

Pre and Post Break Parameter Inference*

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Abstract

This paper discusses inference about the pre and post break value of a scalar parameter in GMM time series models with a single break at an unknown point in time. We show that treating the break date estimated by least squares as the true break date leads to substantially oversized tests and confidence intervals unless the breaks is large. We develop an alternative test that controls size uniformly and that is approximately efficient in some well defined sense.

JEL Classification: C32, C12

Keywords: Nuisance Parameters; Least Favorable Distribution; Size of Test; Monte Carlo; Uniformity

1 Introduction

This paper is concerned with testing hypotheses about the pre and post break value of a parameter in a time series model with a single break. By reversing the time ordering, inference about the post break value becomes inference about the pre break value, so that for simplicity, we will exclusively discuss the post break case in the remainder of this paper. If

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the break date is known, inference is straightforward, as standard results apply by restricting attention to the stable post break data. The effect of an incorrectly chosen break date crucially depends on whether the chosen date is smaller or larger than the true break date. On the one hand, if the chosen date is larger, then a restriction to the presumed post break data still yields a stable model, and inference remains valid. There is a cost of efficiency, though, since more post break data could have been used. On the other hand, if the chosen date is smaller than the true break date, then the presumed post break data stems from an unstable model. Parameter estimators in this unstable model tend to yield estimates of the average parameter value, which is different from the true post break value, so standard inference yields tests and confidence intervals with distorted size.

These distortionary effects are small if the chosen break date is sufficiently close to the true break date. With the break date unknown, this requires a precise break date estimator. As formally shown by Bai (1994, 1997) and Bai and Perron (1998) for linear regressions, and Hall, Han, and Boldea (2008) for two stage least squares, the least squares break date estimator is indeed sufficiently precise for these distortionary effects to become asymptotically negligible *if* the parameter shift is sufficiently pronounced. Formally, these papers study asymptotics in which the break magnitude, while possibly shrinking, is outside the local $T^{-1/2}$ neighborhood. Relative the sampling uncertainty of the parameter estimator from a stable model, the parameter shift thus becomes infinitely large. We consider the behavior of 5% nominal level two-sided test based on the least squares break date estimator under local asymptotics in Section 2.4 below. With the break restricted to the middle 70% of the sample, a break magnitude of less than 5 standard deviations leads to effective size of more than 10% for all break dates, and a break of 11 standard deviations still yields size greater than 10% for some break dates. These distortions are further exacerbated by an attempt to pre-test for parameter stability. In other words, for any sample size T , there exist a break magnitude for which these standard methods yield incorrect inference, so they are not uniformly valid.

It is instructive to consider these issues in the context of an example. Productivity growth in most developed nations appears to have suffered shifts in its mean a number of times. Most researchers agree on a downward shift in productivity following the oil crisis of 1973, typically dated at the third quarter. In post 1973 data, there is a suggestion of a mid 1990's upward shift in average productivity growth and speculation about its cause. A recent

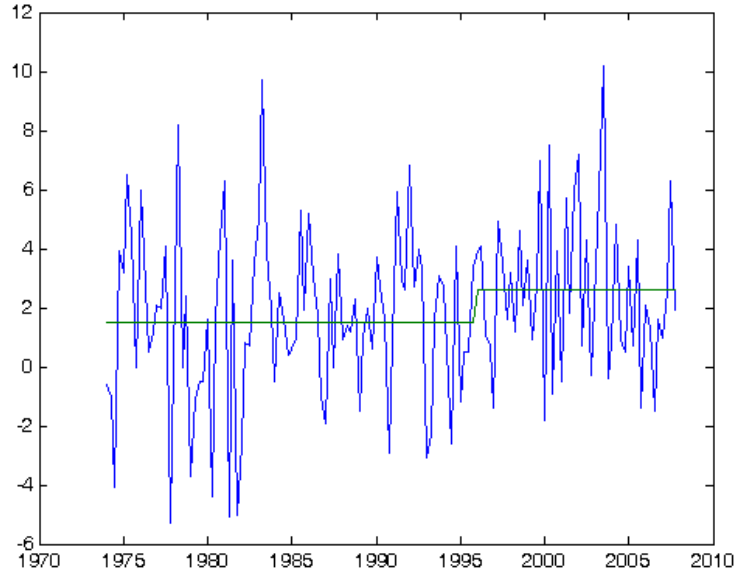


Figure 1: Quarterly growth in non farm business productivity growth and average growth pre and post 1995 Q4 (BLS series PRS85006092)

summary appears in Jorgenson, Ho, and Stiroh (2008), who informally suggest a break in the fourth quarter of 1995. Gordon (2003) also dates the increase to 1995. Anderson and Klieson (2006) refer to the increase in productivity as the 'defining economic event of the past decade'.

Figure 1 plots quarterly observations of non farm business productivity growth¹, along with the pre and post break means using Jorgenson, Ho, and Stiroh (2008) break date of 1995 Q4. Based on this data, US quarterly productivity growth increased by 1.1% in 1995 Q4. The least-squares estimate of the break date is slightly later at 1997 Q1. The first three columns of Table 1 contain 95% confidence intervals for current US productivity growth, conditional on these two break dates. The qll test of Elliott and Mueller (1996) returns a statistic of -5.88 which fails to reject the null of stability at the 5% level.

With a (long-run) standard deviation of productivity growth of approximately 3% and a sample of $T = 136$ observations, 5-11 standard deviations of the full sample estimator correspond to 1.3-2.8% shift in productivity growth. Absent prior knowledge that the break date is indeed 1995 Q4, or that the change of productivity is at least of the order of 1.5%, nothing suggests that either interval has close to its nominal confidence level. More gener-

¹(PRS85006092) from the BLS

Table 1: Inference about current US productivity growth

Method	95% Confidence Interval
Break date chosen	[1.36,3.42]
Least squares break date estimator	[1.97,3.27]
Test of this paper	[1.99,3.37]

ally, it seems hard to argue that break magnitudes of less than 11 standard deviations of the full sample estimator are not part of the empirically relevant parameter space in most applications. One way to see this that shift of, say, 8 standard deviations tend to induce highly significant rejections of the null hypothesis of parameter stability with high probability—see Elliott and Müller (2007). Yet there is continued debate about the stability of monetary policy, for instance, with Orphanides (2004) arguing for rather stable relationships, while Cogley and Sargent (2005) find instabilities that they deem ‘economically important’, but of a magnitude that would be detected by a formal parameter stability test less than 25% of the time.

Simple adjustments to the standard procedure to ensure size control over all break magnitudes do not deliver reasonable tests. For instance, in the set-up mentioned above and described in detail in Section 2.4, the critical value for the 5% two-sided t-statistic would need to be increased from 1.96 to approximately 4.9 to ensure uniform size control, with obvious adverse effects on power.² A Bonferroni procedure based on uniformly valid confidence sets for the break date developed by Elliott and Müller (2007) performs well for large breaks, but has poor power for breaks of moderate magnitude. The main contribution of this paper is thus the construction of a powerful test about the post break parameter value in a general GMM time series model with unknown break date, that controls size uniformly over the break magnitude. This test follows a switching rule: if there is strong evidence for a large break, then the test essentially reduces to standard inference using post break data determined by the least squares break date estimator, with a slight increase of the 5% critical value from 1.96 to 2.01. In absence of strong evidence for a large break, the test switches to a likelihood ratio test. This likelihood ratio test is carefully constructed to ensure both

²This adjustment to the critical value is also the end result of the hybrid subsampling method recently advanced by Andrews and Guggenberger (2007a, 2007b) for *any* break magnitude in the local $T^{-1/2}$ neighborhood.

overall size control and approximate efficiency in the sense that (local asymptotic) weighted average power of the suggested test is demonstrably at most 2.5% smaller than the weighted average power of any test that controls size.

From a statistical point of view, the problem is one of constructing powerful tests in the presence of a nuisance parameter under the null hypothesis—the nuisance parameter being the break data and break magnitude, neither of which can be consistently estimated under local asymptotics. We appropriately modify the algorithm developed in Müller and Watson (2008) to identify the approximate least favorable distribution for this nuisance parameter. This approximate least favorable distribution identifies the null density for the likelihood ratio part of the test statistic, and, using the insight of Müller and Watson (2008) and Andrews, Moreira, and Stock (2007), allows the construction of an upper bound on the power of all tests that control size.³ Beyond overcoming the substantial ‘engineering’ challenges in this example with two nuisance parameters, exacerbated by the impossibility of accurately representing continuous time Gaussian processes on a discrete computer, a further contribution of this paper to this methodology is a careful consideration of size control. A test controls size if the rejection probability, viewed as a function of the null parameter space, is always below the nominal level. The standard approach is to evaluate the rejection probability for a grid of parameter points by Monte Carlo, and to conclude size is controlled if none of the estimates is larger than the nominal level. We develop an algorithm that, while still based on Monte Carlo estimation, instead estimates (a bound on) the whole rejection probability function, ensuring that no additional implicit smoothness assumptions are necessary.

In many ways, the analysis in this paper mirrors the developments in the weak instrument literature over the last decades: Just like Staiger and Stock (1997), we consider an alternative asymptotic embedding that formalizes the lack of uniform validity of the standard method. Akin to their rule of thumb that standard asymptotics are approximately reliable when the first stage F-statistic is sufficiently large, we suggest a formal switching rule to the least squares break date based inference when a parameter stability test is highly significant. Finally, just as Andrews, Moreira and Stock (2006, 2007), we consider tests that are efficient

³By reframing the issue of identifying a powerful test with uniform size control in general decision theoretic terms (see Section 3.2 below), this power bound is closely related to the bound on the minimax value generated by the least favorable distribution over a subset of possible distributions, as analyzed by Chamberlain (2000).

relative to a weighted average power criterion, and identify an approximately optimal test.

Returning to the US productivity example, the last row in Table 1 shows the 95% confidence interval constructed by inverting the test suggested here. As one might have thought, the confidence interval is somewhat wider than the other two intervals, but not to the point of being entirely uninformative about current US productivity.

The remainder of the paper is organized as follows. The next section defines the asymptotic analogues to the small sample post break parameter inference problem, quantifies the size control issues of least squares break date based inference, and discusses the relationship of the small sample problem in parametric and GMM models to this limiting problem. In Section 3, we discuss in detail the construction of the test, the power bound and the size determination. The test statistic is defined in terms of partial sample GMM statistics in Section 4, where we also consider its asymptotic and small sample properties. Proofs not given in the main text are collected in an appendix.

2 Limiting Problems and Underlying Models

The natural large sample analogue to the small sample post break parameter inference is an inference problem about the drift component of a Gaussian process on the unit interval. The scalar version of this limiting problem is discussed next, along with the properties of the solution we derive. The next subsection numerically investigates the corresponding asymptotic properties of least squares break date based inference, motivating our construction of an alternative test. Subsection 3 considers the multivariate limiting problem, and shows in which sense the approximate efficiency of our solution in the scalar case carries over to the multivariate case. Finally, subsection 4 provides a formal link between the underlying small sample problem to the multivariate limiting problem for both parametric and GMM time series models.

2.1 Scalar Limiting Problem

The main focus of this paper is the following hypothesis testing problem: Suppose we observe the scalar Gaussian process $G(\cdot)$ on the unit interval (a random element with values in $D_{[0,1]}$,

the space of cadlag function on the unit interval)

$$G(s) = W(s) + \beta s + \delta \min(\rho, s), \quad s \in [0, 1] \quad (1)$$

where $W(\cdot)$ is standard Wiener process and $\beta, \delta \in \mathbb{R}$, and $\rho \in [0.15, 0.85]$. The process $G(s)$ is a Wiener process with drift $\beta + \delta$ for $s < \rho$, and drift β for $s \geq \rho$. The testing problem is $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$, so that geometrically, the question is whether the slope of the deterministic component $\beta s + \delta \min(\rho, s)$ is equal to zero after the (potential) kink at $s = \rho$. The location ρ and magnitude δ of the kink are nuisance parameters. This composite nature of the hypotheses can be made more explicit by introducing the notation $\theta = (\beta, \delta, \rho)$, $\Theta_0 = \{\theta = (0, \delta, \rho) : \delta \in \mathbb{R} \text{ and } \rho \in [0.15, 0.85]\}$ and $\Theta_1 = \{\theta = (\beta, \delta, \rho) : \beta \neq 0, \delta \in \mathbb{R} \text{ and } \rho \in [0.15, 0.85]\}$, so that the hypothesis testing problem becomes

$$H_0 : \theta = \Theta_0 \quad \text{against} \quad H_1 : \theta \in \Theta_1. \quad (2)$$

Let $\varphi : D_{[0,1]} \mapsto [0, 1]$ be a possibly randomized test of (2), where $\varphi(g)$ indicates the conditional probability of rejecting the null hypothesis test conditional on observing $G = g$. If the range of φ only consists of the two values $\{0, 1\}$, then φ is not randomized. The unconditional rejection probability is then simply the expectation of $\varphi(G)$, which we write as $E_\theta[\varphi(G)]$ to indicate that this expectation depends on the value of θ . With these definitions, a test φ is of level 5% if

$$E_\theta[\varphi(G)] \leq 5\% \text{ for all } \theta \in \Theta_0. \quad (3)$$

As an initial motivation of this testing problem, consider observations $\{y_t\}_{t=1}^T$ from a scalar Gaussian model with a break in the mean at time τ

$$y_t = \mu + \Delta \mathbf{1}[t \leq \tau] + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1). \quad (4)$$

We then have the following equality in distribution

$$T^{-1/2} \sum_{t=1}^{sT} y_t \sim G(s) \quad \text{for any } s \in \left\{ \frac{t}{T} \right\}_{t=1}^T$$

where $\rho = \tau/T$, $\beta = T^{1/2}\mu$ and $\delta = T^{1/2}\Delta$. Except for the scaling by $T^{-1/2}$ and the discretization $s \in \left\{ \frac{t}{T} \right\}_{t=1}^T$, the testing problem (2) involving the observation G is therefore identical to inference about the post-break mean in Gaussian location model (4), with the break date restricted to be in the middle 70% of all observations.

We argue below how the testing problem (2) based on the observation $G(\cdot)$ naturally arises as the asymptotically relevant problem in more complicated settings. For this discussion, however, it is useful to first summarize the essential properties of the test of (2) derived here.

It is easy to see that no uniformly most powerful test exists for inference about β in absence of knowledge of ρ and δ . Some tests will have good power for certain ranges of value of ρ and δ , while performing poorly for other values. To obtain a reasonable assessment of the overall quality of tests, we therefore adopt a weighted average power criterion, as, for instance, in Wald (1943), Andrews and Ploberger (1994), Andrews (1998), etc. Specifically, we determine a 5% level test φ that comes close to maximizing

$$WAP(\varphi) = \int E_{\theta}[\varphi(G)]dF(\theta) \quad (5)$$

where the weighting function F is chosen as follows.

Condition 1 *Under F , ρ is distributed uniform over $[0.15, 0.85]$ and (β, δ) is bivariate normal independent of ρ , with $\beta \sim \mathcal{N}(0, \sigma_{\beta}^2)$, $\delta \sim \mathcal{N}(0, \sigma_{\delta}^2)$ and β independent of $\beta + \delta$, where $\sigma_{\beta}^2 = 22$ and $\sigma_{\delta}^2 = 400$.*

The Gaussianity of the marginal distribution of β and δ in F under Condition 1 are chosen for analytical convenience, and the independence of β and $\beta + \delta$ mirrors the independence of the pre and post break sample information $G(\rho)$ and $G(1) - G(\rho)$. The value of $\sigma_{\beta}^2 = 22$ is motivated by King's (1988) discussion of overall reasonable point-optimal tests, since for $\sigma_{\beta}^2 = 22$, the best 5% level test has approximately 50% weighted average power. The uniform weighting over ρ accords to the choice in Andrews, Lee, and Ploberger (1996) and represent a desire to have reasonable power for all possible break dates. Finally, the value $\sigma_{\delta}^2 = 400$ is motivated as follows: For large δ , (say, $|\delta| > 12$ or so) there is a lot of information about the break date ρ , and good tests can come close to performing as well as if ρ was known. By putting enough weight on rather large values of δ , the choice $\sigma_{\delta}^2 = 400$ ensures that tests that perform well according to WAP share this desirable feature. At the same time, the distribution $\mathcal{N}(0, 400)$ concentrates about half of its mass on $|\delta| < 12$, so tests with high WAP must also perform reasonably in the arguably empirically relevant region for δ where the lack of knowledge of the break date severely complicates good inference about β .

The two main contributions of this paper regarding the hypothesis testing problem (2) based on the observation $G(\cdot)$ is summarized as follows.

Proposition 1 *Under Condition 1,*

(i) *any 5% level test φ of (2) has $WAP(\varphi) \leq \bar{\pi} \simeq 51.5\%$;*

(ii) *the (nonrandomized) test φ^* defined in the appendix is of level $\alpha^* \simeq 5\%$, and has $WAP(\varphi^*) = \pi^* \simeq 49.0\%$.*

The numbers for $\bar{\pi}$, α^* and π^* in Proposition 1 were estimated based on 50,000 independent Monte Carlo draws . As discussed in detail in Section 3, they are *not* subject to any other qualifications, such as a replacement of the level requirement (3) by a finite set of values for (δ, ρ) or a finite step Random Walk approximation for draws of G .

2.2 Pretest and Least-Square Based Inference

As discussed in the introduction, Bai (1994, 1997) and Bai and Perron (1998) suggest and analyze the following procedure for conducting inference about the post break value of a coefficient in a linear regression that is subject to a break: Estimate the break date by least-squares, construct a post break dummy using this estimate, and perform the usual t-test on that dummy. These papers show that this procedure results in asymptotically correct inference for break magnitudes diverge to infinity when multiplied by the square root of the sample size. This corresponds to $|\delta| \rightarrow \infty$ in the notation developed here. Another procedure, arguably most prevalent in applied work, first performs a pretest for a break on some conventional significance level, followed by standard full sample inference about the post break value if no significant break is found, and performs least-squares based post break inference as above if the pretest rejects.

In the limiting problem (1) and (8), the nominally 5% level least squares and pretest procedure (based on the supF statistic) correspond to the test

$$\mathbf{1}[\text{supF} > \text{cv}_{pre}(\alpha_{pre})] \mathbf{1}[\hat{t} > \Phi(0.975)] + \mathbf{1}[\text{supF} < \text{cv}_{pre}(\alpha_{pre})] \mathbf{1}[|G(1)| > \Phi(0.975)] \quad (6)$$

where $\text{cv}_{pre}(\alpha_{pre})$ is the critical value of the supF statistic of level α_{pre} as tabulated in Andrews (1993), (and $\text{cv}_{pre}(1) = 0$ corresponds to always using the least squares break date estimator, with no pretest), and

$$\hat{\rho} = \underset{r \in [0.15, 0.85]}{\text{argmax}} \frac{(G(r) - rG(1))^2}{r(1-r)} \quad \text{supF} = \frac{(G(\hat{\rho}) - \hat{\rho}G(1))^2}{\hat{\rho}(1-\hat{\rho})} \quad \hat{t} = \frac{G(1) - G(\hat{\rho})}{\sqrt{1-\hat{\rho}}}. \quad (7)$$

Under standard assumptions, the large sample properties of the small sample pretest and least square based tests in a linear regression with a single coefficient converge in distribution

to those in (6) for breaks of magnitude $T^{-1/2}$ —this follows, for instance, from Proposition 1 in Elliott and Müller (2007).

Figure 1 displays the null rejection probability of (6) for $\alpha_{pre} = 1$ (no pretest), $\alpha_{pre} = 0.10$, $\alpha_{pre} = 0.05$ and $\alpha_{pre} = 0.01$ as a function of δ for selected values of ρ , based on 50,000 Monte Carlo draws and 1,000 step random walk approximations to Wiener processes. None of these tests comes close to control size uniformly over δ , with null rejection probabilities of at least 20% for some value of (ρ, δ) . Pre-testing for a break seems to both substantially exacerbate and shift the size control problem to larger values of δ .

