such that $x_j > y_j$ for all $j \in w(S_i)$ and $i \in \mathcal{S}$. With strict superadditivity, there is some excess, ϵ, generated by the union of any two sub-grand coalitions in the partition, i.e. $v(T \cup H) - [v(T) + v(H)] = \epsilon$. Therefore, we can let $x_{ij} = y_{ij} + \epsilon_{ij}$ where $\epsilon_i j > 0$ and $\sum_i \sum_j \epsilon_{ij} = \epsilon$. This satisfies the first requirement.

The second requirement of formability is that the winning coalitions $w(S_i)$ be stable. Because the decision rule is unanimity, the set of winning coalitions in S_i is a singleton. Therefore, any subset of $w(S_i)$ cannot join another winning coalition for $\mathcal{S}$. By the above analysis, no subset of $w(S_i)$ can make a profitable deviation. Therefore, $w(S_i)$ is stable. $\square$

This result should be straightforward. It says that if there are gains to combining coalitions, then unanimity is a political process that will allow us to do so. The next result relates the upper balanced condition to the core of cores.

Proposition 2 *In a fully transferrable SCG where each coalition S_i employs unanimity as its decision rule, the cores of cores exists if the game is upper balanced.*

Proof: First, under unanimity we only need to have one member of each coalition prefer the grand coalition over a deviation. With upper transferrable utility, we can transform the solid coalitional game into a basic game with I representative players, one from each coalition S_i. Then by the Bondareva-Shapley Theorem no subset of those I players can increase their payoffs by deviating from the grand coalition to join a sub-grand coalition T. Since each coalition needs unanimity for a winning coalition, the grand coalition is stable. □

Because upper balanced is a weaker requirement than strict superadditivity, this result taken together with the last suggests that there are situations in which the grand coalition is stable but not formable. The next result relates majority voting to transferrable games.

Proposition 3 *In a lower transferrable SCG where each coalition employs majority voting as its decision rule, no partition is formable.*

Proof: The second requirement for formability is violated. That is, the winning coalition is not stable. To see this, note that with lower transferrable payoffs, we can normalize the formation of a winning coalition to the simple majority game. First, let x_i be the allocation to S_i when it joins the sub-grand coalition T. We define the majority voting game that arises as,

$$v(w(S_i)) = \begin{cases} x_i & \text{if } |w(S_i)| > \frac{N_i}{2} \\ 0 & \text{otherwise} \end{cases}$$

Osbourne and Rubinstein (1994) show that the core of this game is empty. Suppose $|w(S_i)| = N_i - 1$. Therefore, $v(w(S_i)) = x_i$ and $\sum_{j \in w(S_i)} x_{ij} \geq x_i$. There are N_i winning coalitions of size $N_i - 1$. So, summing over all coalitions of size $N_i - 1$ we should get $\sum_{\{w(S_i):|w(S_i)|=N_i-1\}} \sum_{j \in w(S_i)} x_{ij} \geq N_i x_i$. However,

$$\begin{aligned} \sum_{\{w(S_i):|w(S_i)|=N_i-1\}} \sum_{j \in w(S_i)} x_{ij} &= \\ \sum_{j \in S_i} \sum_{\{w(S_i):|w(S_i)|=N_i-1, j \ni w(S_i)\}} x_{ij} &= \\ \sum_{j \in S_i} (N_i - 1) x_{ij} &= \\ x_i(N_i - 1). \end{aligned}$$

This is a contradiction. □

This result is worrisome at first glance, but it need not be. It says that the grand coalition (or any other partition for that matter) cannot form when decisions are made by majority rule. However, this result relies on lower transferrable payoffs, which are a theoretical idealization and are rarely attainable in practical problems. In the real world, we observe many cases where a grand coalition has formed from smaller coalitions by majority vote. Therefore, proposition 3 suggests that the set of available consequences is somehow restricted.

