Replicator dynamics in a Cobweb model with rationally heterogeneous expectations

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Abstract

This paper extends the adaptively rational equilibrium dynamics of Brock and Hommes [Brock, W.A., Hommes, C.H., 1997. A rational route to randomness. Econometrica 65, 1059–1160] by introducing a generalized version of the replicator dynamic. The replicator equilibrium dynamics (RED) couples the price dynamics of a Cobweb model with predictor selection governed by an evolutionary replicator dynamic. We show that the RED supports the conclusion of Sethi and Franke [Sethi, R., Franke, R., 1995. Behavioural heterogeneity under evolutionary pressure: macroeconomic implications of costly optimisation. The Economic Journal, 105, 583–600] that costly rational beliefs persist, though unlike them, these results obtain in the deterministic case. Numerical evidence shows that complex dynamics exist, as in Brock and Hommes, even though no weight is placed on strictly dominated predictors in an RED steady-state.
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1. Introduction

The continuing dominance of the rational expectations hypothesis in dynamic macro-economic models over other, primarily adaptive approaches to expectation formation can be attributed to analytic tractability and the appeal of optimally formed beliefs. Rational expectations are optimal

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in the sense that, in equilibrium, they produce non-systematic forecast errors. A recent literature examines instances when it might not be optimal, from a utility maximization standpoint, for agents to form expectations rationally. Models of *rationally heterogeneous expectations* address this issue by modeling expectation formation as a conscious choice between costly competing belief-formation alternatives. In these models agents may have systematic forecast errors but the cost to improving upon them is prohibitive.

Evans and Ramey (1992, 1998) examine the issue of costly expectation formation through a process of 'expectation calculation'. In these papers, agents have an algorithm that can be iterated subject to a cost per iteration. They find that under certain conditions, the outcome of agents' choices may converge to the rational expectations equilibrium. Brock and Hommes (1997) and Sethi and Franke (1995) extend the 'expectation calculation' model so that agents choose between various costly predictor functions. Agents weigh the potential relative benefits and costs of predictor use in order to make their belief-formation decision. Brock and Hommes (1997) define the adaptively rational equilibrium dynamics (ARED) as an equilibrium for a system in which predictor choice follows a multinomial logit. Brock and Hommes (1997, 1998) and Brock et al. (2005) show that if agents react strongly enough to changes in relative net benefits, then the dynamics may become complicated. Sethi and Franke, on the other hand, find that when predictor selection evolves according to a replicator dynamic, as used extensively in evolutionary game theory, the asymptotic distribution of predictor types is invariant and places positive weight on the rational predictor.[1] Further, the Sethi–Franke model addresses one possible criticism of Brock and Hommes (1997): the ARED selection model yields a positive proportion of agents using the strictly dominated rational expectations predictor even in a steady-state.[2] In Sethi and Franke (1995) strictly dominated predictors vanish asymptotically. A drawback to the Sethi–Franke approach, though, is that they require exogenous disturbances in order for rational expectations to be chosen asymptotically.

These pioneering papers leave two important questions unresolved. In the deterministic case, Sethi–Franke find that the proportion of agents using a costly sophisticated predictor vanishes, though they may coexist with simpler less-accurate forecast methods in a stochastic environment. On the other hand, Araujo and Sandroni (1999) and Sandroni (2000) find that in models with perfect capital markets 'boundedly rational' agents will vanish. Finally, Brock and Hommes obtain coexistence in steady-state, though the selection mechanism underlying their predictor dynamics is based on a random utility model, and, thus implicitly stochastic. The first question is whether coexistence can obtain in a purely deterministic environment.

The second question concerns the effects of heterogeneity on the dynamics of the model.[3] Brock and Hommes found that for certain values of the 'intensity of choice' parameter, cycles of high order and chaos can emerge. It is natural then to wonder whether this type of behavior is specific to their model of predictor proportion dynamics, or whether it obtains more generally. This leads to the second question: can a replicator similar to the one studied by Sethi and Franke result in complex dynamics similar to those found by Brock and Hommes?