In fact, to obtain a 5% level test in the no pretest case, one must employ a critical value of approximately 4.9 instead of $\Phi(0.975) \simeq 1.96$ for the t-statistic. As one might expect, such a test has quite low power: the weighted average power of such a test is approximately xx percent.

2.3 Multivariate Limiting Problem

Now assume that in the hypothesis testing problem (2) we observe a multivariate version of (1), with

$$\begin{pmatrix} G(s) \\ \tilde{G}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} W(s) \\ \tilde{W}_k(s) \end{pmatrix} + \int_0^s \begin{pmatrix} \beta + \delta[\lambda \leq \rho] \\ \tilde{f}(\lambda) \end{pmatrix} d\lambda \quad (8)$$

where \tilde{G} is $k \times 1$, the $k \times 1$ and $k \times k$ matrices A_{12} and A_{22} are known, \tilde{W}_k is a k vector standard Wiener process independent of W , and $\tilde{f}(\lambda)$ is a $D_{[0,1]} \mapsto \mathbb{R}^k$ cadlag function, so that $(G, \tilde{G}')' \in D_{[0,1]}^{k+1}$. With $(G, \tilde{G}')'$ as the observation, tests φ are now mappings from $D_{[0,1]}^{k+1} \mapsto [0, 1]$, and the rejection probability is given by $E_{\theta, \tilde{f}}[\varphi(G, \tilde{G})]$ (the rejection probability also depends on A_{21} and A_{22} , but we omit this dependence for ease of notation).

As an initial motivation for (8), think again of a Gaussian location problem

$$\begin{pmatrix} y_t \\ \tilde{y}_t \end{pmatrix} = \begin{pmatrix} \mu + \Delta \mathbf{1}[t \leq \tau] \\ \tilde{\mu}_t \end{pmatrix} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \Sigma) \quad \text{with } \Sigma = \begin{pmatrix} 1 & A'_{21} \\ A_{21} & A_{21}A'_{21} + A_{22}A'_{22} \end{pmatrix} \quad (9)$$

where $\tilde{y}_t, \tilde{\mu}_t \in \mathbb{R}^k$, so that partial sums $T^{-1/2} \sum_{t=1}^s (y_t, \tilde{y}_t)'$ have the same distribution as $(G, \tilde{G}')'$ for $s \in \{t/T\}_{t=1}^T$, where $\rho = \tau/T$, $\beta = T^{1/2}\mu$, $\delta = T^{1/2}\Delta$ and $f(t/T) = T^{1/2}\tilde{\mu}_t$.

The first element in (8), G , is distributed exactly as in the scalar problem (1) above. Thus, if we define φ^* to be the same test as above, that is, φ^* is a function of G only, then

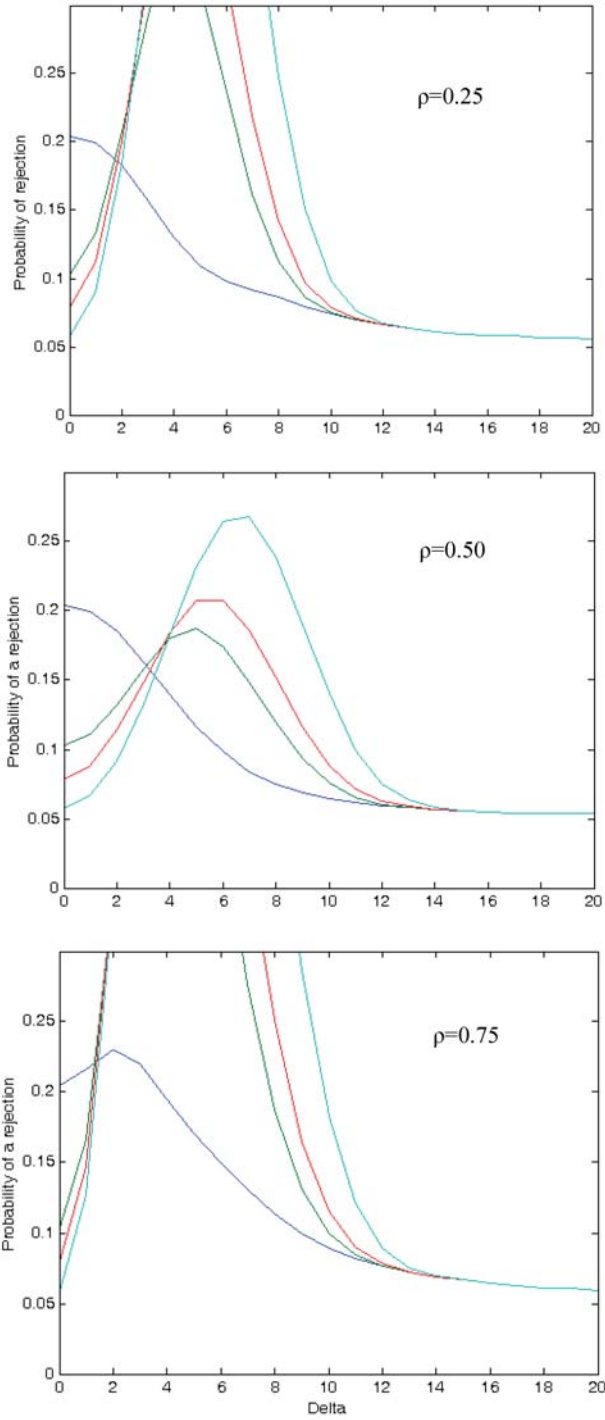


Figure 2: Null rejection probabilities of nominal 5% level tests for post break parameter value based on least squares break date estimator. Blue: no parameter stability pretest, Green: 10% level pretest, Red: 5% level pretest, Magenta: 1% level pretest.

φ^* continues to have the properties indicated in Proposition 1. The rejection probability of φ^* obviously does not depend on the distribution of \tilde{G} , so that in particular

$$E_{\theta, \tilde{f}}[\varphi^*(G)] \leq \alpha^* \text{ for all } \tilde{f} \text{ and } \theta \in \Theta_0 \quad (10)$$

$$\inf_{\tilde{f}(\cdot) \in D_{[0,1]}^k} WAP_{\tilde{f}}(\varphi^*) = \sup_{\tilde{f}(\cdot) \in D_{[0,1]}^k} WAP_{\tilde{f}}(\varphi^*) = \pi^* \quad (11)$$

where for a generic test $\varphi : D_{[0,1]}^{k+1} \mapsto [0, 1]$, $WAP_{\tilde{f}}(\varphi) = \int E_{\theta, \tilde{f}}[\varphi(G, \tilde{G})] dF(\theta)$. In other words, φ^* is a robust test of $H_0 : \theta \in \Theta_0$ from observing (8) in the sense that its size and (weighted average) power does not depend on the nuisance parameter \tilde{f} .

In the following, we restrict attention to the case where \tilde{f} is known to be of the form $\tilde{f}(\lambda) = \tilde{\beta} + \tilde{\delta} \mathbf{1}[\lambda \leq \rho]$ for $\tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k$. This corresponds to a Gaussian location problem (9) where the last k elements of Y_t undergo a single mean shift at the same time as a the first element. Write $E_{\theta, \tilde{\beta}, \tilde{\delta}}$ for the expectation with respect to the distribution of $(G, \tilde{G})'$ with \tilde{f} of this form, so that a 5% level test $\varphi : D_{[0,1]}^{k+1} \mapsto [0, 1]$ now satisfies $E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi(G, \tilde{G})] \leq 0.05$ for all $\theta \in \Theta_0$, $\tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k$.⁴

Proposition 2 (i) For any unbiased 5% level test φ ,

$$E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_\rho(G)] - E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi(G, \tilde{G})] \geq 0 \text{ for all } \theta \in \Theta_1, \tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k$$

where $\varphi_\rho(G) = \mathbf{1}[|\frac{G(1)-G(\rho)}{\sqrt{1-\rho}}| > \Phi(0.975)]$ and $\Phi(0.975) = 1.96\dots$ is 97.5th percentile of the c.d.f. of a standard normal.

(ii) Let \tilde{F} be the probability distribution for $(\theta, \tilde{\beta}, \tilde{\delta})$ with the same marginal distribution for θ as under Condition 1, and $(\tilde{\beta}, \tilde{\delta}) = (\beta A_{21}, \delta A_{21})$ conditional on θ . For any 5% level test

$$\int E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi(G, \tilde{G})] d\tilde{F}(\theta, \tilde{\beta}, \tilde{\delta}) \leq \bar{\pi}.$$

(iii) Suppose in addition that $\tilde{\delta} = 0$ known, so that tests $\varphi : D_{[0,1]}^{k+1} \mapsto [0, 1]$ are of 5% level if $E_{\theta, \tilde{\beta}, 0}[\varphi(G, \tilde{G})] \leq 0.05$ for all $\theta \in \Theta_0$ and $\tilde{\beta} \in \mathbb{R}^k$. Consider the group of transformations

$$\begin{pmatrix} G(s) \\ \tilde{G}(s) \end{pmatrix} \rightarrow \begin{pmatrix} G(s) \\ \tilde{G}(s) + \tilde{b}s \end{pmatrix}, \tilde{b} \in \mathbb{R}^k. \quad (12)$$

For any 5% level test φ invariant to (12), the rejection probability $E_{\theta, \tilde{\beta}, 0}[\varphi(G, \tilde{G})]$ does not depend on $\tilde{\beta}$, and under Condition 1,

$$WAP_{inv}(\varphi) = \int E_{\theta, \tilde{\beta}, 0}[\varphi(G, \tilde{G})] dF(\theta) \leq \bar{\pi}_{inv}(\bar{R}^2)$$

⁴All following propositions also hold for the scalar case ($k = 0$) with the natural interpretation of notation.

where $\bar{R}^2 \geq R^2 = A'_{21}(A_{21}A'_{21} + A_{22}A'_{22})^{-1}A_{21}$, and $\bar{\pi}_{inv}(0) = \bar{\pi} \simeq xx$, $\bar{\pi}_{inv}(0.2) \simeq xx$, $\bar{\pi}_{inv}(0.4) \simeq xx$, $\bar{\pi}_{inv}(0.6) \simeq xx$, and $\bar{\pi}_{inv}(0.8) \simeq xx$.

Part (i) of Proposition 2 establishes that the usual two-sided t-test using post break data $G(1) - G(\rho)$, φ_ρ , is the uniformly most powerful unbiased test in the multiparameter problem with (8) observed. With ρ unknown this test is infeasible. It nevertheless provides an arguably relevant benchmark to assess the performance of other tests. In particular, note that φ_ρ , just like φ^* , does not depend on \tilde{G} , but is a function of G only. We compare the power of φ^* to that of φ_ρ in detail in Section 4 below, and find that for large $|\delta|$, the power becomes quite close. Thus, for large $|\delta|$, φ^* has similar rejection properties as the uniformly most powerful unbiased test, so that there is at most a small cost in terms of power for ignoring the additional information in \tilde{G} . (Although φ^* is not unbiased, as it is not similar—see Section 3 below.)

Part (ii) of Proposition 2 shows that for a particular weighting function \tilde{F} with the same marginal distribution as F on θ , no 5% level test can have weighted average power larger than $\bar{\pi}$. At the same time, (11) implies that the weighted average power of φ^* with respect to \tilde{F} is equal to π^* , and, by Proposition 1, π^* is only slightly smaller than $\bar{\pi}$. In this sense, φ^* is close to being admissible also in the hypothesis testing problem with (8) observed. Thus, any test with "substantially" larger power than φ^* for some $(\theta, \tilde{\beta}, \tilde{\delta})$ is less powerful for some other $(\theta, \tilde{\beta}, \tilde{\delta})$. Given this inevitable trade-off, the choice of φ^* of approximately maximizing weighted average power relative to \tilde{F} has the perhaps appealing implication of yielding a test whose weighted average power over θ does not depend on the nuisance parameters $(\tilde{\beta}, \tilde{\delta})$.

Finally, part (iii) of Proposition 2 provides bounds on the gains in weighted average power that are possible if it is known that the other parameters do not undergo a break, $\tilde{\delta} = 0$ (or, equivalently, if $\tilde{\delta}$ was known) under an invariance requirement that corresponds to changes of the mean of the additional k observations. A maximal invariant to this group is given by $(G(s), \tilde{G}(s) - s\tilde{G}(1))$. Intuitively, these power gains are possible because if \tilde{G} is correlated with G ($R^2 > 0$), observing $\tilde{G}(s) - s\tilde{G}(1) = \tilde{W}(s) - s\tilde{W}(1)$ provides information about $W(s) - sW(1)$, which is useful for learning about ρ and δ . In the extreme case of perfect correlation ($R^2 = 1$), $W(s) - sW(1)$ is effectively observed, and by comparing $G(s) - sG(1)$ to $W(s) - sW(1)$, one can back out ρ and δ and obtain as powerful inference about β as if there was no break in the model. The numbers for $\bar{\pi}_{inv}$ were estimated using 50,000 independent Monte Carlo draws. They show that rather large values of R^2 are necessary

before knowledge of $\tilde{\delta} = 0$ could potentially be used to generate tests with substantially larger weighted average power than φ^* .

In summary, the test φ^* that disregards the additional observation \tilde{G} in (8) is (i) robust in the sense of providing reliable inference about β regardless of the nuisance function \tilde{f} ; (ii) for large δ has a rejection profile that is close to that of the uniformly most powerful unbiased test; (iii) is close to admissible in a model where other parameters undergo a break at the same time; (iv) approximately maximizes weighted average power even when it is known that only the parameter of interest is subject to a break as long as R^2 is small. For these reasons, the remainder of this paper focusses on φ^* as a reasonable test also in the multiparameter context of (8). We note, however, that it should be possible in principle to use the numerical methods discussed in Section 3 to also identify 5% level feasible test that comes close to maximizing weighted average power by efficiently exploiting the additional observation \tilde{G} when it is known that only the parameter of interest is subject to a break, for any particular value of R^2 .

2.4 Asymptotic Efficiency Implications for Parametric and GMM Models

In this subsection, we discuss how the scalar and multiparameter limiting problems (1) and (8) arise as the relevant asymptotic problems in standard small sample post break parameter inference. We consider two distinct approaches to asymptotic efficiency of tests: On the one hand, we rely on LeCam's Limits of Experiments theory to derive upper bounds on power in correctly specified likelihood models. On the other hand, we exploit the recent results of Müller (2007) and provide upper bounds on post break parameter tests in time series GMM models under an asymptotic robustness constraint.

The result of this subsection are (asymptotic) upper bounds on power, analogous to Proposition 2. We assume that the parameter value is localized around a known parameter value Γ_0 ; while such knowledge is potentially unrealistic, this is without loss of generality, since it can only increase the achievable power. Let $X_T = (x_{T,1}, \dots, x_{T,T}) \in \mathbb{R}^{qT}$ be the available data in a sample of size T , so that tests φ_T are sequences of (measurable) functions $\varphi_T : \mathbb{R}^{qT} \mapsto [0, 1]$.

We first discuss parametric models. The density of X_T , relative to some σ -finite measure

μ_T , is given by the product $\prod_{t=1}^T f_{T,t}(\Gamma_t)$, when the parameter $\Gamma \in \mathbb{R}^{k+1}$ takes on the value Γ_t at time t . This form of the likelihood arises naturally with $f_{T,t}(\Gamma_t)$ the density of $x_{T,t}$ conditional on $\mathfrak{F}_{T,t-1}$, the σ -field generated by $\{x_{T,s}\}_{s=1}^{t-1}$. We will refer to the model with density $\prod_{t=1}^T f_{T,t}(\Gamma_0)$ as the 'stable' model. Define $l_{T,t}(\Gamma) = \ln f_{T,t}(\Gamma)$, $s_{T,t}(\Gamma) = \partial l_{T,t}(\Gamma)/\partial \Gamma$ and $h_{T,t} = \partial s_{T,t}(\Gamma)/\partial \Gamma'$, and suppose $T^{-1} \sum h_{T,t} \xrightarrow{p} -H$ in the stable model for some positive definite and known $(k+1) \times (k+1)$ matrix H , so that H^{-1} is the Fisher information. In the unstable model, the parameter vector Γ evolves as

$$\Gamma_{T,t} = \Gamma_0 + T^{-1/2} \begin{pmatrix} \beta\omega \\ \tilde{\beta} \end{pmatrix} + T^{-1/2} \begin{pmatrix} \delta\omega \\ \tilde{\delta} \end{pmatrix} \mathbf{1}[t \leq \rho T] \quad (13)$$

where ω^2 is the 1-1 element of H , $\beta, \delta, \rho \in \mathbb{R}$ and $\tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k$, and $\rho \in [0.15; 0.85]$. The hypothesis of interest continuous to concern the value of the post break parameter

$$H_0 : \beta = 0 \quad \text{against} \quad H_1 : \beta \neq 0. \quad (14)$$

The factor $\omega > 0$ in (13) ensures that asymptotically, the small sample problem (14) maps to the limiting problem (8) where the disturbance in G is a *standard* Wiener process. Tests $\varphi_T : \mathbb{R}^{qT} \mapsto [0, 1]$ of (14) have unconditional rejection probability $E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_T(X_T)]$ in a sample of size T , and φ_T is defined to be of asymptotic level 5% if $\limsup_{T \rightarrow \infty} E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_T(X_T)] \leq 5\%$ for all $\theta \in \Theta_0$, $\tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k$.

Under sufficient regularity conditions about the parametric model, one can show that the likelihood ratio statistic LR_T of the model with parameter evolution described by (13) and the stable model satisfies

$$\begin{aligned} \ln \text{LR}_T &= T^{-1/2} \sum_{t=1}^T (\Gamma_{T,t} - \Gamma_0)' s(\Gamma_0) + \frac{1}{2} T^{-1} \sum_{t=1}^T (\Gamma_{T,t} - \Gamma_0)' h_t(\Gamma_0) (\Gamma_{T,t} - \Gamma_0) + o_p(1) \\ &\Rightarrow \int_0^1 \begin{pmatrix} \beta + \delta \mathbf{1}[\lambda \leq \rho] \\ \tilde{\beta} + \tilde{\delta} \mathbf{1}[\lambda \leq \rho] \end{pmatrix}' \Sigma^{-1} d \begin{pmatrix} G(\lambda) \\ \tilde{G}(\lambda) \end{pmatrix} - \frac{1}{2} \int_0^1 \begin{pmatrix} \beta + \delta \mathbf{1}[\lambda \leq \rho] \\ \tilde{\beta} + \tilde{\delta} \mathbf{1}[\lambda \leq \rho] \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \beta + \delta \mathbf{1}[\lambda \leq \rho] \\ \tilde{\beta} + \tilde{\delta} \mathbf{1}[\lambda \leq \rho] \end{pmatrix} d\lambda \end{aligned} \quad (15)$$

where $\Sigma = \omega H^{-1}$. The r.h.s. of (15) may be recognized as log of the Radon-Nikodym derivative of the distribution of $(G, \tilde{G})'$ with respect to the distribution of the Wiener processes $(W, (A_{21}W + A_{22}\tilde{W}_k)')$. This suggests that the information regarding $(\theta, \tilde{\beta}, \tilde{\delta})$ from observing X_T converges in large samples to that contained in $(G, \tilde{G})'$. This intuition is made formally precise in the Limit of Experiments theory pioneered by LeCam; see, for instance, van der Vaart (1998) for an introduction.

We state a precise regularity assumption that entail convergence of the experiments in this sense in Condition 2 in the appendix. The Asymptotic Representation Theorem (see, for instance, Theorem 9.3 of van der Vaart (1998)) and Prohorov's Theorem then ensure that for any test $\varphi_T : \mathbb{R}^{q_T} \mapsto [0, 1]$ and any subsequence T' of T , there exists a further subsequence T'' of T' and a test $\varphi : D_{[0,1]}^{k+1} \mapsto [0, 1]$ such that

$$E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_{T''}(X_{T''})] \rightarrow E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi(G, \tilde{G})] \text{ for all } \theta \in \Theta_0 \cup \Theta_1, \tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k. \quad (16)$$

Thus, along any such subsequence T'' , the asymptotic properties of φ_T are exactly equivalent to a particular test φ in the limiting problem discussed in section 2.2. above. Define a test φ_T as asymptotically unbiased if any test φ satisfying (16) is unbiased. Similarly, define a test φ_T as asymptotically invariant if any test φ satisfying (16) is invariant to (12).