Trade agreements are a good example of where this issue might arise. Well known results by Samuelson and Dixit confirm that there are gains from trade. Economists commonly suggest that political entities can use lump-sum transfers to divide the gains from trade among a winning political coalition (i.e. majority) in order to pass trade agreements into law. However, proposition 3 says that lump-sum transfers are not politically feasible. It suggests that the set of allocations of the gains from trade must be further restricted for political equilibrium. The next result is a corollary that relates the above result to the core of cores.

Corollary 1 *In a lower transferrable SCG where each coalition employs majority voting as its decision rule, every allocation that is feasible for the partition* $\tilde{\mathcal{S}}$ *is in the core of cores.*

The proof is obvious from the definition of the core of cores and proposition 3. That is, winning coalitions are unstable in lower transferrable SCG's. Taking the above two results together suggests that a partition of $\mathcal{S}$ is stable if and only if there is no partition formable from that partition.

5 Examples

Here I will present a simple example of a game with hierarchical coalition structures. Various forms of this game will shed light on the various aspects of the core of cores concept.

5.1 Market for an Indivisible Good

A set of buyers, B, and a set of sellers L meet in a market for an indivisible good. Each seller has one unit of the indivisible good and reservation price 0. The buyers all have a reservation price of 1. It is clear from the difference in reservation prices that this is a viable market which should support transactions. To model this as a coalitional game with transferable payoff, let $v(S) = min\{|S \cap L|, |S \cap B|\}$ for any coalition S. Therefore, with our normalized reservation prices, the value to any coalition S is simply the minimum number of transactions available.

Result 1 *(a) The core of this game when $|B| < |L|$ is a price of 0. More specifically, each seller gets 1 and each buyer gets 0. (b) The core when $|L| < |B|$ is a price of 1.*[2]

Proof: Let b be the buyer whose payoff is minimal among the buyers and l be the seller whose payoff is minimal among the sellers. Any core allocation must satisfy $x_b + x_l \geq v(b, l) = 1$. Since $|L| = v(N) = x(N) \geq |B|x_b + |L|x_l \geq |B|x_b + |L|(1 - x_b) = (|B| - |L|)x_b + |L|$ $x_b = 0$ and $x_l \geq 1$. Using $v(N) = |L|$ and the fact that l is the seller with lowest payoff, all sellers get 1. □

Result 2 *When $|B| = |L|$ the core is defined by all x such that $x_i = \alpha \geq 0$ for all $i \in L$ and $x_j = \beta \geq 0$ for all $j \in B$ and $\alpha + \beta = 1$.*

[2]Result 1 and its proof come directly from Obsborne and Rubinstein (1994).

Proof: Suppose x is in the core. Let $\bar{x}_b$ be the highest payoff received by a buyer and $\underline{x}_b$ be the lowest payoff received by a buyer. Similarly define $\bar{x}_l$ and $\underline{x}_l$. Since $\bar{x}_b+\underline{x}_b+\bar{x}_l+\underline{x}_l \geq v(\bar{b},\underline{b},\bar{l},\underline{l}) = 2$, therefore $\bar{x}_b+\bar{x}_l \geq 1$ which implies $\underline{x}_b + \underline{x}_l \leq 1$. However, $\underline{x}_b + \underline{x}_l \geq v(\underline{b},\underline{l}) = 1$. Hence, $\underline{x}_b + \underline{x}_l = 1 = \bar{x}_b + \bar{x}_l$. This implies that $x_i + x_j = 1$ for any $i \in L$ and $j \in B$. Equivalently, $x_i = \alpha$ for all $i \in L$ and $x_j = \beta$ for all $j \in B$. □

This result is similar to the common exchange economy result that says all agents of the same type receive the same allocation in the core.