We address these questions by incorporating into a Cobweb model a replicator dynamic based on Sethi–Franke and studying the resulting replicator equilibrium dynamics (RED). In our model, price depends on heterogeneous expectations, which are assumed to be formed as an economic

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[1] Droste et al. (2002) examine a Cournot duopoly model where decisions follow a replicator dynamic.

[2] In a special case of their model where agents react fully to profit differences there is no heterogeneity and no rational agents.

[3] In a different context Calvet et al. (2004) consider the implications of heterogeneity on state dynamics.

choice between costly predictor function alternatives. These choices are, in turn, made by adapting to the past costs and benefits of the various predictors and result in predictor proportions that follow a law of motion closely resembling the replicator dynamic in evolutionary game theory. We compare the dynamics in the RED to the ARED where the law of motion for predictor selection follows a multinomial logit as in Brock and Hommes (1997).

Analysis of the RED first requires specification of an appropriate replicator dynamic. The usual replicator, developed by Weibull (1995) and others, does not extend to the Cobweb model with dynamic predictor selection because the fitness measure is realized profits and these are not bounded above zero. Sethi and Franke consider a Boolean decision, and the method they use does not directly generalize to a setting that includes more than two options. In order to study the vanishing of strictly dominated predictors in a general setting, we alter their replicator dynamics in a natural way to incorporate models with any finite number of predictors.[4] We then show that in the RED strictly dominated predictors vanish, so in a steady-state, a zero proportion of agents will use the costly rational expectations predictor. This is in contrast to the steady-state of Brock and Hommes in which a positive proportion of agents will use the rational predictor even though it costs more to do so.

To obtain specific analytic results, we focus on the two predictor case by assuming that agents choose between rational and naive expectations. We demonstrate that the instability result of Brock and Hommes, which states that the steady-state will be unstable if it is unstable when agents mass on the cheapest predictor, holds in the RED as well. Further, we corroborate the Sethi and Franke finding that sophisticated predictors do not vanish asymptotically. Importantly, this result obtains even when there is no noise in the model. This leads us to conclude that stochastic models are not necessary for costly sophisticated predictors to coexist with simpler, cheaper predictors. This result is particularly significant since it holds even in models that asymptotically place zero weight on strictly dominated predictors.

We also find support for the complicated dynamics exhibited by the model in Brock and Hommes (1997). Brock and Hommes show that as the 'intensity of choice' between predictors increases (but remains finite), the equilibrium trajectories become aperiodic and converge to a strange attractor. We argue through numerical simulations that the RED also may generate periodic and aperiodic trajectories depending upon the sensitivity of agents' predictor choices. The existence of complex dynamics in the RED is not obvious. Because strictly dominated predictors vanish under the RED one might expect that the model will be stable. The complex dynamics under this alternative law of motion instead highlights the importance of the interaction between the stability properties of the steady-state and evolution of predictor proportions. The steady-state of the Cobweb model tends to repel all trajectories under naive expectations and attract them under rational expectations. When dynamic predictor proportions are coupled with the Cobweb model via either ARED or RED, complicated interactions between the steady-state's dual role as an attractor and a repellor create complex dynamics. These results suggest that models with steady-states that have this attractor–repellor property may generate complicated dynamics regardless of the predictor selection map.

This paper proceeds as follows. Section 2 presents the Cobweb model with rationally heterogeneous expectations. Section 3 presents analytic results on the stability of the system; careful

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[4] Brock et al. define a model with a large type limit as one where there may be an infinite number of predictors. These models are an attempt to provide an analytic approach to the Santa Fe Institute's Agent-based Modeling approach, which is primarily computational.

contrasts are drawn with the ARED of Brock and Hommes (1997). Section 4 presents a detailed numerical argument in favor of complex dynamics. Finally, Section 5 concludes.

2. Model and replicator

This section introduces the Cobweb model with rationally heterogeneous expectations. The Cobweb model is used frequently in the literature on learning because of its simple structure and its wide array of dynamic properties. The model assumes a production lag that forces supply decisions to be made one period in advance. Because of this lag, supply is determined by firms' expectations of next period's price. Depending on the relative slope values of supply and demand and the exact nature of the expectation formation device, the steady-state may be unstable. Carlson (1968) and Branch (2002) showed that if adaptive expectations place a large enough weight on past prices, the steady-state will be locally stable. However, under naive expectations (i.e. the belief that last period's price will prevail this period), the steady-state may be unstable. Our analysis focuses on when the destabilizing predictor also carries the lowest cost.