Proposition 3 (i) *For any asymptotically unbiased test φ_T of asymptotic level 5%,*

$$\limsup_{T \rightarrow \infty} E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_T(X_T)] \leq E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_\rho(G)] \text{ for all } \theta \in \Theta_1, \tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k$$

(ii) *For any test φ_T of asymptotic level 5%,*

$$\limsup_{T \rightarrow \infty} \int E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_T(X_T)] d\tilde{F}(\theta) \leq \bar{\pi}.$$

(iii) *For any asymptotically invariant test φ_T of asymptotic level 5%,*

$$\limsup_{T \rightarrow \infty} \int E_{\theta, \tilde{\beta}, 0}[\varphi_T(X_T)] dF(\theta) \leq \pi_{inv}(\bar{R}^2) \text{ for all } \tilde{\beta} \in \mathbb{R}^k.$$

Proposition 3 almost exactly mirrors the results of Proposition 2 above. Furthermore, we argue in Section 4 below that the test $\hat{\varphi}^* : \mathbb{R}^{q_T} \mapsto [0, 1]$ has the same asymptotic rejection properties as the test φ^* . Thus, to the extent that Proposition 2 implies φ^* to be an attractive test in the context of the multivariate limiting problem (8), $\hat{\varphi}^*$ is correspondingly an attractive test in the context of post break inference in a correctly specified parametric model that is subject to a local break.

We now turn to an application of the asymptotic efficiency concept introduced by Müller (2007), which we use here to state asymptotic power bounds on post break parameter inference in a class of GMM models. Let the moment condition be the \mathbb{R}^l -valued function $g(\cdot, \cdot)$, so that $E[g(x_{T,t}, \Gamma_0)] = 0$ when the true parameter at date t is given by $\Gamma = \Gamma_0$. Let the $\mathbb{R}^{k \times l}$ valued function $\Upsilon(\cdot, \Gamma)$ be the partial derivative of g with respect to Γ , and write

$g_t(\Gamma) = g(x_{T,t}, \Gamma)$ and $\Upsilon_t(\Gamma) = \Upsilon(x_{T,t}, \Gamma)$ for notational simplicity. Under the parameter evolution (13), conditional on $\tau = [\rho T]$, the natural estimators for the pre and post break value of Γ are given by the GMM estimator using pre and post break data,

$$\begin{aligned}\hat{\Gamma}_{pre}(\tau) &= \arg \min_{\Gamma} \left(\sum_{t=1}^{\tau} g_t(\Gamma) \right)' \hat{V}_{pre} \left(\sum_{t=1}^{\tau} g_t(\Gamma) \right) \\ \hat{\Gamma}_{post}(\tau) &= \arg \min_{\Gamma} \left(\sum_{t=\tau+1}^T g_t(\Gamma) \right)' \hat{V}_{post} \left(\sum_{t=\tau+1}^T g_t(\Gamma) \right)\end{aligned}$$

with \hat{V}_{pre} and \hat{V}_{post} possibly data dependent $l \times l$ positive definite weighting matrices, and associated covariance matrix estimators $\hat{\Omega}_{pre}(\tau)$ and $\hat{\Omega}_{post}(\tau)$ of $\hat{\Gamma}_{pre}(\tau)$ and $\hat{\Gamma}_{post}(\tau)$, respectively. A natural starting point for inference about the post break parameter value in absence of knowledge of $\rho \in [0.15; 0.85]$ are thus the processes $\{\hat{\Gamma}_{pre}([\cdot T]), \hat{\Gamma}_{post}([\cdot T]), \hat{\Omega}_{pre}([\cdot T]), \hat{\Omega}_{post}([\cdot T])\} \in D_{[0.15, 0.85]}^{4(k+1)}$. Assuming that $T^{-1/2} \sum_{t=1}^{[\cdot T]} g_t(\Gamma_0) \Rightarrow \Sigma_g^{1/2} W_{k+1}(\cdot)$ for some positive definite matrix Σ_g and $T^{-1} \sum_{t=1}^{[\cdot T]} \Upsilon_t(\Gamma_0) \xrightarrow{p} \cdot \tilde{\Upsilon}$ for some full column rank matrix $\tilde{\Upsilon}$, the usual Taylor expansion arguments yield for efficient GMM estimation that

$$\begin{aligned}\{T^{1/2} \hat{\Gamma}_{pre}([sT]), T^{1/2} \hat{\Gamma}_{post}([sT]), T \hat{\Omega}_{pre}([sT]), T \hat{\Omega}_{post}([sT])\} \\ \Rightarrow \left\{ \begin{pmatrix} G(s) \\ \tilde{G}(s) \end{pmatrix}, \begin{pmatrix} G(1) - G(s) \\ \tilde{G}(1) - \tilde{G}(s) \end{pmatrix}, s\omega^2 \Sigma, (1-s)\omega^2 \Sigma \right\}\end{aligned}\quad (17)$$

in $D_{[0.15, 0.85]}^{(k+1) \times 4}$, where now $\omega^2 \Sigma = (\tilde{\Upsilon}' \Omega_g^{-1} \tilde{\Upsilon})^{-1}$ (cf. Andrews (1993)).

Consider now the set of data generating processes for X_T that satisfy (17). One might want to choose the tests φ_T of (14) in a way that whenever (17) holds with $\beta = 0$, the test does not overreject asymptotically, that is $\limsup_{T \rightarrow \infty} E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_T(X_T)] \leq 0.05$ for all $\theta \in \Theta_0$, $\tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k$ for *any* sequence of distributions of X_T that satisfies (17). Müller (2007) shows that under this robustness constraint, the best small sample test becomes the best test in the limiting problem (that is, with the r.h.s. of (17) assumed observed), evaluated at sample analogues (that is, $\{T^{1/2} \hat{\Gamma}_{pre}([\cdot T]), T^{1/2} \hat{\Gamma}_{post}([\cdot T]), T \hat{\Omega}_{pre}([\cdot T]), T \hat{\Omega}_{post}([\cdot T])\}$). Proposition 2 above shows in which sense φ^* may be considered approximately best in the equivalent problem of observing $(G(\cdot), \tilde{G}(\cdot))' \in D_{[0,1]}^{k+1}$ directly. Thus, part (ii) of Proposition 2 implies, for instance, that the weighted average asymptotic power (defined as in part (ii) of Proposition 3) of any 5% level test is no larger than $\bar{\pi}$. We discuss in detail in Section 4 below how to construct the small sample analogue $\hat{\varphi}^*$, and show in Proposition 4 that its asymptotic

rejection profile is equal to the rejection profile of φ^* whenever (17) holds, so its asymptotic weighted average power is equal to π^* . Thus, $\hat{\varphi}^*$ is the approximately best test among all robust tests in the sense of Müller (2007). We omit a formal statement analogous Proposition 3 to conserve space.

3 Determination of an Approximately Efficient Test with Uniform Size Control

This section describes the methods and algorithm that underlie the claim of Proposition 1, that is we discuss the determination of a 5% level test of (2) based on the observation G as in (1) that approximately maximizes weighted average power with a weighting function as described in Condition 1.

In the first subsection, we briefly review Andrews, Moreira and Stock's (2007) and Müller and Watson's (2008) central idea of identifying an approximate least favorable distribution to obtain tests that demonstrably come close to maximizing (weighted average) power in the presence of nuisance parameters under the null hypothesis. The second subsection provides details on the numerical algorithm for obtaining the approximately least favorable prior, which is suitably modified version of what is developed in Müller and Watson (2008) for the problem at hand. In the third subsection, we introduce a new approach to the numerical study of size control of tests of a composite null hypothesis, which underlies our claim of uniform size control of the test φ^* . Finally, the fourth subsection contains details on the Monte Carlo determination of the power bound, which is complicated by the impossibility of generating and representing (pseudo) random Wiener process draws on a discrete computer.

3.1 Hypothesis Tests with Nuisance Parameters under Null Hypothesis

By Girsanov's Theorem, the Radon-Nikodym derivative of the measure of G in (1) with parameter $\theta \in \Theta_0 \cup \Theta_1$ relative to the measure ν of the standard Wiener process W , evaluated at G , is given by

$$f_\theta(G) = \exp[\delta G(\rho) + \beta G(1) - \frac{1}{2}(\beta^2 + 2\beta\delta\rho + \delta^2\rho)]. \quad (18)$$

Expressed in terms of the density f_θ , the hypothesis test (2) becomes

$$\begin{aligned} H_0 &: \text{The density of } G \text{ is } f_\theta, \theta \in \Theta_0 \\ H_1 &: \text{The density of } G \text{ is } f_\theta, \theta \in \Theta_1 \end{aligned} \tag{19}$$

and, as noted in subsection 2.1, both the null and alternative hypothesis involve the two nuisance parameters δ and ρ . Note, however, that weighted average power can be rewritten as

$$WAP(\varphi) = \int E_\theta[\varphi(G)]dF(\theta) = \int \int \varphi(G)f_\theta dF(\theta)d\nu$$

as Fubini's Theorem justifies the change of order of integration. Thus, maximizing weighted average power is equivalent to maximizing power against the single alternative

$$H_F : \text{The density of } G \text{ is } h = \int f_\theta dF(\theta) \tag{20}$$

and the remaining challenge is to deal with the composite nature of the null hypothesis. The key insight underlying the following computations is Lemma 2 of Müller and Watson (2008), which we reproduce here for convenience.

Lemma 1 *Let Λ be a probability distribution on Θ_0 , and let φ_Λ be the best level α test of the null hypothesis H_Λ : The density of G is $\int f_\theta d\Lambda(\theta)$ against H_F . Then for any level α test φ of H_0 against H_F , $\int \varphi_\Lambda h d\nu \geq \int \varphi h d\nu$.*

Proof. Since φ is a level α test of H_0 , $\int \varphi f_\theta d\nu \leq \alpha$ for all $\theta \in \Theta_0$. Therefore, $\int \int \varphi f_\theta d\nu d\Lambda(\theta) = \int \int \varphi f_\theta d\Lambda(\theta) d\nu \leq \alpha$ (where the change in the order of integration is allowed by Fubini's Theorem), so that φ is also a level α test of H_Λ against H_1 . The result follows by the definition of a best test. ■

Lemma 1 formalizes the perfectly intuitive result that replacing the composite null hypothesis H_0 with the single mixture null hypothesis H_Λ can only simplify the testing problem in the sense of allowing for more powerful tests. Its appeal lies in the fact that the best test of H_Λ against H_F is easily constructed: by the Neyman-Pearson Lemma, the best test rejects for large values of the likelihood ratio statistic $h/\int f_\theta d\Lambda(\theta)$. Thus, Lemma 1 provides a set of easily constructed power bounds on the original problem, indexed by the distribution Λ .

Lemma 1 can usefully be thought of as generalizing a standard result concerning tests with a composite null hypothesis; see, for instance, Theorem 3.8.1 of Lehmann and Romano (2005): A distribution Λ^{**} is *least favorable* if the best level α test of $H_{\Lambda^{**}}$ against the single

alternative H_F is also of level α in the testing problem with the composite null hypothesis H_0 against H_F , so that—using the same reasoning as in the proof of Lemma 1—this test is also the best test of H_0 against H_F . In contrast to this standard result, Lemma 1 is formulated without any restriction on the probability distribution Λ . This is useful because in many contexts, it is difficult to identify the least favorable distribution Λ^{**} (and it may not even exist).

The strategy suggested by Lemma 1 is to work instead with a numerically determined *approximately least favorable* distribution Λ^* : Suppose we knew of a Λ^* such that (i) the best 5% level test of H_{Λ^*} against H_F , φ_{Λ^*} , has power 51.5% and (ii) the best 4.x% level test $\tilde{\varphi}_{\Lambda^*}$ of H_{Λ^*} against H_F has power 49.5% and $\tilde{\varphi}_{\Lambda^*}$ is a 5% level test of H_0 . This would prove Proposition 1, and Λ^* would be approximately least favorable in the sense that the resulting power bound is close to the least upper bound (as the least upper bound cannot be below the power of a 4.x% level test of H_{Λ^*} against H_F), and also in the sense that a slight adjustment of the level in the problem H_{Λ^*} against H_F yields a best test of level 5% under H_0 .

This is the basic idea underlying the following subsections. It is important to keep in mind, however, that the Neyman-Pearson structure of the tests is critical only for the construction of the power bound of 51.5% via Lemma 1. The existence of *any* feasible 5% level test φ of H_0 against H_F with 49.5% power proves part (ii) of Proposition 1. For reasons discussed below, the test φ^* determined here is only roughly equivalent to a Neyman-Pearson test relative to an approximately least favorable distribution.

3.2 Approximately Least Favorable Distribution

The guiding principle for the construction of an appropriate approximately least favorable distribution Λ^* is the second property discussed above: for a distribution Λ to be approximately least favorable, it must be true that a slight adjustment of the critical value (or, equivalently, of the level of the best test of H_{Λ} against H_F) yields a 5% level test under H_0 . A violation of this property guides how Λ should be amended, with more mass on those values of θ where the overrejection occurred. By suitably iterating changes of Λ with computations of its rejection profile under H_0 , an approximately least favorable Λ^* can be determined numerically.

Müller and Watson (2008) directly implement this idea by considering distributions Λ that have mass on a finite number of points $\{\theta_1, \dots, \theta_\ell\} \subset \Theta_0$. If for a given Λ , φ_Λ overrejects at $\theta' \in \Theta_0$, then $\theta_{\ell+1} = \theta'$ is added to the list. A new Λ is determined by assigning the appropriate probabilities on $\{\theta_1, \dots, \theta_{\ell+1}\}$ to ensure exact size control at $\{\theta_1, \dots, \theta_{\ell+1}\}$, and the algorithm iterates these steps until an appropriate least favorable distribution is found. The numerical feasibility of this procedure depends on ℓ not to become too large (say, $\ell < 40$ or so).

For the problem studied here, this strategy turns out to not work well: If φ_Λ overrejects for a moderately large value of $|\delta|$ and a (possibly small) interval of values for ρ , then the inclusion of any single point $\theta' = (0, \delta', \rho') \in \Theta_0$ in Λ does not remedy the overrejection problem for all values of ρ in the interval. The reason is that for moderately large values of $|\delta|$, the distribution of G with ρ'' slightly different from ρ' is already too dissimilar for size control at ρ' to imply approximately size control at ρ'' . Roughly speaking, with large breaks, different break dates lead to almost singular probability distributions for G , so that a very large number of points would be required to ensure approximately overall size control.

Instead, we note that for Λ to be approximately least favorable, an equivalent condition is that (the slightly level adjusted) test φ_Λ is of 5% level under any H_Ψ : The density of G is $\int f_\theta d\Psi(\theta)$, for all probability distributions Ψ on Θ_0 . By initially restricting this condition to a finite set $\mathcal{P} = \{\Psi_1, \dots, \Psi_M\}$ with moderately large M , the above algorithm can be successfully implemented to identify Λ_M^* , a mixture of \mathcal{P} , such that $\varphi_{\Lambda_M^*}$ controls size for all H_Ψ , $\Psi \in \mathcal{P}$. Collect the distributions Ψ_j that receive positive mass by Λ_M^* in $\mathcal{P}^* = \{\Psi_1^*, \dots, \Psi_M^*\} \subset \mathcal{P}$. Using Λ_M^* as the new starting point, consider a further set $\mathcal{P}' = \{\Psi_1', \dots, \Psi_{M'}'\}$ and again identify $\Lambda_{M'}^*$, a mixture of \mathcal{P}^* and \mathcal{P}' , so that $\varphi_{\Lambda_{M'}^*}$ controls size for all H_Ψ , $\Psi \in \mathcal{P}^* \cup \mathcal{P}'$. This procedure is then iterated, and by considering sets of probability distributions that more and more closely approximate point masses in Θ_0 (although they still average over short intervals of ρ), a suitable Λ^* can be determined. In our implementation, we choose Ψ 's with (not necessarily mean zero) Gaussian distributions over δ conditional on ρ , with smaller and smaller variances in later steps of the iteration.

We relegate details on this algorithm to the appendix, but discuss here two additional issues that are relevant to the determination of the approximate least favorable distribution Λ^* .

On the one hand, the observation G is a transformation of a Wiener process, so that

computation of the Neyman-Pearson tests and their rejection profile requires the simulation of (pseudo) random continuous time processes. Discrete time Gaussian random walk approximation based on, say, 1000 steps are computationally expensive and of uncertain accuracy. For this reason, we approximate all integrals over ρ by sums with $\rho \in \{1/100, 2/100, \dots, 1\}$. The rejection probability of the (approximate) Neyman-Pearson test thus becomes a function of

$$\left\{G\left(\frac{l}{100}\right)\right\}_{l=1}^{100}. \quad (21)$$

The 100 random variables (21) can be simulated without any approximation error, as they are exactly jointly Gaussian. What is more, the fact that φ^* only depends on G through (21) significantly simplifies the determination of the size of φ^* discussed in the following subsection.

A power bound on tests that are functions of (21) is not, however, in general a valid power bound on the original problem involving the observation G . The correct implementation of the power bound via Lemma 1 requires evaluation of the actual density (18) of G , a complication we address in subsection 3.4. In the end, the comparison of this power bound with the power of the feasible test φ^* reveals whether working with (21) instead of the whole process G might have led to substantial inaccuracies in the resulting approximately least favorable distribution, and Proposition 1 shows that these are small.

On the other hand, we chose to construct the feasible test φ^* by combining two different tests, with the supF statistic based on the n observations (21) determining to which one it switches: For large values of supF, that is strong evidence for a break, φ^* is essentially identical to the usual two-sided t-test on least-squares estimated post break data, \hat{t} in (7), whereas for small values of supF, φ^* is equal to the Neyman-Pearson test of H_{Λ^*} against H_F . The motivation for this switching rule is threefold: First, it formally justifies the "rule of thumb" that for highly significant realizations of a structural break test statistic, least-squares based inference is just fine, with a patch only necessary for moderate break magnitudes δ . Second, it ensures that even for very large breaks ($|\delta| \rightarrow \infty$), the test has attractive properties—this cannot be ensured by construction by any weighted average power criterion with integrable weighting function, as any such weighting concentrates almost all of its mass on a compact interval. Third, the switch to an analytically easily tractable test statistic facilitates the study of the size properties of φ^* for the (unbounded) set of large δ discussed in the next subsection.

The adoption of this switching requires a determination of Λ^* such that the overall test φ^* is of level 5%, rather than the critical value adjusted Neyman-Pearson test $\tilde{\varphi}_{\Lambda^*}$ of H_{Λ^*} against H_F in isolation. While it is true that for large $|\delta|$, the t-test and $\tilde{\varphi}_{\Lambda^*}$ behave quite similarly (after all, for $|\delta|$ large, there is ample information about ρ , so that both $\tilde{\varphi}_{\Lambda^*}$ and the t-test are approximately equal to the best infeasible test with ρ known), we choose a cut-off value for supF that is just large enough to ensure that the t-test controls size when the probability of switching is close to one. Thus, for values of $|\delta|$ where the switch to the t-test only occurs, say, half the time, the t-test has null rejection probability larger than 5%. Given the switching rule, Λ^* must therefore induce a compensating underrejection in this region of δ . In the above algorithm, this is achieved by requiring that the rejection probability of φ_{Λ^*} is substantially lower than 5% under H_{Ψ} for Ψ 's that concentrate on these regions (which in turn leads to a Λ^* with substantial mass on these Ψ 's).