5.2 Multiple Markets

To link the market for an indivisible good to the solid coalitional framework, imagine there are I markets S_i, each with a set of buyers B_i and sellers L_i. It is important that buyers and sellers do not move freely among markets. This might be due to national boundaries, as would be the case in a closed economy. Presumably, the leaders of each market or a third party recognize that the value of these markets united is greater than the sum of the values of the markets divided. In other words, if a trade agreement can be reached, at least as many transactions will take place, possibly more. The problem is, can we set a market price that will simultaneously satisfy each market's decision rule to join the other markets? Notice that we are specifying a market price rather than freely transferring payoffs. The market price will in turn define the payoffs under any sub-grand coalition. This restriction on the set of allocations will help avoid the negative result of proposition 3.

Take for instance markets 1 and 2. Let $|B_1| = P$ and $|L_1| = P - 1$ for any integer $P > 1$. Also assume $|B_2| = Z - 1$ and $|L_2| = Z$ for any integer $Z > 1$. Our transferrable payoff value function $v(S)$ gives $v(S_1) = N - 1$ and $v(S_2) = Z - 1$. Therefore, the core allocation in market 1, x_1, has $x_{b,1} = 0$ and $x_{l,1} = 1$. The core is the opposite for market 2. Now let I be any number. Define $\mathcal{S} = \cup_{i=1}^{I} S_i$ as the grand coalition (the coalition of all markets). A sub-grand coalition is a coalition among $T \subset \mathcal{S}$ of the markets. Whenever two or more coalitions unite, they must do so under a price agreement p^*. We want to see what types of decision rules support the formation of the grand coalition and given a decision rule $\mathcal{R}$ what must be the market price p^*. Let's first look at unanimity as a decision rule.

Result 3 *With I markets, $p^* \in [0, 1]$ defines the core of cores, x, under unanimity.*

The proof of this result is straightforward. It lies in the fact that any price is pareto efficient. Specifically, increasing the price hurts buyers and decreasing the price hurts sellers. When the price changes someone will oppose it. Therefore, there can be no strict preference for a sub-grand coalition by all types.

Result 4 *With I markets, there is no p^* such that the grand coalition forms from the individual coalitions under unanimity.*

Given that the status quo in each coalition is the core, which we know to be pareto efficient, no coalition strictly prefers the grand coalition under unanimity. Taking the previous two results together, the core of cores cannot be beaten, but it also cannot form from the status quo. What about a majority rule?

Result 5 *With I markets, $p^* \in (0, 1)$ defines the core of cores, x, under majority rule.*

Proof: Notice that the majority prefers the grand coalition to the core of the individual coalition by the results from the single market. With a slight abuse of notation, let $v(\mathcal{S})$ be the number of transactions available to the grand coalition. We then know that $v(\mathcal{S}) \geq v(T) + v(H)$ for any two sub-grand coalitions $T \cup H = \mathcal{S}$. This inequality is strict if and only if either $|B_T| > |L_T|$ and $|B_H| < |L_H|$ or vice versa. This means that any change in price will be opposed by a majority in one or the other markets. In the case where both sub-grand coalitions have at least as many buyers as sellers, there will be at least $|B| - |L| \geq 0$ buyers excluded from transactions under a sub-grand coalition. Then those buyers do not have strict preference for a sub-grand coalition, and the buyers that are involved in a transaction do not have a majority. □

This results says that given that the grand coalition is formed, there is no $y \in V(T)$ for any sub-grand coalition $T \subset \mathcal{S}$ and winning coalitions $w(S_i)$ such that $y \succ_j x$ for all $j \in w(S_i)$ and all $i \in T$. Therefore, the grand coalition is stable. The next result deals with the formation of the grand coalition under majority rule. But first we must introduce some more notation. Let $\tilde{L}$ and $\tilde{B}$ be the set of markets that contain a majority of sellers and a majority of buyers respectively.

Result 6 *(a) The grand coalition can not form from the* I *markets if* $|B_i| = |L_i|$ *for any market* i. *(b) If no market has an equal number of buyers and sellers, then any* $p^* \in (0,1)$ *supports formation of the grand coalition under the majority decision rule if and only if the following two conditions hold:*

$$\sum_{i \in \tilde{L}} |w(S_i)| \leq v(\mathcal{S}) \quad \textit{and} \quad \sum_{i \in \tilde{H}} |w(S_i)| \leq v(\mathcal{S})$$

for any winning coalition in market i.