2.1. The Cobweb model with rational heterogeneity

Our construction of the Cobweb model follows Brock and Hommes (1997). Facing a production lag, firms make supply decisions based on expectations of future prices:

$$S(p_{t+1}^e) = \underset{Q}{\mathrm{argmax}}\, p_{t+1}^e Q - c(Q) = (c')^{-1} p_{t+1}^e. \tag{1}$$

In this model we will not *a priori* impose homogeneous expectations. Instead, we assume there exists a large number of firms who choose from a set of predictor functions

$$H(p^t) = (H_1(p^t), \ldots, H_K(p^t))$$

to form their expectation p_{t+1}^e using the information set $p^t = (p_t, p_{t-1}, \ldots, p_{t-L},)$. Assume that $H_j : \mathbb{R}^L \to \mathbb{R}$ maps past prices into a forecast of future price so that

$$H_j(p, \ldots, p) = H_k(p, \ldots, p)$$

for all j and k. Demand is given by the function $D(p_{t+1})$. We maintain generality in this section, as in Brock and Hommes (1997), in order to compare our results to theirs. This restriction on the set of predictors is identical to theirs. Below we focus on the special case of perfect foresight versus naive expectations. Brock and Hommes (1997) note that the perfect foresight predictor does not satisfy the assumption above that predictors map past and current prices into a future expectation. Following Brock and Hommes, we relax this assumption in the particular case of rational versus naive expectations. The two predictor example below, though, is still a special case of the general model here.[5]

Define $q_{j,t}$ as the proportion of agents that use H_j in time t. Market equilibrium is given by

$$D(p_{t+1}) = \sum_{j=1}^{K} q_{j,t} S(H_j(p^t)). \tag{2}$$

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[5] For further discussion of this issue, see Brock and Hommes (1997).

For appropriately defined $D(\cdot)$, $S(\cdot)$ (see Brock and Hommes, 1997), there exists a solution to Eq. (2), which we write as

$$p_{t+1} = D^{-1}\left(\sum_{j=1}^{K} q_{j,t} S(H_j(p^t))\right). \tag{3}$$

This equation of motion describes the price dynamics.

2.2. The replicator dynamic

To close the model we must specify the evolution of the proportions $q_{j,t}$. We consider a law of motion inspired by Sethi and Franke (1995), who assume that q_j is increasing in its excess mean payoff, but we extend the replicator to a setting that includes more than two predictors. To construct this extension, we deviate slightly from their method by imposing that q_j *decreases* in its *deficit* mean payoff. The reason for this deviation will be made clear in the sequel.

2.2.1. Adaptively rational equilibrium dynamics

For future comparisons it will be instructive to review first the predictor dynamic of Brock and Hommes (1997). They assume that $q_{j,t}$ follows the law of motion

$$q_{j,t} = \frac{\exp\{\beta U_{j,t}\}}{\sum_{k=1}^{K} \exp\{\beta U_{k,t}\}}, \tag{4}$$

where $U_{j,t}$ is any fitness measure for predictor j. The form of Eq. (4) is a multinomial logit (MNL) and is used frequently in the discrete choice literature. It has the nice feature that predictor choice is directed towards those predictors that return higher payoffs. Its main drawback is that it is derived from a random-utility model under very specific assumptions about the underlying stochastic process. Brock and Hommes (1997) refer to β as the 'intensity of choice' parameter. For larger values of β agents will react more strongly to changes in relative net benefits.