From a decision theoretic perspective, the least favorable prior has a minimax interpretation in the problem of distinguishing between H_0 against H_F . Suppose a false rejection of H_0 induces loss 1, a false acceptance of H_F induces loss $L_F > 0$, and a correct decision has loss 0. Then risk is given by

$$R(\theta, \varphi) = \mathbf{1}[\theta \in \Theta_0]E_{\theta}[\varphi(G)] + L_F \mathbf{1}[\theta \in \Theta_1] \int E_t[1 - \varphi(G)]dF(t). \quad (22)$$

The α level test based on the least favorable prior Λ^{**} (supposing it exists) minimizes $\sup_{\theta \in \Theta} R(\theta, \varphi)$ among all tests φ for $L_F = \alpha/(1 - WAP(\varphi_{\Lambda^{**}}))$, and achieves $\sup_{\theta \in \Theta} R(\theta, \varphi) = \alpha$. The α^* -level test based on the *approximately* least favorable prior Λ^* achieves $\sup_{\theta \in \Theta} R(\theta, \varphi^*) = \alpha^*$ for $L_F = \alpha^*/(1 - \pi^*)$, and $\inf_{\varphi} \sup_{\theta \in \Theta} R(\theta, \varphi) \geq \alpha^*(1 - \bar{\pi})/(1 - \pi^*) \simeq 0.951\alpha^*$ by Proposition 1, so φ^* is an approximately minimax decision rule. In this reasoning, the usual requirement of size control becomes an endogenous solution to the desire to find a maximin decision rule under risk (22).⁵

In assessing the appeal of (inherently pessimistic) minimax rules, it is instructive to consider the reasonableness of the implied least favorable distribution. Figure 1 plots the approximately least favorable distribution Λ^* determined by the above algorithm for $\delta \geq 0$; by construction, the distribution is symmetric in the sign of δ . Overall, apart from some peculiarities close to the endpoint for the break date at $\rho = 0.85$, Λ^* does not seem particularly bizarre.

⁵And Lemma xx is recognized as a special case of Chamberlain xx.

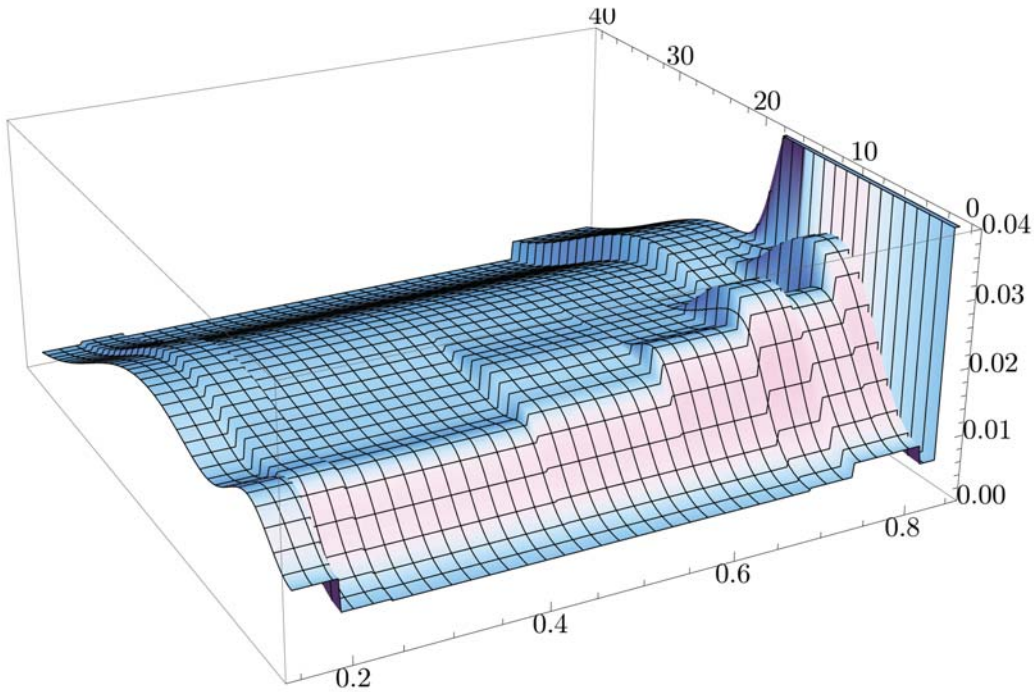


Figure 3: Approximate Least Favorable Distribution Λ^*

From a Bayesian perspective, one might want to decide between H_0 and H_F (or H_1) by computing posterior odds. With a prior of F on Θ_1 , and a prior of Λ^* on Θ_0 , φ_{Λ^*} rejects for large values of the resulting posterior odds, or, equivalently, for large values of the Bayes factor. Apart from the switching to a t-test discussed above, which in this context may be regarded as an approximation to the posterior, an analysis based on φ^* thus also has a straightforward Bayesian interpretation. The prior Λ^* on H_0 and the cut-off value $\simeq 2.56$ are endogenously determined so that Bayes factors above 2.56—that is, Bayes factors that occur for at most 5% of draws under H_0 for *all* values of $\theta \in \Theta_0$ —are as frequent as possible under H_F . In this way, the rule of rejecting for a large Bayes factor also has attractive frequentist properties.

It might be instructive to contrast this to the properties of the test that rejects for a large Bayes factor when the prior on the null hypothesis is not endogenized in this fashion. For instance, in this problem, a natural choice for the prior distribution Λ on Θ_0 is a uniform distribution for ρ on $[0.15, 0.85]$, and $\delta \sim \mathcal{N}(0, 400)$ independent of ρ ; this mirrors the prior F of Condition 1 on the alternative Θ_1 , except that $\sigma_\beta^2 = 0$. With that prior, one must choose a cut-off value of $\simeq 10$ to ensure that only 5% of the time, one observes Bayes factors

larger than the cut-off for *all* values of $\theta \in \Theta_0$. Correspondingly, the probability of rejection under H_F falls to $\simeq 36.7\%$.

3.3 Size Control

With a candidate test φ^* determined as outlined in the last subsection, the question is whether it actually controls size, i.e. whether

$$\sup_{\theta \in \Theta_0} E_{\theta}[\varphi^*(G)] \leq 0.05. \quad (23)$$

By construction of φ^* , this is "approximately" true, but we now discuss how to establish (23) more rigorously, that is part (ii) of Proposition 1.

The problem we face is a standard one: What is the size of a given test under a composite null hypothesis? For some nonstandard hypothesis testing problems, the form of the test greatly simplifies this issue. For instance, the conditional likelihood ratio statistic of Moreira and Andrews, Moreira, and Stock (2006), or the statistic suggested in Jansson and Moreira (2006), have by construction an (asymptotic) null rejection probability that does not depend on the nuisance parameter. For other tests or testing problems, however, essentially nothing analytical is known about the rejection probability. The usual approach then is to resort to a Monte Carlo grid search: Choose a finite set $\Theta_{grid} \subset \Theta_0$, estimate the rejection probability by Monte Carlo for each $\theta \in \Theta_{grid}$, and conclude that the test controls size if the largest of these rejection probabilities is smaller or equal to the nominal level. Examples for this approach include Stock and Watson (1996), Bunzel and Vogelsang (2005), Srianthakumar and King (2006), Andrews, Moreira, and Stock (2007), Andrews and Guggenberger (2007a, 2007b, 2007c), Guggenberger (2008), and Müller and Watson (2008), among others.⁶ For the specific problem here, this approach would amount to computing the Monte Carlo estimate of the null rejection probability

$$\hat{p}(\theta) = N^{-1} \sum_{i=1}^N \varphi^*(G_i) \quad \text{with} \quad G_i(s) = W_i(s) + \delta \min(\rho, s) \quad (24)$$

for all $\theta = (0, \delta, \rho) \in \Theta_{grid}$, where W_i are N (pseudo) random draws of a standard Wiener process⁷ (which may or may not be the same across different θ), and to conclude that φ^* is

⁶Dufour (2006) suggests searching for the maximum Monte Carlo rejection probability directly using simulated annealing techniques, which may also fail to yield the global maximum.

⁷As noted above, suffices to generate $G(t/100)$, which is actually feasible.

of level 5% if $\sup_{\theta \in \Theta_{grid}} \hat{p}(\theta) < 0.05$.

Clearly, though, this approach is not fully satisfying. Without some knowledge about the smoothness of $\hat{p}(\theta)$, even a fine grid search of this form does not provide any upper bound on $\sup_{\theta \in \Theta_0} \hat{p}(\theta)$, simply because the rejection probability $\hat{p}(\theta)$ could be very different between the grid points. What is more, Θ_0 is typically unbounded, so that it is impossible from the start to finely cover Θ_0 . We now describe an approach that, while still based on Monte Carlo estimation, can handle both these difficulties by bounding the function $\hat{p}(\theta)$ on the whole set Θ_0 .

The key idea is the following change of perspective: Rather than to only regard the average $\hat{p}(\theta)$ in (24) as a function of θ , consider instead $\varphi^*(G_i)$ as a function of $\theta = (0, \delta, \rho) \in \Theta_0$, conditional on W_i . The actual data G_i is a function of the (pseudo) randomness W_i and the nuisance parameter θ , so that for each realization of W_i , the test statistic $\varphi^*(G_i)$ may be regarded as a function of θ , $\varphi^*(G_i) = S_i(\theta)$ with $S_i : \Theta_0 \mapsto \{0, 1\}$ (as long as φ^* is non-randomized). The function $S_i(\theta)$ is, of course, perfectly known, and one can analytically study its properties. So suppose initially that it was possible to exactly determine and store the "critical regions" $C_i = \{\theta : S_i(\theta) = 1\}$, $i = 1, \dots, N$, that is the range of values for θ for which the test $\varphi^*(G_i)$ rejects.⁸ For instance, if the nuisance parameter space was one dimensional, C_i would typically be a finite union of intervals, and it would suffice to analytically determine their endpoints (possibly including $\pm\infty$). Clearly, the overall Monte Carlo estimate of the null rejection probability function $\hat{p} : \Theta_0 \mapsto [0, 1]$ is simply the average of these critical regions

$$\hat{p}(\theta) = N^{-1} \sum_{i=1}^N \mathbf{1}[\theta \in C_i]$$

and size control amounts to $\sup_{\theta \in \Theta_0} \hat{p}(\theta) \leq 0.05$.

Now for our problem, the nuisance parameter space is two dimensional, and the form of φ^* does not make it feasible to exactly describe the sets C_i analytically. So consider instead a finite partition $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_K\}$ of Θ_0 , such as the *rectangles* of a grid, and suppose study of S_i allows one to conclude that the sets $\{Q_j : j \in J_i\} \subset \mathcal{Q}$ with $J_i \subset \{1, 2, \dots, K\}$ cover C_i , i.e. $C_i \subset \bigcup_{j \in J_i} Q_j$. It then suffices to keep track of the index sets J_i , $i = 1, \dots, N$

⁸This terminology is meant to be suggestive only; the actual critical region, of course, is the subset of the sample space $D_{[0,1]}$ for which φ^* rejects.

to establish the bound

$$\sup_{\theta \in \Theta_0} \hat{p}(\theta) \leq \bar{p} = \sup_{\theta \in \Theta_0} N^{-1} \sum_{i=1}^N \mathbf{1}[\theta \in \bigcup_{j \in J_i} Q_j] = \max_j N^{-1} \sum_{i=1}^N \mathbf{1}[j \in J_i] \quad (25)$$

and one can conclude that φ^* is a 5% level test if $\bar{p} \leq 0.05$. Note that this bound remains valid (although it becomes more conservative) when the sets J_i are larger than necessary, that is if $j \in J_i$ despite $C_i \cap Q_j = \emptyset$. The left panel of Figure 3 illustrates the cover for a finite partition of the nuisance parameter space $\mathbb{R} \times [0.15, 0.85]$ of (δ, ρ) into rectangles (some of which are necessarily unbounded).

This approach benefits from a finer partition \mathcal{Q} of Θ_0 with K large in two ways. On the one hand, a fine \mathcal{Q} allows for a relatively accurate description of the actual critical region C_i , thus making the bound (25) less conservative. On the other hand, it will typically become easier to decide analytically whether or not $C_i \cap Q_j = \emptyset$, for each $j = 1, \dots, K$. But the computational cost of making K large is also substantial, as the brute force implementation requires a total of NK such evaluations. For our problem, it turns out that such a brute force implementation is not practical on today's PCs, as K needs to be chosen of the order of magnitude of 10^7 for the bound (25) to become sufficiently sharp.

Thus, instead of trying to decide individually whether $C_i \cap Q_j = \emptyset$ for each $Q_j \in \mathcal{Q}$ and $i = 1, \dots, N$, we implement the following divide and conquer algorithm: For each i , initially try to decide for the whole set Θ_0 whether or not $C_i \cap \Theta_0 = \emptyset$. If such a determination can be made, set $J_i = \emptyset$ or $J_i = \{1, 2, \dots, K\}$, respectively. Otherwise, divide Θ_0 into a coarse partition of 4 subsets $\Theta_0 = \{Q_1^c, Q_2^c, Q_3^c, Q_4^c\}$, where each Q_j^c , $j = 1, \dots, 4$ can be covered exactly by a subset of \mathcal{Q} . Try to decide for each Q_j^c whether or not $C_i \cap Q_j^c = \emptyset$. If for any Q_j^c such a determination can be made, include (or not) the appropriate indices in J_i , and do not consider the subspace Q_j^c further. Otherwise, partition Q_j^c again into 4 subsets that can be covered exactly by a subset of \mathcal{Q} , and iterate until either the determination could be made, or the partition is as fine as \mathcal{Q} . The right hand panel of Figure 3 illustrates this approach. The advantage of this algorithm is that it "zooms in" only when necessary. In our implementation, it cuts the number of required evaluations from 10^7 to around 10^4 to 10^6 , depending on the draw of W_i .

We again relegate the details of the algorithm and the arguments employed for deciding whether or not $C_i \cap Q_j = \emptyset$ to the appendix, where we also discuss the (slightly) different treatment of unbounded subsets $Q_j \in \mathcal{Q}$.

The bound \bar{p} in (25), of course, is still subject to Monte Carlo error, and one might wonder about the effect of the supremum over Θ_0 . Let P^* be the distribution of the N Monte Carlo draws of W_i , $i = 1, \dots, N$.

Lemma 2 For any $x \in \mathbb{R}$, $P^*(\sup_{\theta \in \Theta_0} \hat{p}(\theta) > x) \geq \sup_{\theta \in \Theta_0} P^*(\hat{p}(\theta) > x)$.

Proof. Write E^* for the expectation with respect to P^* . Then, applying Jensen's inequality, $P^*(\sup_{\theta \in \Theta_0} \hat{p}(\theta) > x) = E^*[\mathbf{1}[\sup_{\theta \in \Theta_0} \hat{p}(\theta) > x]] = E^*[\sup_{\theta \in \Theta_0} \mathbf{1}[\hat{p}(\theta) > x]] \geq \sup_{\theta \in \Theta_0} E^*[\mathbf{1}[\hat{p}(\theta) > x]] = \sup_{\theta \in \Theta_0} P^*(\hat{p}(\theta) > x)$. ■

Lemma 2 shows that the P^* distribution of $\sup_{\theta \in \Theta_0} \hat{p}(\theta)$ (weakly) stochastically dominates that of $\hat{p}(\theta)$ for any fixed $\theta \in \Theta_0$. Thus, the P^* probability of observing $\bar{p} \leq 0.05$ is at most as large as the probability of observing $\hat{p}(\theta) \leq 0.05$, even if $\theta \in \Theta_0$ is chosen to maximize actual null rejection probability $E_\theta[\varphi^*(G)]$. Both $\sup_{\theta \in \Theta_0} \hat{p}(\theta)$ and \bar{p} are, in general, upward biased estimates of the size of the test φ^* . A finding of $\bar{p} \leq 0.05$ is thus *stronger* evidence that φ^* is of level 5% compared to any other 5% nominal level test of a single null hypothesis whose critical value was determined by N Monte Carlo draws.

3.4 Power Bound

We now turn to numerical issues that arise in the computation of the power bound via Lemma 1. Without the approximation of integrals by sums as in (21), the Neyman-Pearson test rejects for large values of a statistic of the form

$$\text{LR} = \frac{\int f_\theta dF(\theta)}{\int f_\theta d\Lambda^*(\theta)} = \frac{\int_{0.15}^{0.85} d_0(r) \exp[A_0(r)G(r)^2 + B_0(r)G(r)G(1) + C_0(r)G(1)^2] dr}{\sum_{i=1}^L \int_{a_i}^{b_i} d_i(r) \exp[A_i(r)G(r)^2 + B_i(r)G(r)] dr} \quad (26)$$

where d_i, A_i, B_i and C_i are functions $[0.15, 0.85] \mapsto \mathbb{R}$, and $a_i, b_i \in [0.15, 0.85]$, $i = 0, \dots, L$. The right hand side in (26) arises from analytically integrating over the Gaussian distribution of (β, δ) in F of Condition 1 in the numerator, and over the Gaussian distribution of δ in the denominator. As mentioned above, proper application of Lemma 1 requires the determination of the power of a 5% level test of H_{Λ^*} : The density of G is $\int f_\theta d\Lambda^*(\theta)$ against the alternative H_F : The density of G is $\int f_\theta dF(\theta)$, based on (26).

The usual approach would be to run a Monte Carlo experiment to estimate the critical value and power, using step function approximations to G and Riemann sums instead of integrals in (26). With enough steps (and Monte Carlo draws), this approach delivers any

degree of accuracy. However, the test statistic (26) is poorly behaved for this approach: For large breaks (large $|\delta|$), the exponents in (26) take on large values for some ρ (of the order of magnitude of $\frac{1}{2}\rho\delta^2$), which renders the integrand extremely peaked. Intuitively, for large $|\delta|$, there is ample information about ρ , reflected in a very peaked likelihood. This leads to potentially very poor properties of Riemann sum approximations, casting doubt on the validity of a power bound computed in this way.

For this reason, we develop a lower and upper bound for LR that can be simulated exactly, and use Monte Carlo draws of the lower bound for the computation of the critical value, and Monte Carlo draws from the upper bound for the power computation. This ensures that the resulting upper bound on the power of the test statistic LR is only subject to the usual Monte Carlo error.

The basic element of the bounds is the following inequality: Let $g : [a, b] \mapsto \mathbb{R}$ for some $a > b$ with $\sup_{a \leq x \leq b} |g(r)| < M_g$, and let $\bar{g} = \frac{1}{b-a} \int_a^b g(r) dr$. Then Jensen's inequality and $e^x \leq x \cosh[M_g]$ for all $|x| < M_g$ implies

$$\exp[\bar{g}] \leq \frac{1}{b-a} \int_a^b \exp[g(r)] dr \leq \exp[\bar{g}] \cosh[M_g]. \quad (27)$$

Write any of the integrals in (26) as a sum of integrals over a fine partition of $[0.15; 0.85]$ into J intervals $[c_{j-1}, c_j]$, $j = 1, \dots, J$ with $c_{j-1} < c_j$ for all j and $c_0 = 0.15$ and $c_J = 0.85$. In terms of $W(s) = G(s) - \beta s - \delta \min(\rho, s)$, the representative short integral over $r \in [c_{j-1}, c_j]$ in the denominator has the form

$$\int_{c_{j-1}}^{c_j} d(r) \exp [A(r)W(r)^2 + B(r)W(r)] dr$$

where now $d(r)$, $A(r)$ and $B(r)$ are also functions of $\theta = (\beta, \delta, \rho)$. Using (27), we have

$$\begin{aligned} \underline{d} \exp \left[\underline{A} \overline{W_j^2} + \underline{B} \bar{W}_j \right] &\leq \frac{1}{c_j - c_{j-1}} \int_{c_{j-1}}^{c_j} d(r) \exp [A(r)W(r)^2 + B(r)W(r)] dr \\ &\leq \bar{d} \exp [\bar{A} \overline{W_j^2} + \bar{B} \bar{W}_j] \cosh[M_j] \end{aligned} \quad (28)$$

where $\overline{W_j^2} = \frac{1}{c_j - c_{j-1}} \int_{c_{j-1}}^{c_j} W(r)^2 dr$, $\bar{W}_j = \frac{1}{c_j - c_{j-1}} \int_{c_{j-1}}^{c_j} W(r) dr$, $S_{1,j} = \sup_{c_{j-1} \leq r \leq c_j} |W(r) - \bar{W}_j|$, $S_{2,j} = \sup_{c_{j-1} \leq r \leq c_j} |W(r)^2 - \overline{W_j^2}|$, $M_j = \bar{A} S_{2,j} + \bar{B} S_{1,j}$, and lower and upper bars on d , A and B indicate maximum and minimum values of $d(r)$, $A(r)$ and $B(r)$ on $c_{j-1} \leq r \leq c_j$, respectively, provided $W(r) > 0$ for $c_{j-1} \leq r \leq c_j$. Similar inequalities hold when $W(r) < 0$, when $W(r)$ changes sign on $c_{j-1} \leq r \leq c_j$, and also for the numerator of (26). A lower bound on LR is then given by replacing integrals in the denominator of (26) with sums

over the upper bound in (28) and to replace the integrals in the numerator with sums over corresponding lower bounds, and vice versa for an upper bound.