Proof: Part (a) is straightforward. Because a price is pareto efficient, markets with equal numbers of buyers and sellers will never have a majority benefiting from any change in the price. Exactly half will benefit and half will not. If there is no price change from the market with equal buyers and sellers to the grand coalition, then all will be indifferent. Since formability requires that winning coalitions strictly prefer the grand coalition, the grand coalition is not formable when any of its individual markets has an equal number of buyers and sellers.

For part (b) remember that each individual of whichever type is in lesser numbers in a market receives the maximum payoff of 1. They will vote against joining the grand coalition for any p^*. Each member of whichever type is in greater numbers in a market receives the minimum payoff, 0. They will only vote to join the grand coalition if their payoff is greater than zero. With a price of $p^* \in (0,1)$ buyers and sellers that are involved in a transaction get payoffs greater than zero.

There are $v(\mathcal{S}) = \min\{|B|, |L|\}$ transactions in the grand coalition. More specifically, $v(\mathcal{S})$ buyers and $v(\mathcal{S})$ sellers are involved in a transaction when the grand coalition forms. If the above conditions hold, then there are enough transactions to include all buyers and sellers from each winning coalition. This is due to the fact that in a market with more sellers, a winning coalition consists of all sellers (vice versa for a market with more buyers). Buyers will earn $p^* > 0$ and sellers will earn $1 - p^* > 0$. Therefore, all members of winning coalitions have strict preference for the grand coalition, and the grand coalition is formable. If the above conditions do not hold, then there are not enough transactions to include all members of each winning coalition. In which case, a member of a winning coalition will earn zero, and that player loses strict preference for the grand coalition. Therefore the grand coalition is not formable. □

This example, markets for an indivisible good, shows that the formation of the grand coalition under a given decision rule and the stability of the grand coalition under that same decision rule are separate issues. The grand coalition may be stable and formable, not stable and not formable, formable and not stable, or stable and not formable. With unanimity, the grand coalition is always stable and never formable. With majority rule, the grand coalition is always stable and sometimes formable.

5.3 Infrastructure Sharing

This section presents an example of an SCG in which three villages are each in need of the same infrastructure investment. As it applies to development, the infrastructure may be a medical clinic, a food storage shed, or a borewell for safe drinking water. The investment is costly, and must be financed through taxation. Taxation may take the form of labor to construct the new infrastructure or an equivalent marginal product of labor in home production. In addition, the location of the investment matters. That is, each citizen has single peaked preferences with respect to the location of the investment. The closer it is to their own house, the lower the transportation costs they incur. If it is a borewell, then heavy buckets of water must be carried between one's house and the source. If it is a medical clinic, then one must travel some distance to receive treatment. In such a case, children and adults alike might often be accompanied by an adult, further detracting from home production. Finally, if it is a grain storage facility, then farmers must periodically transport their crop to and from the facility.

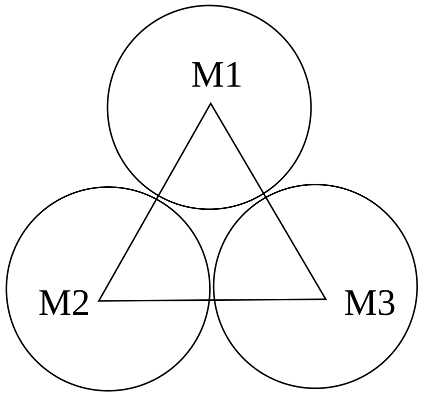
Up to this point, the setup is similar to Haimonko, LeBreton and Weber (2003) in which they analyze the location and funding of a central government in the framework of Aumann and Dreze (1974). It is also similar to Haeringer (2000) in which the author analyzes a finite public goods economy to find the funding and allocation schemes that give rise to a stable structure.