We follow Brock and Hommes (1997) in assuming $U_{j,t} = \pi_{j,t}$ where $\pi_{j,t}$ is period t's realized net profits for predictor j, and is calculated as

$$\pi_{j,t} = p_t S(H_j(p^{t-1})) - c(S(H_j(p^{t-1}))) - C_j. \tag{5}$$

This equation of motion describes the dynamics of net profits, and throughout the paper payoffs are calculated accordingly.[6] The variable C_j is the cost of using predictor j. In Brock and Hommes (1997) and Branch (2002), C_j plays a large role in the stability conditions. If $C_j = 0$ for all j, then in a steady-state the agents are evenly distributed across all predictors and the steady-state is locally stable. Notice that in a steady-state $\pi_j - \pi_k = C_k - C_j$. If β and C_j are finite, then in a steady-state all predictors receive positive weight. This is a feature particular to the selection method employed by Brock and Hommes and is the result of the random utility model underlying the multinomial logit law of motion (4). This feature is a drawback to the MNL approach because in a steady-state with a finite 'intensity of choice', some agents will pay a higher price for a predictor that returns the same forecast as a less costly alternative. We are now ready to define the first equilibrium concept.

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[6] Brock and Hommes (1997) initially allow $U_{j,t}$ to be a weighted average of all past profits. Branch and Evans (2006) consider a stochastic framework where agents choose predictors based on unconditional mean profits.

Definition. Adaptively rational equilibrium dynamics (ARED) is a process (p_t, q_t) that satisfies Eqs. (3)–(5).

2.2.2. Replicator equilibrium dynamics

Sethi and Franke consider an evolutionary approach. Using a predictor proportion law of motion that closely resembles the replicator dynamic in evolutionary game theory, they examine the case of rational and adaptive expectations in a model of strategic complementarity. In the deterministic version of the model rational agents are driven from the market asymptotically. A goal of this paper is to see if this result extends to the Cobweb model. We are also interested in whether the replicator dynamic can overcome the aforementioned persistence of strictly dominated predictors that occur in the ARED.

In order to find a replicator that applies to selection between more than two choices, we must guarantee that the new proportions are in the unit interval and sum to one. A straightforward generalization of Sethi and Franke appears impossible.

Instead we propose an alternative approach that is inspired by Sethi and Franke. Our approach imposes that any increase in a predictor proportion is offset by an analogous decrease in other proportions. We require the following definition:

Definition. A vector of predictor proportions, $q_t = [q_{1,t}, \ldots, q_{K,t}]'$, is an element of the $K-1$-*simplex* $\Delta S^{K-1} \subset \mathbb{R}^K$, that is, $q_{i,t} \in [0,1]$ and $\sum_i q_{i,t} = 1$.

Recall that $\pi_{j,t}$ denotes the profit received by agents using predictor j in time t, given the predictor proportions q_{t-1}. At this point it is useful to summarize the timing of the Cobweb model with rationally heterogeneous expectations. Agents make supply decisions at $t-1$ based on their forecast of price at time t. After the realization of price at time t, agents then calculate realized profits $\pi_{i,t}$, which depend on the proportions q_{t-1}. Finally, agents update their predictor choice by weighing the realized profits at time t against the average profit across predictors, and the associated price forecasts then determine supply.

For a given vector of proportions, q_t, define mean utility at time $t+1$ to be the average profit, that is

$$\bar{\pi}_{t+1} = \frac{1}{K} \sum_j \pi_{j,t+1}.$$

Next, define the following sets:

$$\hat{K} = \{1, \ldots, K\}, \qquad \hat{G}(q_t) = \{i \in \hat{K} | \pi_{i,t+1} \geq \bar{\pi}_{t+1}\}, \qquad \hat{B}(q_t) = \{i \in \hat{K} | \pi_{i,t+1} < \bar{\pi}_{t+1}\}.$$

Notice $\hat{G}$ is the complement of $\hat{B}$ in $\hat{K}$. As the timing protocol described above emphasizes, the appropriate *ex post* metric on a predictor j from time t is $\pi_{j,t+1}$.