A Monte Carlo draw from these bounds for LR requires generation of the $4J$ random variables $\{\overline{W}_j^2, \bar{W}_j, S_{1,j}, S_{2,j}\}_{j=1}^J$. Although $\{\bar{W}_j\}_{j=1}^J$ is jointly Gaussian, this is non-trivial, as even the marginal distributions of $\overline{W}_j^2, S_{1,j}$ and $S_{2,j}$ are nonstandard. Fortunately, as detailed in the appendix, it is possible to rely on existing analytical results on the distribution of related statistic of a Wiener process to generate draws $\{\overline{W}_j^{2^l}, \overline{W}_j^{2^u}, \bar{W}_j, S_{1,j}^u, S_{2,j}^u\}_{j=1}^J$ where with very high probability, $\overline{W}_j^{2^l} \leq \overline{W}_j^2 \leq \overline{W}_j^{2^u}$, $S_{1,j} \leq S_{1,j}^u$ and $S_{2,j} \leq S_{2,j}^u$ for all $j = 1, \dots, J$. By amending the above bounds accordingly, this is sufficient for the computation of a Monte Carlo estimate of an upper bound on the power of LR, entirely avoiding any qualifications that arise through the impossibility of generating exact draws of LR under $\int f_\theta d\Lambda^*(\theta)$ and $\int f_\theta dF(\theta)$.

4 Test Statistic and Properties

This section defines the test we suggest for applied work, $\hat{\varphi}^*$, and discusses its properties. In particular, we compare its asymptotic power to the infeasible two-sided t-test with the known break date, and the (approximately) size corrected versions of the least-squares based tests considered in subsection 2.4.

4.1 Definition of Test Statistic

We define the statistic for a general method of moments framework with the $k+1$ dimensional parameter $\Gamma = (\gamma, \tilde{\gamma})'$, with $\gamma \in \mathbb{R}$ and $\tilde{\gamma} \in \mathbb{R}^k$. The data in a sample of size T is given by $X_T = (x_1, \dots, x_T)$,⁹ and the population moment condition is $E[g(X_t, \Gamma_0)] = 0$ when the true parameter at date t is given by $\Gamma = \Gamma_0$ for some known, \mathbb{R}^m valued function $g(\cdot, \cdot)$. Write $g_t(\Gamma) = g(X_t, \Gamma)$ for notational simplicity. The parameter Γ changes its value from $\Gamma_{pre} = (\gamma_{pre}, \tilde{\gamma}'_{pre})'$ to $\Gamma_{post} = (\gamma_{post}, \tilde{\gamma}'_{post})'$ at some unknown date $\tau \in \mathbb{N}$ in the middle 70% of the sample, $\tau \in [0.15T, 0.85T]$. The hypotheses of interest concern the post break value of the first element γ of Γ ,

$$H_0 : \gamma_{post} = \gamma_{post,0} \quad \text{against} \quad H_1 : \gamma_{post} \neq \gamma_{post,0}. \quad (29)$$

⁹We allow for double arrays in the data generating process, but omit an explicit dependence on T to ease notation.

Denote by $\hat{\Gamma}_{pre}(t)$ and $\hat{\Gamma}_{post}(t)$ standard GMM estimators of Γ using data $\{X_s\}_{s=1}^t$ and $\{X_s\}_{s=t+1}^T$, respectively, and denote by $\hat{\Omega}_{pre}(t)$ and $\hat{\Omega}_{post}(t)$ the estimators of the covariance matrix of $\hat{\Gamma}_{pre}(t)$ and $\hat{\Gamma}_{post}(t)$, respectively. These covariance matrix estimators, as well as the weighting matrix for efficient GMM estimation in the overidentified case, are based on data $\{X_s\}_{s=1}^t$ and $\{X_s\}_{s=t+1}^T$ for the pre and post break estimators, respectively, and are consistent for a stable model. When $\{g_t(\Gamma_0)\}$ is serially uncorrelated, this is typically achieved by the usual White form of the covariance matrix estimator, whereas time series correlation in $\{g_t(\Gamma_0)\}$ requires a correction as in Newey and West (1987) or Andrews (1991), for instance.

Our test statistic of (29) only requires evaluation of $\{\hat{\Gamma}_{pre}(t), \hat{\Gamma}_{post}(t), \hat{\Omega}_{pre}(t), \hat{\Omega}_{post}(t)\}$ at the 71 numbers $t = \lfloor lT/100 \rfloor$ for $l = 15, 16, \dots, 85$, where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. Let $\hat{\gamma}_{pre}(l)$ and $\hat{\gamma}_{post}(l)$ be the first element of $\hat{\Gamma}_{pre}(\lfloor lT/100 \rfloor)$ and $\hat{\Gamma}_{post}(\lfloor lT/100 \rfloor)$, and denote by $\hat{\omega}_{pre}^2(l)$ and $\hat{\omega}_{post}^2(l)$ the 1-1 element of $\hat{\Omega}_{pre}(\lfloor lT/100 \rfloor)$ and $\hat{\Omega}_{post}(\lfloor lT/100 \rfloor)$, respectively. Define

$$\begin{aligned}
\widehat{\text{supF}} &= \max_{16 \leq l \leq 85} \frac{(\hat{\gamma}_{post}(l) - \hat{\gamma}_{pre}(l-1))^2}{\hat{\omega}_{post}^2(l) + \hat{\omega}_{pre}^2(l-1)} \\
\hat{\Delta}_{pre}(l) &= l\hat{\gamma}_{pre}(l) - (l-1)\hat{\gamma}_{pre}(l-1), \quad \hat{\Delta}_{post}(l) = (101-l)\hat{\gamma}_{post}(l-1) - (100-l)\hat{\gamma}_{post}(l) \\
\hat{l} &= \arg \min_{16 \leq l \leq 85} \left(\sum_{j=16}^{l-1} \hat{\Delta}_{pre}(j)^2 - (l-1)\hat{\gamma}_{pre}(l-1)^2 + \sum_{j=l+1}^{85} \hat{\Delta}_{post}(j)^2 - (100-l)\hat{\gamma}_{post}(l)^2 \right) \\
\hat{t}_{post} &= \frac{\hat{\gamma}_{post}(\min(\hat{l}+1, 85)) - \gamma_{post,0}}{\sqrt{\hat{\omega}_{post}^2(\min(\hat{l}+1, 85))}} \\
\hat{\omega}^2 &= \frac{(\hat{l}-1)^2}{9900} \hat{\omega}_{pre}^2(\hat{l}-1) + \frac{(100-\hat{l})^2}{9900} \hat{\omega}_{post}^2(\hat{l}) \\
\widehat{\text{LR}} &= \frac{\sum_{l=15}^{85} \frac{\exp[\frac{1}{2} \frac{\sigma_{pre}^2(\hat{\gamma}_{pre}(l) - \gamma_{post,0})^2 l^2}{100^2 \hat{\omega}^2 v(l, \sigma_{pre}^2)}] + \frac{1}{2} \frac{\sigma_{\beta}^2(\hat{\gamma}_{post}(l) - \gamma_{post,0})^2 (100-l)^2}{100^2 \hat{\omega}^2 v(100-l, \sigma_{\beta}^2)}]}{71 \sqrt{v(l, \sigma_{pre}^2) v(100-l, \sigma_{\beta}^2)}}}{\sum_{j=1}^{18} \sum_{l=a_j}^{b_j} \frac{p_j \exp[-\frac{1}{2} \frac{\mu_{\delta,j}^2 l}{100 v(l, \sigma_{\delta,j}^2)} + \frac{1}{2} \frac{\sigma_{\delta,j}^2(\hat{\gamma}_{pre}(l) - \gamma_{post,0})^2 l^2}{100^2 \hat{\omega}^2 v(l, \sigma_{\delta,j}^2)}] \cosh[\frac{(\hat{\gamma}_{pre}(l) - \gamma_{post,0}) \mu_{j-1}}{100 v(l, \sigma_{\delta,j}^2) \hat{\omega}}]}{(b_j - a_j + 1) \sqrt{v(l, \sigma_{\delta,j}^2)}}}
\end{aligned} \tag{30}$$

where $v(l, \sigma^2) = 1 + \sigma^2 l/100$, $\sigma_{pre}^2 = 378$, $\sigma_{\beta}^2 = 22$, and a_j, b_j, p_j, μ_j and $\sigma_{\delta,j}^2$ are defined in Table 2. The test $\hat{\varphi}^*$ of (29) rejects H_0 if and only if either (i) $\widehat{\text{supF}} > 90$ and $|\hat{t}_{post}| > 2.01$, or (ii) $\widehat{\text{supF}} \leq 90$ and $\widehat{\text{LR}} > 2.41$.¹⁰

¹⁰One obtains the test φ^* of Proposition 1 and defined in the appendix by setting $\hat{\gamma}_{pre}(l) = 100G(l/100)/l$, $\hat{\gamma}_{post}(l) = (G(1) - G(l/100))/(1 - l/100)$, $\hat{\omega}_{pre}^2(l) = 100/l$ and $\hat{\omega}_{post}^2(l) = 100/(100 - l)$, respectively.

Table 2: Weights, Intervals, Means and Variances of the Approximate Least Favorable Distribution

p_j	a_j	b_j	σ_j^2	μ_j
0.006	15	24	200	28
0.038	75	85	200	28
0.588	15	85	100	20
0.067	85	85	4	3
0.057	20	74	300	16
0.123	85	85	10	5
0.008	15	19	10	5
0.032	20	74	10	9
0.020	75	82	10	7
0.002	83	84	10	13
0.001	85	85	3	15.5
0.026	20	74	3	6
0.001	75	82	3	13
0.005	60	69	10	12
0.009	45	59	10	11
0.004	80	82	10	11
0.009	70	74	10	9
0.004	60	69	3	8

The intuition for these computations is roughly as follows. The statistic $\widehat{\text{supF}}$ is the largest F-statistic of the null hypothesis that the value of γ in the first $(l - 1)\%$ is equal to the value of γ in the last $(100 - l)\%$, maximized over $16 \leq l \leq 85$. By leaving out the middle 1% of observations, it is ensured that for any true break fraction within the middle 70% of the sample, one of these F-statistics (often, the largest) only involves estimators $(\hat{\gamma}_{pre}(l), \hat{\omega}_{pre}^2(l))$ and $(\hat{\gamma}_{post}(l), \hat{\omega}_{post}^2(l))$ from stable periods. When $\widehat{\text{supF}} > 90$, that is, when there is strong evidence for the occurrence of a break, the test $\hat{\varphi}^*$ rejects if the usual t-statistic \hat{t}_{post} is larger than 2.01 (rather than the usual 1.96) in absolute value, where \hat{t}_{post} uses \hat{l} to determine the appropriate post-break data.

In (approximately) linear and stationary models, $\hat{\gamma}_{pre}(l)$ is centred around the average parameter value of the first $l\%$ of the sample, $[lT/100]^{-1} \sum_{t=1}^{\lfloor lT/100 \rfloor} (\gamma_{pre} + (\gamma_{post} - \gamma_{pre})\mathbf{1}[t > \tau])$, and similarly, $\hat{\gamma}_{post}(l)$ is centred about the average parameter value of the last $(100 - l)\%$ of the sample. The statistics $\hat{\Delta}_{pre}(l)$ and $\hat{\Delta}_{post}(l)$ thus approximately estimate the value of the parameter in the l th percent of the sample. If the true break τ is in the l_0 th percent of the sample, then $\hat{\Delta}_{pre}(l)$ should be approximately equal to $\hat{\gamma}_{pre}(l_0)$ for all $l < l_0$, and similarly for $\hat{\Delta}_{post}(l)$. The break fraction estimator \hat{l} is determined by minimizing a corresponding least squares criterion, again dropping the middle one percent. By adding one to \hat{l} , it is ensured that with high probability, $\hat{l} + 1$ is at least as large as the true break fraction in percentage points, so that \hat{t}_{post} is based on estimates from a stable post-break period. The advantage of the least squares criterion (30) based on $(\hat{\Delta}_{pre}(l), \hat{\Delta}_{post}(l))$ over, say, an analogue to the F-statistics that underlie $\widehat{\text{supF}}$, is that \hat{l} in (30) has appealing properties under relatively weak conditions also when the break magnitude is not small—cf. Proposition 4 below.

In absence of strong evidence for a break, that is $\widehat{\text{supF}} \leq 90$, the test $\hat{\varphi}^*$ switches to deciding the null hypothesis based on a likelihood ratio statistic. The numerator of this statistic is the result of the weighting over alternative values for γ_{post} , break dates and break magnitudes of Condition 1. The denominator is the result of a weighting over break dates and break magnitudes that make detection of this alternative as difficult as possible, the numerically determined approximate least favorable distribution discussed in Section 3, ensuring size control of $\hat{\varphi}^*$ even when information about the true break fraction is scarce. In this likelihood ratio statistic, the scale of $\hat{\gamma}_{pre}(l)$ and $\hat{\gamma}_{post}(l)$ is normalized by $\hat{\omega}$. If the break data τ satisfies $(\hat{l} - 1)/100 \leq \tau/T \leq \hat{l}/100$, then the estimator $\hat{\omega}$ is based on variance estimators of stable models, which improves the small sample properties of $\widehat{\text{LR}}$.

4.2 Asymptotic Properties

Proposition 4 (i) Suppose $T^{1/2}(\gamma_{pre} - \gamma_{post}) = \omega\delta \in \mathbb{R}$, $\tau = \lfloor \rho T \rfloor$ with $\rho \in [0.15, 0.85]$ and

$$\begin{aligned} & \{T^{1/2}(\hat{\gamma}_{pre}(l) - \gamma_{post}), T^{1/2}(\hat{\gamma}_{post}(l) - \gamma_{post}), T\hat{\omega}_{pre}^2(l), T\hat{\omega}_{post}^2(l)\}_{l=15}^{85} \\ & \Rightarrow \left\{ \omega \frac{G_0(l/100)}{l/100}, \omega \frac{G_0(1) - G_0(l/100)}{1 - l/100}, \frac{\omega^2}{l/100}, \frac{\omega^2}{1 - l/100} \right\}_{l=15}^{85} \end{aligned} \quad (31)$$

where $G_0(s) = W(s) + \delta \min(\rho, s)$, and $\omega > 0$. (i.a) If $T^{1/2}(\gamma_{post} - \gamma_{post,0}) = \omega\beta \in \mathbb{R}$, then $\hat{\varphi}^*(X_T) \Rightarrow \varphi^*(G)$. (i.b) If $T^{1/2}(\gamma_{post} - \gamma_{post,0}) \rightarrow \pm\infty$, then $\hat{\varphi}^*(Y_T) \xrightarrow{p} 1$.

(ii) Suppose $T^{1/2}(\gamma_{pre} - \gamma_{post}) \rightarrow \pm\infty$, $\tau = \lfloor \rho T \rfloor$ with $\rho \in [0.15, 0.85]$, and, for some $\omega_{pre}, \omega_{post} > 0$,

$$\begin{aligned} & \{T^{1/2}(\hat{\gamma}_{pre}(l) - \gamma_{pre}), T\hat{\omega}_{pre}^2(l)\}_{l=15}^{\lfloor 100\rho \rfloor} \Rightarrow \left\{ \omega_{pre} \frac{G_0(l/100)}{l/100}, \frac{\omega_{pre}^2}{l/100} \right\}_{l=15}^{\lfloor 100\rho \rfloor} \\ & \{T^{1/2}(\hat{\gamma}_{post}(l) - \gamma_{post}), T\hat{\omega}_{post}^2(l)\}_{l=\lceil 100\rho \rceil}^{85} \Rightarrow \left\{ \omega_{post} \frac{G_0(1) - G_0(l/100)}{1 - l/100}, \frac{\omega_{post}^2}{1 - l/100} \right\}_{l=\lceil 100\rho \rceil}^{85} \\ & P(T^{1/2}(\hat{\gamma}_{pre}(\lfloor 100\rho \rfloor + 1) - \gamma_{pre}) > M) \rightarrow 1, P(T^{1/2}(\hat{\gamma}_{post}(\lceil 100\rho \rceil - 1) - \gamma_{post}) > M) \rightarrow 1 \end{aligned} \quad (32)$$

for any $M \in \mathbb{R}$, where $\lceil \cdot \rceil$ denotes the smallest larger integer function. (ii.a) If $\gamma_{post} = \gamma_{post,0}$, then $\limsup_{T \rightarrow \infty} E[\hat{\varphi}^*(X_T)] \leq 5\%$. (ii.b) If $T^{1/2}(\gamma_{post} - \gamma_{post,0}) = \omega_{post}\beta \in \mathbb{R}$ and $\rho \notin R_{grid} = \{0.15, 0.16, \dots, 0.84\}$, then $\lim_{T \rightarrow \infty} \hat{\varphi}^*(X_T) = \mathbf{1}[\lceil Z + \beta\sqrt{0.98 - \lfloor 100\rho \rfloor / 100} \rceil > 2.01]$. (iii.c) If $T^{1/2}(\gamma_{post} - \gamma_{post,0}) \rightarrow \pm\infty$, then $\hat{\varphi}^*(X_T) \xrightarrow{p} 1$.

(iii) Suppose $\gamma_{post} = \gamma_{post,0}$; (31) holds for all sequences $(\rho_T, \delta_T) = (\tau/T, T^{1/2}(\gamma_{pre} - \gamma_{post})) \rightarrow (\rho, \delta) \in [0.15, 0.85] \times \mathbb{R}$; (32) holds for all sequences $(\rho_T, \delta_T) \rightarrow (\rho, \delta) \in ([0.15, 0.85] \setminus R_{grid}) \times \{+\infty, -\infty\}$; and for all for all sequences $(\rho_T, \delta_T) \rightarrow (\rho, \delta) \in R_{grid} \times \{+\infty, -\infty\}$, there exists a subsequence $(\rho_{T'}, \delta_{T'}) \rightarrow (\rho, \delta)$ such that along this subsequence, (32) holds except for

$$\begin{aligned} & (T^{1/2}(\hat{\gamma}_{pre}(l_0) - \gamma_{pre}), T\hat{\omega}_{pre}^2(l_0)) \Rightarrow (\omega_{pre} \frac{G_0(l_0/100)}{l_0/100} + \nu_{pre}(l_0), \frac{\omega_{pre}^2}{l/100}) \\ & (T^{1/2}(\hat{\gamma}_{post}(l_0) - \gamma_{pre}), T\hat{\omega}_{pre}^2(l_0)) \Rightarrow (\omega_{post} \frac{G_0(1) - G_0(l_0/100)}{1 - l_0/100} + \nu_{post}(l_0), \frac{\omega_{post}^2}{1 - l_0/100}) \end{aligned} \quad (33)$$

where $l_0 = 100\rho$ and $\nu_{pre}(l_0), \nu_{post}(l_0) \in \mathbb{R} \cup \{-\infty, \infty\}$ are nonrandom constants, of which at most one is nonzero. Then $\limsup_{T \rightarrow \infty} \sup_{\theta \in \Theta_0} E_\theta[\hat{\varphi}^*(X_T)] \leq \alpha^* \simeq 5\%$.