The current analysis departs in several important ways. Whereas the above papers take a single group of individuals and form a stable coalitional structure from them, this analysis starts with three groups (villages) that comprise a solid coalitional structure. Since each of the three villages is in need of the same costly infrastructure investment, there exists the possibility of sharing the investment and the financial burden with one or both of the other villages. The analysis will focus on which partitions of the SCG are stable and formable when the taxation and decision schemes are given.

The solid coalitional structure, $\mathcal{M} = \{M_1, M_2, M_3\}$, consists of three villages. Each M_i is a subset of $\mathbb{R}^2$ such that $M_i \cap M_j = \emptyset$. The citizens of M_i have single-peaked preferences identified by the points $x \in M_i$. When the infrastructure is located at point p, each individual incurs transportation costs $d(p, x) = \|p - x\|$, the Euclidean distance from their ideal point to the infrastructure. The distribution of ideal points is given by a cumulative distribution function F_i, defined over M_i. This gives rise to the continuous density function f_i such that $1 = \int_{M_i} \int f_i(x) dx_1 dx_2$.

In order to simplify the analysis, I make the following assumptions. First, assume that each of the M_i is a unit disk and that they are each centered at a vertex of an equilateral triangle with sides length 2 (note: you can think of this as $2+\epsilon$ where ϵ is arbitrarily small so that the disks are not overlapping).

Furthermore, the ideal points are distributed uniformly over the unit disk, so that $f(x) = \frac{1}{\pi}$ for $x \in M_i$. The cost of the infrastructure is I. If the



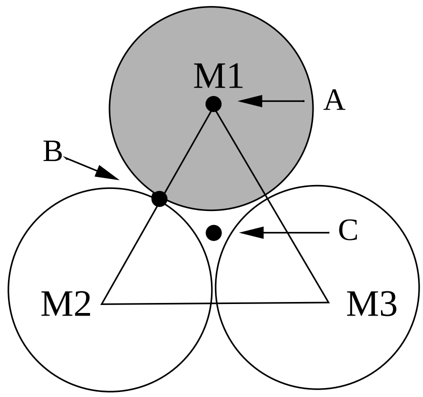
burden of the investment is to be shared evenly among the citizenry, we must have t^* solve $\int_{M_i}\int t^* \frac{1}{\pi} dx_1 dx_2 = I$, therefore $t^* = I$. In the above-mentioned literature, this is referred to as the *laissez faire* cost allocation. As the sum of the investment costs and the transportation costs, the aggregate costs are $I + D_i(p)$ where

$$D_i(p) = \int_{M_i}\int \|p - x\| \frac{1}{\pi} dx_1 dx_2.$$

Because the villages are assumed to be symmetric, we can focus the analysis on M_1. When M_1 does not share the investment with any other villages the borewell, medical clinic, or grain storage facility is located at point A. The justification is that point A is the location of the median voter, and it minimizes the aggregate transportation costs. That is, $p = A$ solves

$$\min_p \int_{M_i}\int \|p - x\| \frac{1}{\pi} dx_1 dx_2.$$

When M_1 joins with M_2, the public good is located at point B. The justification is the same. Point B minimizes the aggregate transportation costs for citizens of M_1 and M_2, and also cannot be defeated by majority vote by any other location. Finally, when all the villages join together for the investment, the public good is located at point C. This point minimizes transportation costs and is the only point that allows for three-way symmetry. That is, for every point $x \in M_i$ there are corresponding points $y \in M_j$ and $z \in M_k$ for which $\|C - x\| = \|C - y\| = \|C - z\|$.



In order to complete the setup of an SCG, we must assign each of the villages a decision rule. This decision rule is the basis upon which the village will decide between coalitional alternatives. That is, each village will use their decision rule to decide whether they want to invest alone, invest with one other village or invest with both of the other villages. For the sake of simplicity, assume that all three villages use majority vote as a decision rule.

Whether or not the grand coalition, $\mathcal{M} = \{M_1, M_2, M_3\}$, forms depends on the investment cost, I. For small values of I, each of the villages prefers to undertake the investment alone rather than sharing with one or both of the other villages. This is simply because the lower transportation costs outweigh the increased taxes. As the cost I increases, there are values under which B is the stable point. That is, the situation in which two of the three villages join together and the other undertakes its own investment is stable. Point B represents a tradeoff between increased transportation costs and a decreased tax burden. Finally, point C, which represents the grand coalition, is stable for even higher values of I. This occurs when the decreased tax burden outweighs the increased transportation costs. All of this information is contained in figure 1 below.

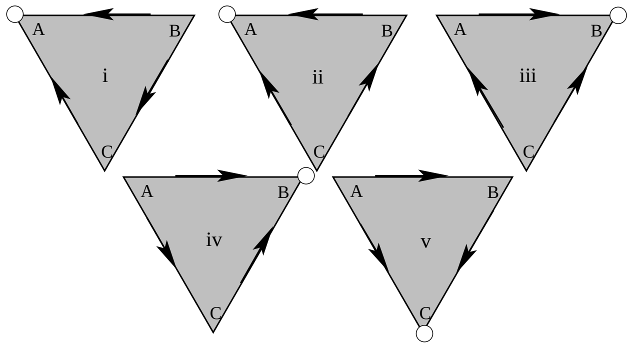


Figure 1: As investment costs increase, the SCG dynamic transitions from i to v.

The five triangles i to v represent the five states of the SCG dynamic. State i corresponds to low levels of I. As I increases, the dynamic progresses from i to ii and so on until state v. The arrows on each of the edges indicates the outcome of a majority vote to decide between the alternatives represented by the corresponding vertices. The stable point is indicated by an open circle. Notice that there is only one stable point per state, and that none of the states contains a cycle.

As is evident, the core of cores exists only after a threshold value $\hat{I}$ of investment costs. That is, the grand coalition in which all three villages share the infrastructure, is only stable with respect to the incentives and decision rule for values of $I \geq \hat{I}$. This is represented by state v, where point C is the grand coalition, $\mathcal{M}$. Also notice that $\mathcal{M}$ is formable from either of the other points, A or B. Finally, while C is not stable in state iv, it is formable. In state iv a majority vote between points A and C yields C as the winner. This example, like the last, shows that formability and stability are separate issues.

Interestingly, the stable point is not always the cost minimizing point. For ranges of I the outcome of the SCG dynamic is a point with higher aggregate costs than an alternative. For example, when $I = 1$, B is the cost-minimizing point for M_1. That is, of the three alternatives $\{A, B, C\}$, $B = p$ minimizes

$$\int_{M_1} \int (\|p - x\| + 1) f(x) dx_1 dx_2.$$

However, the dynamic is as in the triangle labelled ii from figure 1 above. Therefore, point A is the only stable point.

A similar situation occurs when I is in the neighborhood of 4.25. Here, C is the cost-minimizing point, however the dynamic is as in iv from above. This means, that while B is the stable point, it is not cost-minimizing. Notice that both examples occur as the dynamic transitions from one stable point to another. That is, the cost-minimizing point transitions ahead of the stable point.

It is possible to partly remedy this disconnect between the cost-minimizing and stable points by allowing a more general model of cost allocation. Whereas up to this point, all citizens of a village bear the tax burden equally, regardless of their transportation costs, it is possible to allow the individual taxes to depend inversely on transportation costs in the following way:

$$t(x) = \mu - \alpha\|p - x\|.$$

This scheme was first proposed by Haimonko *et al* (2003) for the one-dimensional case but easily applies to higher dimensional settings. As a citizen's transportation costs increase, the tax burden decreases. Naturally, when $\alpha = 0$ the cost allocation is again the *laissez faire* allocation from above, with no reduction in taxes for increased transportation costs. When $\alpha = 1$ there is full compensation, otherwise known as the Rawlsian allocation.