Part (i) of Proposition 4 considers the case where the pre and post break value of γ only differ of the order $T^{-1/2}$. In that neighborhood, the GMM estimators do not contain enough information to pin down the true break fraction exactly, even asymptotically. This asymptotic embedding represents the small sample problem with substantial uncertainty

about the true break date. The proposition establishes that for local alternatives, where correspondingly, the true value γ_{post} differs by the order $T^{-1/2}$ from the hypothesized value $\gamma_{post,0}$, the asymptotic properties of $\hat{\varphi}^*$ are just like those of φ^* in the limiting problem discussed in Section 2.1. So in particular, by Proposition 1, the test $\hat{\varphi}^*$ has asymptotic level $\alpha^* \simeq 5\%$, and it has asymptotic weighted average power under Condition 1 of $\pi^* \simeq 49\%$. Furthermore, $\hat{\varphi}^*$ is consistent in this scenario against any alternative where γ_{post} is outside the $T^{-1/2}$ neighborhood of the hypothesized value.

These results rely on the high level condition (31). The sequence of statistics $\{\hat{\gamma}_{pre}(l), \hat{\gamma}_{post}(l), \hat{\omega}_{pre}(l), \hat{\omega}_{post}(l)\}$ are a special case of the partial-sum GMM estimators analyzed by Andrews (1993). In particular, his primitive Assumption 1 with an appropriate modification to account for the local break in the parameter value imply (31)—see the appendix for details. Alternatively, the approach of Li and Müller (2007) could be amended to yield (31) under a different set of assumption. Also, for the special case of maximum likelihood estimation, Condition 2 can be shown to imply (31) for both the average Hessian and outer product of scores covariance estimators $\hat{\Omega}_{pre}$ and $\hat{\Omega}_{post}$. The conclusions of part (i) thus hold for a wide range of models.

Part (ii) of Proposition 4 establishes the asymptotic properties of $\hat{\varphi}^*$ when the break is (very) large relative to the sample information about γ . The test again controls size and is consistent against any non-local alternative. Under local alternatives $T^{1/2}(\gamma_{post} - \gamma_{post,0}) = \beta \neq 0$, it asymptotically corresponds to a two-sided t-test about unit variance normal with non-centrality parameter of $\sqrt{0.98 - [100\rho]/100}$ and critical value 2.01, at least as long $\rho \neq R_{grid}$.¹¹ In comparison, the two-sided 5% level t-test based on post break data (assuming the break date was known) converges to $\varphi_\rho(G) = \mathbf{1}[|\frac{G(1)-G(\rho)}{\sqrt{1-\rho}}| > 1.96]$, and thus has a non-centrality parameter of $\beta\sqrt{1-\rho}$. With $\beta \sim \mathcal{N}(0, 22)$ as in Condition 1, the average asymptotic power loss of $\hat{\varphi}^*$ relative to φ_ρ over $\rho \in [0.15, 0.85]$ is approximately 1.7 percentage points, with a largest difference of 3.6 percentage points occurring at $\rho = 0.83$.

The results in part (ii) of Proposition 4 require two sets of assumptions. On the one hand, the statistics $\{\hat{\gamma}_{pre}(l), \hat{\omega}_{pre}^2(l)\}$ and $\{\hat{\gamma}_{post}(l), \hat{\omega}_{post}^2(l)\}$ have to behave in the usual way *over the stable pre and post break periods*, respectively, and be asymptotically independent. Note

¹¹This qualification is necessary because if $\rho \in R_{grid}$, \hat{l} potentially takes on the two values $[100\rho] + 1$ and $[100\rho] + 2$ with positive probability, even asymptotically, despite the diverging break magnitude $T^{1/2}(\gamma_{pre} - \gamma_{post})$.

that the limiting variance is may change at the parameter break date; this accommodates, say, changes in the variance of the AR(1) coefficient estimator that are induced by a break in the AR(1) coefficient. One can again invoke the primitive conditions of Andrews (1993) to justify these convergences. On the other hand, the estimators $\hat{\gamma}_{pre}(l)$ and $\hat{\gamma}_{post}(l)$ must diverge from the $T^{-1/2}$ neighborhood of the pre and post parameter values γ_{pre} and γ_{post} for two values of l that involve a positive fraction of post and pre break data, respectively.¹² For a non-local but shrinking break magnitude $|\gamma_{pre} - \gamma_{post}| \rightarrow 0$ and $T^{1/2}(\gamma_{pre} - \gamma_{post}) \rightarrow \pm\infty$, $\hat{\gamma}_{pre}(l)$ typically estimates the average parameter value in the first $l\%$ of the sample, that is

$$\frac{\hat{\gamma}_{pre}(l) - \gamma_{pre}}{\gamma_{post} - \gamma_{pre}} \xrightarrow{p} \mathbf{1}[l > \lfloor 100\rho \rfloor] \frac{l - \lfloor 100\rho \rfloor}{l} \neq 0, \quad (34)$$

and similarly for $\hat{\gamma}_{post}(l)$, which is clearly sufficient for the purposes of Proposition 4 (ii). The convergence (34) can be shown to hold, for instance, under the high-level Condition 1 of Li and Müller (2007) by proceeding as in their Theorem 1. For a fixed break magnitude $\gamma_{post} - \gamma_{pre}$, the distortionary effects of the break becomes even stronger, of course, and barring pathological cancellations, one would expect the condition in Proposition (ii) to be satisfied. We refrain from a detailed discussion of sufficient primitive conditions for the sake of brevity.

Finally, part (iii) of Proposition 4 shows that the test $\hat{\varphi}^*$ controls size uniformly in large samples—for T large enough, no choice of $\theta \in \Theta_0$ leads to a rejection probability substantially above 5%. For local breaks $\gamma_{pre} - \gamma_{post} = O(T^{-1/2})$, the assumption is only that the convergences in part (i) hold for all converging sequences $(\rho_T, \delta_T) \rightarrow (\rho, \delta)$. We show in the appendix that the primitive assumptions of Andrews (1993) are again sufficient. For non-local breaks, care must be taken for the case $\rho_T \rightarrow \rho \in R_{grid}$. If $\rho_T \rightarrow \rho \in R_{grid}$ with $\rho_T \leq \rho$, then $\hat{\gamma}_{pre}(100\rho)$ is an estimator from an unstable model, with a shrinking fraction of the data stemming from the post break model. Depending on the rate at which $\delta_T \rightarrow \infty$, this contamination shifts the center of the asymptotic distribution of $\hat{\gamma}_{pre}(100\rho)$ by $\nu_{pre}(100\rho)$ —typically, given the arguments surrounding (34), one would expect $\nu_{pre}(100\rho) \in \mathbb{R}$ if $\delta_T(\rho_T - \rho) \rightarrow \xi \in \mathbb{R}$.¹³

¹²Note, however, that nothing is assumed about the behavior of the variance estimators $\{\hat{\omega}_{pre}(l), \hat{\omega}_{post}(l)\}$ that involve subsets of unstable periods.

¹³Inspection of the proof of Proposition 4 shows that these assumptions are only needed to ensure that $(\{\hat{\gamma}_{pre}(l)\}_{l=1}^{l_0}, \hat{\Delta}_{post}(l_0+1))$ are asymptotically independent of $(\{\hat{\gamma}_{post}(l), \hat{\omega}_{post}^2(l)\}_{l=l_0+1}^{l_0+2})$, as this independence

4.3 Monte Carlo Analysis

to be added

5 Conclusion

to be added

6 Appendix

Definition of φ^* :

$\varphi^*(G) = \mathbf{1}[\text{supF}_0 > 90] \mathbf{1}[|\hat{t}_0| > 2.00] + \mathbf{1}[\text{supF}_0 \leq 90] \mathbf{1}[\text{LR}_0 > 2.56]$, where

$$\begin{aligned} \text{supF}_0 &= \max_{16 \leq l \leq 85} \frac{100 \left((100-l)G\left(\frac{l-1}{100}\right) - (l-1)(G(1) - G\left(\frac{l}{100}\right)) \right)^2}{99(100-l)(l-1)} \\ S_0(l) &= \frac{100(100G\left(\frac{l}{100}\right) - lG(1))^2}{l(100-l)} + 100^2 \frac{(lG\left(\frac{l-1}{100}\right) - (l-1)G\left(\frac{l}{100}\right))^2}{l(l-1)} \\ \hat{l}_0 &= \arg \max_{16 \leq l \leq 85} S_0(l) \\ \hat{t}_0 &= \frac{G(1) - G(\min(\hat{l}_0 + 1, 85)/100)}{\sqrt{1 - \min(\hat{l}_0 + 1, 85)/100}} \\ \text{LR}_0 &= \frac{\sum_{l=15}^{85} v(l, \sigma_{pre}^2)^{-1/2} v(100-l, \sigma_\beta^2)^{-1/2} \exp\left[\frac{1}{2} \frac{\sigma_{pre}^2 G(l/100)^2}{v(l, \sigma_{pre}^2)} + \frac{1}{2} \frac{\sigma_\beta^2 (G(1) - G(l/100))^2}{v(100-l, \sigma_\beta^2)}\right]}{\sum_{j=1}^{18} \sum_{l=a_j}^{b_j} \frac{p_j}{b_j - a_j + 1} v(l, \sigma_{\delta,j}^2)^{-1/2} \exp\left[-\frac{1}{2} \frac{\mu_{\delta,j}^2 l}{100v(l, \sigma_{\delta,j}^2)} + \frac{1}{2} \frac{\sigma_{\delta,j}^2 G(l/100)^2}{v(l, \sigma_{\delta,j}^2)}\right] \cosh\left[\frac{\mu_j G(l/100)}{v(l, \sigma_{\delta,j}^2)}\right]} \end{aligned}$$

and $v(l, \sigma^2) = 1 + \sigma^2 l/100$, $\sigma_{pre}^2 = 378$, $\sigma_\beta^2 = 22$, and $p_j, \sigma_{\delta,j}^2, \mu_j$ are defined in Table 2.

Proof of Proposition 2:

By Girsanov's Theorem, the log of the Radon-Nikodym (RN) derivative of the distribution of $(G, \tilde{G})'$ with respect to the distribution of $(W, (A_{21}W + A_{22}\tilde{W}_k)')$ is the r.h.s. of (15).

(i) Fix arbitrary values $\rho_0 \in [0.15, 0.85]$, $\delta_0 \in \mathbb{R}$, $\tilde{\beta}_0, \tilde{\delta}_0 \in \mathbb{R}^k$, and consider the construction of the most powerful unbiased test against the specific alternative $H_1^s(b) : \beta = b, \delta = \delta_0 - b, \tilde{\beta} = \tilde{\beta}_0 + bA_{21}, \tilde{\delta} = \tilde{\delta}_0 - bA_{21}$ $\rho = \rho_0$, which is indexed by $b \in \mathbb{R}$. Any 5% level test for the unrestricted null hypothesis is also of level 5% under the specific null hypothesis $H_0 : \beta = 0, \delta = \delta_0, \tilde{\beta} =$

is enough to establish that the mixture of the two \hat{t}_{post} statistics with $\hat{l} = l_0 + 1$ and $\hat{l} = l_0 + 2$ controls size. The condition could be dispensed with entirely by increasing the critical value of \hat{t}_{post} to 2.07, since $P(|Z_1| < 2.07 \text{ and } |Z_2| < 2.07) > 0.95$ for all bivariate normal Z_1, Z_2 with $Z_1 \sim Z_2 \sim \mathcal{N}(0, 1)$ and correlation of at least $\sqrt{15/16}$, so that no mixture of the two \hat{t}_{post} -statistics can induce overrejections.

$\tilde{\beta}_0, \tilde{\delta} = \tilde{\delta}_0, \rho = \rho_0$. The log of the RN derivative of the distribution of $(G, \tilde{G})'$ under $H_1^s(b)$ with respect to the distribution under H_0^s then becomes $b(G(1) - G(\rho_0)) - \frac{1}{2}(1 - \rho_0)b^2$. The experiment of testing H_0^s against H_1^s based on the observation $(G, \tilde{G})'$ is therefore *equivalent* (in the sense of equivalence of statistics experiments) to inference about b in the Gaussian shift experiment of observing $X \sim \mathcal{N}(b, 1 - \rho_0)$. In particular, for any test of H_0^s against $H_1^s(b)$, there exists a test $\varphi_X : \mathbb{R} \mapsto [0, 1]$ that has the same distribution for all $b \in \mathbb{R}$ (this follows, for instance, from Theorem 3.1 of van der Vaart (1998)). It is well known that the best 5% level unbiased test in the Gaussian shift experiment is $\varphi_X^* = \mathbf{1}[|X/\sqrt{1 - \rho_0}| > \Phi(0.975)]$, so that the best unbiased 5% level test of H_0^s against $H_1^s(b)$ is $\varphi_{\rho_0}(G)$. Since $\varphi_{\rho_0}(G)$ does not depend on $\delta_0, \tilde{\beta}_0, \tilde{\delta}_0$, and $\rho_0, \delta_0, \tilde{\beta}_0, \tilde{\delta}_0$ were arbitrary, this proves the claim.

(ii) The bound $\bar{\pi}$ in Proposition 1 is constructed via Lemma 1, i.e., $\bar{\pi}$ is the power of the Neyman-Pearson test φ_{Λ^*} of H_{Λ^*} against H_F , treating G as the observation, with Λ^* the approximately least favorable distribution Λ^* determined as described in Section 3.2.

Now let $\tilde{\Lambda}^*$ be the probability distribution for $(\theta, \tilde{\beta}, \tilde{\delta})$ with the same marginal distribution of θ as Λ^* , and $(\tilde{\beta}, \tilde{\delta}) = (\beta A_{21}, \delta A_{21})$ conditional on θ . The Neyman-Pearson test of $H_{\tilde{\Lambda}^*}$ against $H_{\tilde{F}}$, based on the observation $(G, \tilde{G})'$, is then seen to be identical to φ_{Λ^*} . The result thus follows from Lemma 1.

(iii) The maximal invariant is $(\Psi, G(1))$, where $\Psi(s) = (G(s) - G(1), \tilde{G}(s)' - s\tilde{G}(1)')'$. Note that the distribution of $(\Psi, G(1))$ does not depend on $\tilde{\beta}$, and that $G(1)$ is independent of Ψ . The log of the RN derivative of $(\Psi, G(1))$ with respect to the distribution of (B_{k+1}, Z) , where B_{k+1} is a $k+1$ Brownian Bridge with covariance matrix Σ and $Z \sim \mathcal{N}(0, 1)$ independent of B_{k+1} , is equal to

$$\delta e_1' \Sigma^{-1} (\Psi(\rho) - \rho \Psi(1)) + (\beta + \delta \rho) G(1) - \frac{1}{2} \rho (1 - \rho) e_1' \Sigma^{-1} e_1 \delta^2 - \frac{1}{2} (\beta + \delta \rho)^2$$

where e_1 is the first column of I_{k+1} . By the same arguments as employed in part (i), this experiment is seen to be equivalent to the observation of $(\bar{\Psi}, G(1))$, where $\bar{\Psi}(s) = W(s) - sW(1) + ((1 - \rho)s + \mathbf{1}[s \leq \rho]s)\delta/\sqrt{1 - R^2}$. A calculation yields that the RN derivative of the distribution of $(\bar{\Psi}, G(1))$ under $\theta \sim F$ with respect to the distribution of $(B_1, W(1))$ (where $B_1(s) = W(s) - sW(1)$) is

$$\frac{1}{0.70} \int_{0.15}^{0.85} \left(\frac{(1 + (1 - r)\sigma_\beta^2)(1 + r(\sigma_\delta^2 - \sigma_\beta^2)) - R^2(1 + \sigma_\beta^2(1 - 2r) + r^2\sigma_\delta^2)}{1 - R^2} \right)^{-1/2} \\ \times \exp \left[\frac{1}{2} v(r)' \begin{pmatrix} 1 + \frac{1}{\sigma_\beta^2} - \frac{r(2-r)rR^2}{1-R^2} & -\frac{(1-r)rR^2}{1-R^2} \\ -\frac{(1-r)rR^2}{1-R^2} & \frac{1}{\sigma_\delta^2 - \sigma_\beta^2} + \frac{r(1-r)R^2}{1-R^2} \end{pmatrix}^{-1} v(r) \right] dr$$

where $v(r) = ((1 - r)G(1) - \bar{\Psi}(r)/\sqrt{1 - R^2}, rG(1) + \bar{\Psi}(r)/\sqrt{1 - R^2})'$, and under $\delta \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and

$\rho = r$, it is

$$\left(1 + \frac{r(1 - R^2 r)\sigma_{\delta,1}^2}{1 - R^2}\right)^{-1/2} \exp \left[-\frac{1}{2} \frac{\mu_1^2}{\sigma_{\delta,1}^2} + \frac{1}{2} \frac{\sigma_{\delta}^2 \left(\frac{\bar{\Psi}(\rho)}{\sqrt{1-R^2}} + rG(1) + \mu_1/\sigma_{\delta,1}^2 \right)^2}{1 + \sigma_{\delta,1}^2 \frac{r(1-R^2 r)}{1-R^2}} \right].$$

The power bounds $\bar{\pi}_{inv}$ were computed as described in Section 3.1 and 3.4 using these densities. The inequality $\bar{\pi}_{inv}(R^2) \leq \bar{\pi}_{inv}(\bar{R}^2)$ for $R^2 \leq \bar{R}^2$ holds because one can always reduce the information of the experiment with \bar{R}^2 to that with R^2 by adding a data independent Brownian Bridge of appropriate variance to $\bar{\Psi}$.

Condition 2 *In the stable model with parameter Γ_0*

(i) *in some neighborhood \mathcal{B}_0 of Γ_0 , $l_{T,t}(\Gamma)$ is twice differentiable a.s. with respect to Γ for $t = 1, \dots, T$, $T \geq 1$;*

(ii) *for all $s \in [0.15, 0.85]$ and $\epsilon > 0$ there exists $K(s, \epsilon) > 0$ such that $P(\sup_{\|\Gamma - \Gamma_0\| \geq \epsilon} T^{-1} \sum_{t=1}^{\lfloor sT \rfloor} (l_{T,t}(\Gamma) - l_{T,t}(\Gamma_0)) < -K(s, \epsilon)) \rightarrow 1$ and $P(\sup_{\|\Gamma - \Gamma_0\| \geq \epsilon} T^{-1} \sum_{t=\lfloor sT \rfloor + 1}^T (l_{T,t}(\Gamma) - l_{T,t}(\Gamma_0)) < -K(s, \epsilon)) \rightarrow 1$;*

(iii) *$\{s_{T,t}(\Gamma_0), \mathfrak{F}_{T,t}\}$ is a square-integrable martingale difference array with $T^{-1} \sum_{t=1}^{\lfloor sT \rfloor} E[s_{T,t}(\Gamma_0) s_{T,t}(\Gamma_0)' | \mathfrak{F}_{T,t-1}] \xrightarrow{p} s \Sigma^{-1}$ for all $0 \leq s \leq 1$ and some $(k+1) \times (k+1)$ dimensional matrix Σ , $T^{-1} \sup_{t \leq T} \|E[s_{T,t}(\Gamma_0) s_{T,t}(\Gamma_0)' | \mathfrak{F}_{T,t-1}]\| \xrightarrow{p} 0$ and there exists $\nu > 0$ such that $T^{-1} \sum_{t=1}^T E[\|s_{T,t}(\Gamma_0)\|^{2+\nu} | \mathfrak{F}_{T,t-1}] = O_p(1)$;*

(iv) *$T^{-1} \sum_{t=1}^T \|h_{T,t}(\Gamma_0)\| = O_p(1)$, $T^{-1} \sup_{t \leq T} \|h_{T,t}(\Gamma_0)\| \xrightarrow{p} 0$ and for any decreasing neighborhood \mathcal{B}_T of Γ_0 contained in \mathcal{B}_0 , $T^{-1} \sum_{t=1}^T \sup_{\Gamma \in \mathcal{B}_T} \|h_{T,t}(\Gamma) - h_{T,t}(\Gamma_0)\| \xrightarrow{p} 0$;*

(v) *For all $0 \leq s \leq 1$, $T^{-1} \sum_{t=1}^{\lfloor sT \rfloor} h_{T,t}(\Gamma_0) \xrightarrow{p} -s \Sigma^{-1}$.*

Part (ii) of Condition 2 is an identification condition that ensures consistency of the pre and post potential break maximum likelihood estimator—cf., for instance, condition 6 on page 436 of Schervish (1995). Parts (ii)-(v) are a special case of Condition 2 in Li and Müller (2007) and Condition 1 in Müller and Petalas (2007), who provide further discussion and references.

Proof of Proposition 3:

We first show convergence of the experiment involving observation X_T as $T \rightarrow \infty$ to that of observing $(G, \tilde{G})'$ in the sense of Definition 9.1 of van der Vaart (1998).

By Lemma 1 of Li and Müller (2007), any unstable model with parameter evolution (13) is contiguous to the stable model. Furthermore, by the same reasoning as employed in the proof of Lemma 1 of Li and Müller (2007), under the stable model,

$$\ln \text{LR}_T = \int_0^1 \begin{pmatrix} \beta + \delta \mathbf{1}[\lambda \leq \rho] \\ \tilde{\beta} + \tilde{\delta} \mathbf{1}[\lambda \leq \rho] \end{pmatrix}' \Sigma^{-1} d\hat{W}_T(\lambda) - \frac{1}{2} \int_0^1 \begin{pmatrix} \beta + \delta \mathbf{1}[\lambda \leq \rho] \\ \tilde{\beta} + \tilde{\delta} \mathbf{1}[\lambda \leq \rho] \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \beta + \delta \mathbf{1}[\lambda \leq \rho] \\ \tilde{\beta} + \tilde{\delta} \mathbf{1}[\lambda \leq \rho] \end{pmatrix} d\lambda + o_p(1)$$

where $\hat{W}_T(\lambda) = T^{-1/2}\Sigma^{-1/2}\sum_{t=1}^{\lfloor \lambda T \rfloor} s_{T,t}(\Gamma_0)$. By Theorem 3.6 in McLeish (1974), under Condition 2 (iii), $\hat{W}_T(\cdot) \Rightarrow (W(\cdot), \tilde{W}_{k+1}(\cdot))$. Thus, by the Continuous Mapping Theorem (CMT), also $(\hat{W}_T(\cdot), \ln \text{LR}_T) \Rightarrow ((W(\cdot), \tilde{W}_{k+1}(\cdot)), \ln \text{LR})$ in the stable model, where $\ln \text{LR}$ is the r.h.s. of (15) with $(G, \tilde{G})'$ replaced by $(W, \tilde{W}'_{k+1})'$. Since LR is the RN derivative of the distribution of $(G, \tilde{G})'$ with respect to $(W, \tilde{W}'_{k+1})'$, the general version of LeCam's Third Lemma (see, for instance, Lemma 27 of Pollard (2001)) thus implies that $\hat{W}_T(\cdot) \Rightarrow (G(\cdot), \tilde{G}(\cdot))'$ in the unstable model (13). The convergence of the experiments now follows from the same arguments as employed in the proof of Theorem 9.4 of van der Vaart (1998).

Now for part (i), fix arbitrary $\theta_1 \in \Theta_1, \tilde{\beta}_1, \tilde{\delta}_1 \in \mathbb{R}^k$. Let T' be a subsequence of T such that $\limsup_{T \rightarrow \infty} E_{\theta_1, \tilde{\beta}_1, \tilde{\delta}_1}[\varphi_T(X_T)] = \lim_{T' \rightarrow \infty} E_{\theta_1, \tilde{\beta}_1, \tilde{\delta}_1}[\varphi_{T'}(X_{T'})]$. By (16), there is a further subsequence T'' and test φ such that $E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi_{T''}(X_{T''})] \rightarrow E_{\theta, \tilde{\beta}, \tilde{\delta}}[\varphi(G, \tilde{G})]$ for all $\theta \in \Theta_0 \cup \Theta_1, \tilde{\beta}, \tilde{\delta} \in \mathbb{R}^k$. Since φ_T is asymptotically unbiased and of asymptotic level 5%, this implies φ to be a 5% level unbiased test. By Proposition 2(i), the most powerful unbiased test is φ_ρ , so that $\limsup_{T \rightarrow \infty} E_{\theta_1, \tilde{\beta}_1, \tilde{\delta}_1}[\varphi_T(X_T)] = E_{\theta_1, \tilde{\beta}_1, \tilde{\delta}_1}[\varphi(G, \tilde{G})] \leq E_{\theta_1, \tilde{\beta}_1, \tilde{\delta}_1}[\varphi_\rho(G)]$.

Parts (ii) and (iii) follow analogously.

Details on Determination of Approximate Least Favorable Distribution:

The distributions Ψ on Θ_0 mentioned in the main text are of the form " ρ is uniform on $[\max(0.15, a/100 - \frac{1}{2}), \min(0.85, b/100 + \frac{1}{2})]$ and δ is an equal probability mixture of $\delta \sim \mathcal{N}(\mu_\delta, \sigma_\delta^2)$ and $\delta \sim \mathcal{N}(-\mu_\delta, \sigma_\delta^2)$ " for some integers $a, b \in \{15, \dots, 85\}$, $b \geq a$, and $\mu_\delta, \sigma_\delta^2 > 0$. Collect the four determining numbers of $\Psi = \Psi_\varsigma$ in the vector $\varsigma = (a, b, \mu_\delta, \sigma_\delta^2)$. The density of $G(\cdot)$ under Ψ_ς , with respect to the distribution of $W(\cdot)$, is given by $\frac{1}{\min(0.85, b/100 + 1/2) - \max(0.15, a/100 - 1/2)} \int_{\max(0.15, a/100 - 1/2)}^{\min(0.85, b/100 + 1/2)} f(r, \mu_\delta, \sigma_\delta^2; G) dr$, where

$$f(r, \mu_\delta, \sigma_\delta^2; G) = (r\sigma_\delta^2 + 1)^{-1/2} \exp\left[\frac{1}{2} \frac{\sigma_\delta^2 G(r)^2 - r\mu_\delta^2}{r\sigma_\delta^2 + 1}\right] \cosh\left[\frac{G(r)^2 \mu_\delta}{r\sigma_\delta^2 + 1}\right]$$

and the density of $G(\cdot)$ with $(\beta, \delta, \rho) \sim F$ as in Condition 1 is given by $\frac{1}{0.70} \int_{0.15}^{0.85} h(r; G) dr$, where

$$h(r; G) = (r\sigma_\delta^2 + 1)^{-1/2} ((1-r)\sigma_\beta^2 + 1)^{-1/2} \exp\left[\frac{1}{2} \frac{\sigma_\delta^2 G(r)^2}{r\sigma_\delta^2 + 1} + \frac{1}{2} \frac{\sigma_\beta^2 (G(1) - G(r))^2}{(1-r)\sigma_\beta^2 + 1}\right].$$

As discussed in the main text, we approximate integrals with sums over the 100 point grid $r \in \{\frac{1}{100}, \frac{2}{100}, \dots, 1\}$. Thus, we replace $\frac{1}{0.70} \int_{0.15}^{0.85} h(r; G) dr$ by $\frac{1}{71} \sum_{l=15}^{85} h(l/100)$, and for the distribution Ψ_ς , we approximate the resulting density by $f(\varsigma; G) = \frac{1}{b-a+1} \sum_{l=a}^b f(l/100, \mu_\delta, \sigma_\delta^2; G)$ for $\varsigma = (a, b, \mu_\delta, \sigma_\delta^2)$. All Λ we consider a mixtures of Ψ_ς 's. Let $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_N)$, $\mathbf{p} = (p_1, \dots, p_N)$ with $\sum_{j=1}^N p_j = 1$. Thus, Neyman-Pearson tests can be written as $\varphi(\boldsymbol{\varsigma}, \mathbf{p}, cv) = \mathbf{1}[h(G)/\sum_{j=1}^N p_j f(\varsigma_j; G) < 1/cv]$ for some $cv \in \mathbb{R}$, with null rejection probability $\pi_0(\boldsymbol{\varsigma}, \mathbf{p}, cv; \varsigma) = \int E_\theta[\varphi(\boldsymbol{\varsigma}, \mathbf{p}, cv; G)] d\Psi_\varsigma(\theta)$ under H_{Ψ_ς} . In the following, write H_ς for H_{Ψ_ς} to ease notation. We

approximate π_0 with the Monte Carlo estimator

$$\hat{\pi}_0(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv}; \varsigma) = \frac{1}{20000} \sum_{i=1}^{20000} \left(1 + \left(\text{cv} \sum_{j=1}^N p_j \frac{f(\varsigma_j; G_i^*)}{h(G_i^*)} \right)^{10} \right)^{-1} \quad (35)$$

where G_i^* are (pseudo) random draws from G under Ψ_ς . Because h and f only depend on $\{G(l/100)\}_{l=1}^{100}$, one can generate G_i^* by suitably transforming Gaussian Random Walks with 100 steps (without the discretization to the 100 step grid of the uniform distributions of ρ under Ψ_ς). In contrast to the standard Monte Carlo estimator based on averaging $\varphi(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv})$ directly, the numerical close analogue (35) is a differentiable function of (\mathbf{p}, cv) , which greatly facilitates the computations below.

The algorithm calls three subroutines SR1, SR2 and SR3.

SR1 The routine takes a set $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_N)$ and a levels $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$ as given, and returns an estimate of the least favorable distribution $\Lambda_\boldsymbol{\varsigma}^*$ and corresponding critical value cv that describes the Neyman-Pearson test of the composite null hypothesis $H_\boldsymbol{\varsigma} : G$ has density $\Psi_\boldsymbol{\varsigma}$ for some $\varsigma \in \boldsymbol{\varsigma}$ against H_F , which is of level α_j under H_{ς_j} , $j = 1, \dots, N$. The distribution $\Lambda_\boldsymbol{\varsigma}^*$ is a mixture of the Ψ_ς 's, $\varsigma \in \boldsymbol{\varsigma}$. By the same argument as in Theorem 3.8.1 of Lehmann and Romano (2005), the least favorable mixing weights $\mathbf{p}^* = (p_1^*, \dots, p_N^*)$ have the two properties (i) $\pi_0(\boldsymbol{\varsigma}, \mathbf{p}^*, \text{cv}; \varsigma) \leq \alpha_l$ for $l = 1, \dots, N$; and (ii) $\pi_0(\boldsymbol{\varsigma}, \mathbf{p}^*, \text{cv}; \varsigma_l) < \alpha$ only if $p_l = 0$. This motivates the joint determination of \mathbf{p} and cv as numerical solutions to

$$\hat{\pi}_0(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv}; \varsigma_l) \leq \alpha_l \quad \text{and} \quad p_l(\hat{\pi}_0(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv}; \varsigma_l) - \alpha_l) = 0 \quad \text{for } l = 1, \dots, N. \quad (36)$$

Specifically, we determine appropriate \mathbf{p} and cv by minimizing the objective function

$$\sum_{l=1}^N (a_0 p_l + \exp[a_1(\hat{\pi}_0(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv}; \varsigma_l) - \alpha_l)])(\hat{\pi}_0(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv}; \varsigma_l) - \alpha_l)^2 \quad (37)$$

where $a_0 = 100$ and $a_1 = 2000$. As a function of \mathbf{p} and cv , (37) is continuous and with known first derivative, so that a standard quasi-Newton optimizer can be employed. Also, the $20000N^2$ numbers $f(\varsigma_l; G_i^*)/h(G_i^*)$ for G_i^* drawn under ς_j for $i = 1, \dots, N$, $l = 1, \dots, N$ and $j = 1, \dots, m$ can be computed and stored once to speed up the the evaluation of $\hat{\pi}_0(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv}; \varsigma_l)$ and its partial derivatives.

SR2 The routine takes $(\boldsymbol{\varsigma}, \mathbf{p}, \boldsymbol{\alpha})$ as inputs and returns $(\boldsymbol{\varsigma}', \mathbf{p}', \boldsymbol{\alpha}')$ of length $N' \leq N$ by eliminating values $(\theta_j, p_j, \alpha_j)$ with p_j approximately equal to zero.

SR3 The routine takes a test φ described by the triple $(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv})$, a set grid of values $\boldsymbol{\varsigma}_g = (\varsigma_{g,1}, \dots, \varsigma_{g,N_g})$ and a nominal level $\alpha_n \in [0, 1]$ as given, and determines whether $\varphi(\boldsymbol{\varsigma}, \mathbf{p}, \text{cv})$

Table 3: Grids for Computation of Approximate Least Favorable Distribution

i	l	μ_δ	σ_δ^2	α_i
1	15, 20, 75, 83, 85	0, 2, \dots , 20	300	4.45%
2	15, 20, 75, 83, 85	0, 2, \dots , 16	50	4.45%
3	15, 20, 75, 83, 85	0, 2, \dots , 22	10	4.50%
4	15, 20, 75, 83, 85	0, 2, \dots , 22	3	4.55%
5	15, 20, 30, 45, 60, 70, 75, 80, 83, 85	0, 2, \dots , 22	10	4.55%
6	83, 84, 85	0, $\frac{1}{2}$, \dots , 18	1	4.60%
7	15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 83, 85	0, 1, \dots , 22	10	4.60%

is of level α_n under $H_{\zeta'}$, $\zeta' \in \mathcal{C}'$. Specifically, it evaluates $\hat{\pi}_0(\zeta, \mathbf{p}, \text{cv}; \zeta')$ for all $\zeta' \in \mathcal{C}'$ in a random order until $\hat{\pi}_0(\zeta, \mathbf{p}, \text{cv}; \zeta') > \alpha_n$, in which case ζ' is returned. Otherwise, the routine returns a flag indicating that the test is of level α_n .

Overall the algorithm iterates between the subroutines as follows:

1. Initialize $\zeta = ((15, 24, 28, 200), (75, 85, 28, 200), (15, 85, 20, 4), (85, 85, 3, 4), (15, 85, 0, 300))$ and $\alpha = (0.033, 0.028, 0.034, 0.0445)$. Call SR1 and SR2 to obtain $(\zeta, \mathbf{p}, \text{cv})$.
2. For $i = 1$ to 7:
 - (a) Call SR3 with $(\zeta, \mathbf{p}, \text{cv})$, $\alpha_n = \alpha_{n,i}$ and $\zeta_g = \zeta_{g,i}$.
 - (b) If SR3 returns ζ , extend ζ by ζ and α by α_i , and call SR1 and SR2 to obtain new $(\zeta, \mathbf{p}, \text{cv})$. Go to step a.
 - (c) If SR3 returns flag that $\varphi(\zeta, \mathbf{p}, \text{cv})$ is of level α_n , increase i by one.

The 7 grids $\zeta_{g,i}$ and $\alpha_{n,i}$ are described in Table 3. The grid contains all possible combinations of $\zeta = (a, b, \mu_\delta, \sigma_\delta^2)$ with a and b consecutive values in column 2, and μ_δ as in column 3. The randomization in SR3 prevents cycles that arise through the thinning operation in SR2.

Details on Size Control Computations:

The test φ^* as defined above only depends on $\{G(l/100)\}_{l=1}^{100}$, and conditional on the Monte Carlo draw of the random walk $\{W_i(l)\}_{l=1}^{100}$, $G_i(l/100) = W_i(l/100) + \delta \min(\rho, l/100)$ is a function of δ and ρ . Write $S_i(\delta, \rho)$ for the induced function $\varphi^*(G_i)$. The approach outlined in Section 4.3 requires that for any given values $0 \leq \delta_1 < \delta_2$ and $0.15 \leq \rho_1 < \rho_2 \leq 0.85$, an algorithm must (attempt to) determine $\min_{\delta_1 \leq \delta < \delta_2, \rho_1 \leq \rho < \rho_2} S_i(\delta, \rho)$ and $\max_{\delta_1 \leq \delta < \delta_2, \rho_1 \leq \rho < \rho_2} S_i(\delta, \rho)$ (it is sufficient to consider non-negative δ , because $\varphi^*(G) = \varphi^*(-G)$). For this determination, we now consider how $\sup F_0$,

\hat{l}_0 , \hat{t}_0 and LR_0 behave as functions of (δ, ρ) on the rectangles $\delta_1 \leq \delta < \delta_2, \rho_1 \leq \rho < \rho_2$ where ρ_1 and ρ_2 are such that $l_0 - 1 \leq 100\rho_1$ and $100\rho_2 < l_0$ for some $l_0 \in \{16, \dots, 85\}$, conditional on W_i .

For supF_0 , note that $((100-l)G(\frac{l-1}{100}) - (l-1)(G(1) - G(\frac{l}{100}))) / \sqrt{(100-l)(l-1)} = a^{\text{supF}}(l, W_i) + \delta \cdot b^{\text{supF}}(l, \rho)$, where $a^{\text{supF}}(l, W_i)$ does not depend on (ρ, δ) , and $b^{\text{supF}}(l, \rho) = b_{pre}^{\text{supF}}(l, \rho) = \sqrt{(l-1)/(100-l)(1-\rho)}$ for $l < l_0$, $b^{\text{supF}}(l, \rho) = b_{mid}^{\text{supF}}(l) = \sqrt{(100-l)(l-1)}/100$ for $l = l_0$ and $b^{\text{supF}}(l, \rho) = b_{post}^{\text{supF}}(l, \rho) = \sqrt{(100-l)/(l-1)}\rho$ for $l > l_0$. Thus, as long as $\frac{100}{99}a(l, W_i)^2 < 90$ (which is true for almost all draws of W_i),

$$\begin{aligned} \delta_1 \left(\max_{16 \leq l < l_0} b_{pre}(l, \rho_2) + b_{mid}(l_0) + \max_{l_0 < l \leq 85} b_{post}(l, \rho_1) \right) &\leq \sqrt{\frac{100}{99}} \text{supF}_0 \\ &\leq \delta_2 \left(\max_{16 \leq l < l_0} b_{pre}(l, \rho_1) + b_{mid}(l_0) + \max_{l_0 < l \leq 85} b_{post}(l, \rho_2) \right). \end{aligned}$$

For \hat{l}_0 , note that with $\xi = \rho\delta$, $S_0(l)$ is of the form $S_0(l) = S_0(l, \delta, \xi) = \sum_{j=1}^2 (a_j^S(l, W_i) + b_{i,j}^S(l)\delta + c_{i,j}^S(l)\xi)^2$ for $i \in \{pre, mid, post\}$ in self-explanatory notation. In particular, $b_{pre,1}^S(l) = l$, $b_{post,1}^S(l) = b_{mid,1}^S(l) = 0$, $b_{pre,2}^S(l) = 0$, $b_{mid,2}^S(l) = (l-1)l/100$, $b_{post,2}^S(l) = 0$, $c_{pre,1}^S(l) = -l$, $c_{post,1}^S(l) = c_{mid,1}^S(l) = 100 - l$, $c_{pre,2}^S(l) = 0$, $c_{mid,2}^S(l) = -(l-1)$, and $c_{post,2}^S(l) = 1$. Let $\bar{\rho} = \frac{1}{2}(\rho_1 + \rho_2)$, $\bar{\delta} = \frac{1}{2}(\delta_1 + \delta_2)$, and $\bar{\xi} = \bar{\rho}\bar{\delta}$, and define $\hat{l} = \arg \max_{16 \leq l \leq 85} S_0(l, \bar{\delta}, \bar{\xi})$. With $S_\Delta(l, \delta, \xi) = S_0(l, \delta, \xi) - S_0(\hat{l}, \delta, \xi)$ and $\mathcal{T} = [\delta_1, \delta_2] \times [\xi_1, \xi_2]$, $\xi_1 = \delta_1\rho_1$, $\xi_2 = \delta_2\rho_2$, the set of possible values for \hat{l}_0 is a (possibly proper) subset of $\hat{L}_0 = \{16 \leq l \leq 85 : \sup_{(\delta, \xi) \in \mathcal{T}} S_\Delta(l, \delta, \xi) \geq 0\}$, because the range of $S_\Delta(l, \delta, \xi)$ on \mathcal{T} is at least as large as the range of $S_\Delta(l, \delta, \rho\delta)$ on $[\delta_1, \delta_2] \times [\rho_1, \rho_2]$, for each l . Since $S_\Delta(l, \delta, \xi)$, viewed as a function of δ and ξ , is a quadratic polynomial, the minimum and maximum occur either in a corner $(\delta, \xi) \in \{(\delta_1, \xi_1), (\delta_1, \xi_2), (\delta_2, \xi_1), (\delta_2, \xi_2)\}$, or, possibly, at a local maximum along one of the four sides, or, possibly, at an interior extremum. The five potential interior local extrema are easily computed by solving the appropriate linear first order conditions.