In order for the cost allocation to meet the requirement that

$$\int_{M_1}\int (\mu + \alpha\|p - x\|)f(x)dx_1dx_2 = I,$$

we must have

$$\mu(\alpha) = I + \alpha\hat{d}$$

where

$$\hat{d} = \int_{M_1}\int \|p - x\|f(x)dx_1dx_2,$$

the average distance to p. For those citizens with above average transportation costs, taxes are discounted according to some proportion α of the extra transportation costs. Those that are located closer than average to p bear a larger tax burden. This ensures a more equal allocation of the costs and benefits of the infrastructure investment.

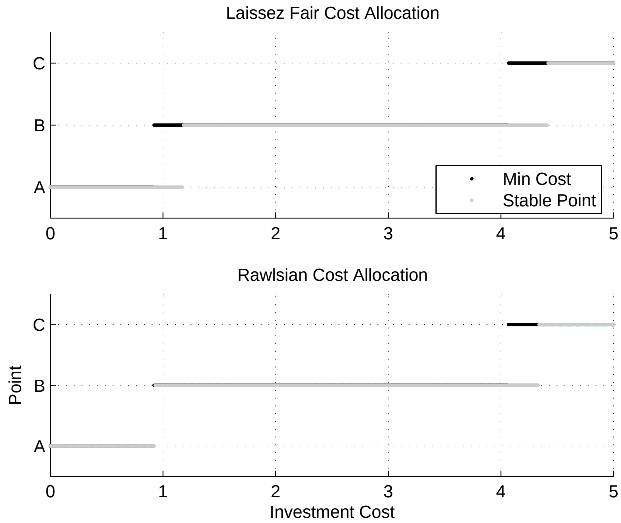


Figure 2: The Rawlsian cost allocation minimizes the regions of I for which the cost-minimizing point is different than the stable point

Increasing α decreases the regions of I for which the stable point does not coincide with the cost-minimizing point. This information is summarized in figure 2. For the *laissez faire* cost allocation ($\alpha = 0$), the regions colored in black represent the values of I for which the cost-minimizing point is not the stable point. Notice that the Rawlsian allocation eliminates this phenomenon for values of I near the transition from stability at A to stability at B. It also reduces the interval of values of I near the transition from stability at B to stability at C for which the cost-minimizing point is different from the stable point.

Another feature of the linear cost allocation scheme is that it reduces the threshold value $\hat{I}$ of investment cost necessary for the formation and stability of the grand coalition $\mathcal{M}$, at point C. In this example, by setting $\alpha = 1$, the core of cores exists and is formable for values of $I \geq \hat{I} \approx 4.3$. As it relates to the application of village infrastructure investment, cooperation between villages is more likely as the number and size of the projects increases and the cost allocation scheme moves towards full compensation.

6 Conclusion

The economic and political worlds are rich with examples of solid coalitional games. Everyday groups or coalitions merge to realize gains such as market power, international trade, development, political power, scientific discovery and environmental protection. Often, these coalitions are divided by legal, national, ethnic, or religious boundaries. And when they merge, they do so under agreement of terms and understanding of consequences.

Coalitions are unlike individuals in the way they make decisions. Sometimes the decision rule is unanimity or a majority rule, and sometimes it is a dictator. As seen in the above analysis, different decision rules have different results. Whether the Sunnis, Shiites and Kurds can form a coalition government in Iraq depends as much on the way the three groups make decisions as it does on the way power is divided. Whether the United States, Mexico and

Canada agreed to join NAFTA depended as much on the political processes in each country as it did on the terms of trade.

The solid coalitional games framework introduced here, including the definitions, properties and results, is the groundwork for future analysis of groups. The general results should be extended, and concepts such as formability and the core of cores should be applied to new problems.

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