Given \hat{L}_0 , it is straightforward to construct upper and lower bounds on \hat{t}_0 .

Finally, for LR_0 , note that the exponents in the numerator is a positive definite quadratic polynomial in δ and $\xi = \rho\delta$. Expanding $\cosh(x) = \frac{1}{2}\exp(x) + \frac{1}{2}\exp(-x)$, also the denominator can be written as a sum of exponentials where each exponent is a positive definite quadratic polynomial in (δ, ξ) . Since positive definite quadratic polynomials are convex, and the sum of log-convex functions is log convex, both the numerator and denominator are log-convex functions of (δ, ξ) , say, $N_{\text{LR}}(\delta, \xi)$ and $D_{\text{LR}}(\delta, \xi)$. Let \bar{v}_D be the 2×1 vector of partial derivatives of $\ln D_{\text{LR}}(\delta, \xi)$ at $(\delta, \xi) = (\bar{\delta}, \bar{\xi})$. Since $\ln D_{\text{LR}}(\delta, \xi) - (\delta - \bar{\delta}, \xi - \bar{\xi})\bar{v}_D$ is convex with zero derivative at $(\delta, \xi) = (\bar{\delta}, \bar{\xi})$, it takes on its minimum at $(\bar{\delta}, \bar{\xi})$, and since $\ln N_{\text{LR}}(\delta, \xi) - (\delta - \bar{\delta}, \xi - \bar{\xi})\bar{v}_D$ is convex, it takes on its maximum in one of the corners. Thus,

$$\max_{\delta_1 \leq \delta < \delta_2, \rho_1 \leq \rho < \rho_2} \text{LR}_0 \leq \frac{\max_{(\delta, \xi) \in \mathcal{T}} N_{\text{LR}}(\delta, \xi) \exp[-(\delta - \bar{\delta}, \xi - \bar{\xi})\bar{v}_D]}{D_{\text{LR}}(\bar{\delta}, \bar{\xi})}$$

and, reversing the role of the numerator and denominator, we can construct an analogous lower bound.

Details on Power Bound Computations:

A standard Wiener process on the unit interval W can be decomposed as $W(s) = D_W^\tau(s) + \alpha_W + (s - 1/2)\beta_W$, where α_W and β_W are the coefficients of a continuous time least squares regression of $W(s)$ on $(1, s - 1/2)$, and $D_W^\tau(s)$ are the residuals. Let $D_W^{\text{sq}} = \int D_W^\tau(s)^2 ds$. With these definitions, a straightforward calculation yields $\int_0^1 W(s)^2 ds = D_W^{\text{sq}} + \frac{1}{3}\beta_W^2 + \alpha_W\beta_W + \alpha_W^2$. Further, define $D_W(s)$ as the demeaned Brownian Bridge $W(s) - sW(1)$, $D_W(s) = W(s) - sW(1) - \alpha_W + \frac{1}{2}W(1)$, and note that $\Delta_W = \sup_{0 \leq s \leq 1} |W(s) - \alpha_W| \leq \frac{1}{2}|W(1)| + \sup_{0 \leq s \leq 1} |D_W(s)|$ and $\Delta_W^{\text{sq}} = \sup_{0 \leq s \leq 1} W(s)^2 \leq 2\Delta_W|\alpha_W| + \Delta_W^2 + |\int_0^1 W(s)^2 ds - \alpha_W^2|$. These expressions are useful because (i) $(W(1), \alpha_W, \beta_W)$ are jointly Gaussian with known covariance matrix and independent of $D_W^\tau(\cdot)$ (and thus D_W^{sq}); (ii) the distribution of D_W^{sq} can be computed exactly by inverting the characteristic function derived in Theorem 9.12 of Tanaka (1996); (iii) Darling (1983) provides a formula for the c.d.f. of $\sup_{0 \leq s \leq 1} D_W(s)$.

Evaluation of the expression in Darling (1983) yields $P(\sup_{0 \leq s \leq 1} D(s) > 2.2) < \varepsilon_D = 5 \cdot 10^{-12}$, so that $P(\sup_{0 \leq s \leq 1} |D_W(s)| > 2.2) < 2\varepsilon_D$. Also, evaluation of the c.d.f. of D_W^{sq} at the 15 points $0.1, 0.2, \dots, 1.5$ by inverting the characteristic function, as discussed in Chapter 6 of Tanaka (1996), permits the straightforward construction of the two random variables $D_W^{\text{sq}l}$ and $D_W^{\text{sq}u}$ such that $P(D_W^{\text{sq}l} \leq D_W^{\text{sq}} \leq D_W^{\text{sq}u}) = 1$ by suitably transforming uniform random variable on the unit interval (with $D_W^{\text{sq}u}$ taking on the value $+\infty$ with probability $P(D_W^{\text{sq}} > 1.5) \simeq 4 \cdot 10^{-14}$). Let $(Z_j, \alpha_{W_j}, \beta_{W_j})$ be J i.i.d. Monte Carlo draws from the distribution of $(W(1), \alpha_W, \beta_W)$, let $(D_j^{\text{sq}l}, D_j^{\text{sq}u})$ be J independent i.i.d. Monte Carlo draws from the distribution of $(D^{\text{sq}l}, D^{\text{sq}u})$. Then define $S_{1,j}^u = 2.2 + 0.5|Z_j|$ and $S_{2,j}^u = 2|\alpha_{W_j}|S_{1,j}^u + (S_{1,j}^u)^2 + |D_{W_j}^{\text{sq}u} + \frac{1}{3}\beta_{W_j}^2 + \alpha_{W_j}\beta_{W_j}|$, $W_{b,j+1} = W_{b,j} + Z_j/\sqrt{J}$, $\bar{W}_j = W_{b,j} + \alpha_W/\sqrt{J}$, $\bar{W}_j^{2l} = (D_{W_j}^{\text{sq}l} + \frac{1}{3}\beta_W^2 + \alpha_W\beta_W + \alpha_W^2)/J + \alpha_W W_{b,j}/\sqrt{J} + W_{b,j}^2$ and $\bar{W}_j^{2u} = \bar{W}_j^{2l} + (D_{W_j}^{\text{sq}u} - D_{W_j}^{\text{sq}l})/J$ for $j = 1, \dots, J$, where $W_{b,1} = 0$. With the implemented number of $J = 800,000$ intervals, a draw $\{\bar{W}_j^{2l}, \bar{W}_j^{2u}, \bar{W}_j, S_{1,j}^u, S_{2,j}^u\}_{j=1}^J$ satisfies $\bar{W}_j^{2l} \leq \bar{W}_j^2 \leq \bar{W}_j^{2u}$, $S_{1,j} \leq S_{1,j}^u$ and $S_{2,j} \leq S_{2,j}^u$ for all $j = 1, \dots, J$ with probability of at least $(1 - 2\varepsilon_D)^J > 1 - 7 \cdot 10^{-5}$ by construction. Thus, very slight corrections to the estimated critical value and Monte Carlo rejection probability yields an unconditionally conservative Monte Carlo estimator of the power bounds.

Proof of Proposition 4:

(i) The CMT, and, in the case of \hat{l} , some tedious algebra, imply that $\hat{l} \Rightarrow \hat{l}_0$ and $\widehat{\text{supF}} \Rightarrow \text{supF}_0$ with \hat{l}_0 and supF_0 as defined the definition of φ^* above (note that replacing G by G_0 leaves \hat{l}_0 and supF_0 unchanged). Define $\beta_T = T^{1/2}(\gamma_{\text{post}} - \gamma_{\text{post},0})$ and $\delta_T = T^{1/2}(\gamma_{\text{pre}} - \gamma_{\text{post}})$.

(i.a) Since $G(s) = G_0(s) + \beta s$, the result follows from the CMT.

(i.b) We have $\hat{t}_{post}/\beta_T \xrightarrow{P} \sqrt{1 - l_0/100}/\sigma$ and a calculation yields $\beta_T^{-2} \ln \widehat{\text{LR}} \xrightarrow{P} c > 0$, so that $P(|\hat{t}_{post}| > 2.01) \rightarrow 1$ and $P(\widehat{\text{LR}} > xx) \rightarrow 1$.

(ii) Note that $\delta_T^{-2} (\hat{\gamma}_{post}(\lceil 100\rho \rceil) - \hat{\gamma}_{pre}(\lceil 100\rho \rceil - 1))^2 \xrightarrow{P} 1$ and $\hat{\omega}_{post}^2(\lceil 100\rho \rceil) \xrightarrow{P} \sigma^2/(1 - l/100)$, so that $P(\widehat{\text{supF}} > 90) \rightarrow 1$. Let

$$\hat{S}(l) = \sum_{j=1}^{l-1} (\hat{\Delta}_{pre}(j) - \hat{\gamma}_{pre}(l-1))^2 + \sum_{j=l+1}^{100} (\hat{\Delta}_{post}(j) - \hat{\gamma}_{post}(l))^2$$

where $\hat{\Delta}_{pre}(j) = \hat{\gamma}_{pre}(15)$ for $j \leq 15$ and $\hat{\Delta}_{post}(j) = \hat{\gamma}_{post}(85)$ for $j > 85$, and note that $\hat{l} = \arg \max_{16 \leq l \leq 85} \hat{S}(l)$. If $100\rho \neq \lfloor 100\rho \rfloor$, then $\hat{S}(\lfloor 100\rho \rfloor + 1) = O_p(T^{-1})$, and for all $l \neq \lfloor 100\rho \rfloor + 1$, $P(T\hat{S}(l) > M) \rightarrow 1$ for any $M \in \mathbb{R}$, so that $\hat{l} \xrightarrow{P} \lfloor 100\rho \rfloor + 1$. Part (ii.b), and part (ii.a) for $\rho \notin R_{grid}$, now follow from the CMT.

Consider thus the proof of part (ii.a) if $100\rho = \lfloor 100\rho \rfloor$. By the same argument, $P(\hat{l} \notin \{l_0, l_0 + 1\}) \rightarrow 0$, where $l_0 = 100\rho$. If $l_0 \in \{84, 85\}$, the result follows again from the CMT. Otherwise, define $t_{post}(l_0 + 1) = (W(1) - W(\frac{l_0+1}{100}))/\sqrt{1 - \frac{l_0+1}{100}} \sim \mathcal{N}(0, 1)$ and $t_{post}(l_0 + 2) = (W(1) - W(\frac{l_0+2}{100}))/\sqrt{1 - \frac{l_0+2}{100}} \sim \mathcal{N}(0, 1)$, and consider

$$\begin{aligned} \hat{S}(l_0) - \hat{S}(l_0 + 1) &= A_{pre} + \hat{\Delta}_{post}(l_0 + 1)^2 - (100 - l_0)\hat{\gamma}_{post}(l_0)^2 + (99 - l_0)\hat{\gamma}_{post}(l_0 + 1)^2 \\ &\Rightarrow \Delta S(l_0) = a \cdot t_{post}(l_0 + 1)^2 + t_{post}(l_0 + 1)B(l_0) + C(l_0) \end{aligned}$$

where A_{pre} is a function of $\{\hat{\gamma}_{pre}(l)\}_{l=15}^{l_0-1}$, the convergence follows from the CMT, a is a positive constant, and $B(l_0)$ and $C(l_0)$ are random variables that are independent of $t_{post}(l_0 + 1)$ and $t_{post}(l_0 + 2)$. Thus, by a further application of the CMT, we obtain

$$\hat{t}_{post} \Rightarrow \mathbf{1}[\Delta S(l_0) \geq 0] \mathbf{1}[|t_{post}(l_0 + 1)| > 2.01] + \mathbf{1}[\Delta S(l_0) < 0] \mathbf{1}[|t_{post}(l_0 + 2)| > 2.01]. \quad (38)$$

Consider now the rejection probability of the r.h.s. of (38) conditional on $B(l_0) = b$ and $C(l_0) = c$. Noting that $\Delta S(l_0)$ is a quadratic function of $t_{post}(l_0 + 1)$ with positive coefficient on the square term, one obtains that the rejecting probability is bounded from above by the replacement $\mathbf{1}[\Delta S(l_0) \geq 0] = \mathbf{1}[t_{post}(l_0 + 1) < -2.01]$, for any value of b, c . A numerical calculation now shows that the conditional rejection probability remains below 5% even in that case for all $l_0 = 15, \dots, 83$, and the result follows.

(iib) Immediate from $P(\hat{l} \notin \{l_0, l_0 + 1\}) \rightarrow 0$ and $\beta_T^{-1}(\hat{\gamma}_{post}(l) - \gamma_{post,0})/\hat{\omega}_{post}(l) \xrightarrow{P} \sqrt{1 - l/100}/\sigma > 0$ for all $l \geq l_0$.

(iii) The claim is essentially a special case of Theorem 1 of Andrews and Guggenberger (2007c). Let θ_T be a parameter sequence such that $\limsup_{T \rightarrow \infty} E_{\theta_T}[\hat{\varphi}^*(Y_T)] = \limsup_{T \rightarrow \infty} \sup_{\theta \in \Theta_0} E_{\theta}[\hat{\varphi}^*(Y_T)]$. Pick a subsequence T' of T such that $\lim_{T' \rightarrow \infty} E_{\theta_{T'}}[\hat{\varphi}^*(Y_{T'})] = \limsup_{T \rightarrow \infty} E_{\theta_T}[\hat{\varphi}^*(Y_T)]$. Since $\bar{\Theta}_0 = \Theta_0 \cup \{(0, \rho, \delta) : \rho \in [0.15, 0.85], \delta \in \{+\infty, -\infty\}\}$ is compact under an appropriate metric, there exists a further subsequence $\theta_{T''}$ of $\theta_{T'}$ such that

$\theta_{T''} \rightarrow \tilde{\theta} = (0, \tilde{\rho}, \tilde{\delta}) \in \bar{\Theta}_0$. If $\tilde{\rho} \notin R_{grid}$, then the result follows as in parts (i) and (ii). If $\tilde{\rho} \in R_{grid}$, then by assumption, there is a further subsequence such that (32) holds, except for (33). Proceeding as in the proof of part (ii.a) now again yields the result (and for the case $\nu_{pre}(l_0) \in \{+\infty, -\infty\}$ or $\nu_{post}(l_0) \in \{+\infty, -\infty\}$, $P(|\Delta S(l_0)| < M) \rightarrow 0$ for all M , so that $\mathbf{1}[\Delta S(l_0) \geq 0] \xrightarrow{p} 0$ or $\mathbf{1}[\Delta S(l_0) \geq 0] \xrightarrow{p} 1$).

Justification of (31) via Andrews (1993):

The true parameters are given by $\Gamma_{T,t} = \Gamma_{post} + T^{-1/2}\mathbf{1}[\tau \leq 100\rho_T]\Delta_T$, where $\Delta_T = -(\omega\delta_T, -\tilde{\delta}_T) \rightarrow \Delta_0 \in \mathbb{R}^{k+1}$ and $\rho_T \rightarrow \rho_0 \in [0.15, 0.85]$, so that $E[g(X_{T,t}, \Gamma_{T,t})] = 0$. Write $g_t(\Gamma)$ for $g(X_{T,t}, \Gamma)$. Our set-up corresponds to what Andrews (1993) refers to as 'pure' structural change. We now show that under Assumption 1 of Andrews (1993), with the assumption in part (b) $E[g_t(\Gamma_0)] = 0$ replaced by $E[g_t(\Gamma_{T,t})] = 0$, (31) holds. These computations closely correspond to what is derived in section 5.4 of Andrews (1993), with two differences: first, we do not assume that $\max_{t \leq T} \sup_{\|\Gamma - \Gamma_{post}\| \leq T^{-1/2}K} \|E[\Upsilon_t(\Gamma)] - \tilde{\Upsilon}\| \rightarrow 0$ for some $K > 0$ (cf. first line on page 832), and second, we consider sequences for Δ_T and ρ_T . (We note that there is a typo in Assumption 1-LP of Andrews (1993); it should read $\sup_{\pi \in \Pi} \|\sqrt{T}E\tilde{m}_T(\theta_0, \pi) - \mu(\pi)\| \rightarrow 0$.)

By the argument in Andrews' proof of Theorem 4 (a), it suffices to show that $\sup_{0 \leq s \leq 1} \|T^{-1/2}E[\sum_{t=1}^{\lfloor sT \rfloor} g_t(\Gamma_{post}) + \tilde{\Upsilon} \int_0^s \eta_0(l)dl]\| \rightarrow 0$. Now clearly $\sup_{0 \leq s \leq 1} \|\lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^{\lfloor sT \rfloor} \eta_T(t/T) - \int_0^s \eta_0(l)dl\| \rightarrow 0$. Also,

$$\begin{aligned} T^{-1/2}E\left[\sum_{t=1}^{\lfloor sT \rfloor} g_t(\Gamma_{post}) + T^{-1/2}\tilde{\Upsilon}\eta_T(t/T)\right] &= T^{-1/2}E\left[\sum_{t=1}^{\lfloor \min(s, \rho_T)T \rfloor} g_t(\Gamma_{post}) + T^{-1/2}\tilde{\Upsilon}\eta_T(t/T)\right] \\ &= (\tilde{\Upsilon} - E[\tilde{\Upsilon}_T(\min(s, \rho_T))])\Delta_T \end{aligned}$$

where $\tilde{\Upsilon}_T(r)$ is equal to $T^{-1} \sum_{t=1}^{\lfloor rT \rfloor} \int_0^1 \Upsilon_t(\Gamma_{post} + \lambda T^{-1/2}\Delta_T)d\lambda$. Furthermore,

$$\begin{aligned} \sup_{0 \leq s \leq 1} \|E[\tilde{\Upsilon}_T(\min(s, \rho_T)) - \tilde{\Upsilon}_T(\min(s, \rho_0))]\| &\leq T^{-1}E \sum_{t=\lfloor \min(\rho_0, \rho_T)T \rfloor}^{\lfloor \max(\rho_0, \rho_T)T \rfloor} \left\| \int_0^1 \Upsilon_t(\Gamma_{post} + \lambda T^{-1/2}\Delta_T)d\lambda \right\| \\ &\leq (\lfloor \max(\rho_0, \rho_T)T \rfloor - \lfloor \min(\rho_0, \rho_T)T \rfloor) \sup_{t \leq T} E[\sup_{\Gamma \in \mathcal{G}_0} \|\Upsilon_t(\Gamma)\|] \rightarrow 0 \end{aligned}$$

where the convergence follows, since by Andrews' assumption 1(f), $\sup_{t \leq T} E[\sup_{\Gamma \in \mathcal{G}_0} \|\Upsilon_t(\Gamma)\|] = O(1)$ for some neighborhood \mathcal{G}_0 of Γ_{post} . Finally, $\sup_{0 \leq s \leq 1} \|\tilde{\Upsilon} - E[\tilde{\Upsilon}_T(\min(s, \rho_0))]\| \rightarrow 0$ follows from Andrews' (A.8).

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