To see the necessity of our approach we first explain why the conventional replicator dynamic, exemplified by Weibull (1995), does not apply in our setting. Section 4.1 of Weibull derives a discrete-time version of the replicator dynamic of the form

$$q_{i,t+1} = \frac{\alpha + \pi_{i,t+1}}{\alpha + \bar{\pi}_{t+1}} q_{i,t}$$

where α represents the 'background birthrate'. Under this dynamic the predictor proportions from our model satisfy $\sum_i q_{i,t+1} = 1$. But a vector of proportions must satisfy two conditions: the sum of its elements must be one, and each individual element must lie in the unit interval. In the usual setting the fitness measure attached to the replicator dynamics is non-negative. In the Cobweb

model with rationally heterogeneous expectations, however, profits can take negative values. This implies that under Weibull's replicator dynamic the time path for q_{t+1} could lie outside the unit simplex. Instead, we seek a generalization of the Sethi–Franke dynamic that guarantees that the proportions will sum to one and each proportion will lie in the unit interval.

We proceed as follows. Given q_t, a reasonable dynamic should decrease $q_{i,t+1}$ if $i \in \hat{B}(q_t)$. To this end, let

$$r : (-\infty, 0) \to (-1, \delta),$$

with $\delta \leq 0$, be so that $r' > 0$. The function r governs the rate at which predictor proportions evolve. For $i \in \hat{B}(q_t)$ we set

$$q_{i,t+1} = (1 + r(\pi_{i,t+1} - \bar{\pi}_{t+1}))q_{i,t}. \tag{6}$$

Since $x < 0$ implies $r(x) < 0$ it follows that $i \in \hat{B}(q_t) \Rightarrow q_{i,t+1} \leq q_{i,t}$, with equality only when $q_{i,t} = 0$. Also, because we have decreased a non-negative number by a percentage, we are guaranteed that the new proportion is non-negative. The benefit to this approach is that if we instead raised those $q_{j,t}$ associated to positive net profits, we could not guarantee the new proportions would be less than one.

Given q_t, if $i \in \hat{G}(q_t)$ then we want the value of $q_{i,t+1}$ to increase. To ensure the proportions maintain their desired properties, we must impose that this increase comes from the decreased $q_{j,t+1}$. Thus for each $j \in \hat{B}(q_t)$, we distribute the amount by which $q_{j,t+1}$ decreased to the $q_{i,t+1}$ for $i \in \hat{G}(q_t)$, and we weight the amount given to a particular $q_{i,t+1}$ by the deviation from the mean of the corresponding net profit; this is natural as through this mechanism, better performing predictors receive more weight. Specifically, for $i \in \hat{G}(q_t)$, let

$$w_i(q_t) = \frac{(\zeta/|\hat{G}(q_t)|) + \pi_{i,t+1} - \bar{\pi}_{t+1}}{\zeta + \displaystyle\sum_{j \in \hat{G}(q_t)} \pi_{j,t+1} - \bar{\pi}_{t+1}},$$

where $|\hat{G}(q_t)|$ denotes the cardinality of the set $\hat{G}(q_t)$. The terms containing ζ are present in the weighting scheme to deal with the possibility that $\hat{B}(q_t)$ is empty and $\pi_{i,t+1} = \bar{\pi}_{t+1}$ for all i. Choosing ζ small limits the increase in the proportion of those agents using a predictor that exactly yields the average.[7] Notice $w_i(q_t) \in (0, 1]$ and $\sum w_i(q_t) = 1$. For $i \in \hat{G}(q_t)$ we may now set

$$q_{i,t+1} = q_{i,t} - w_i(q_t) \sum_{j \in \hat{B}(q_t)} r(\pi_{j,t+1} - \bar{\pi}_{t+1})q_{j,t}, \tag{7}$$

where the negative sign comes from the fact that $j \in \hat{B}(q_t)$ implies $r(\pi_{j,t+1} - \bar{\pi}_{t+1}) < 0$. Notice that if $\hat{B}(q_t)$ is empty then $\pi_{i,t+1} = \bar{\pi}_{t+1}$ for all i and we necessarily have that $q_{i,t+1} = q_{i,t}$.[8] We have the following lemma.

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[7] This replicator dynamic has a jump discontinuity in the following sense: if $\pi_j < \bar{\pi}$ then q_j decreases, but if $\pi_j = \bar{\pi}$ (and there is some i so that $\pi_i \neq \bar{\pi}$) then q_j increases. This jump can be made arbitrarily small by choosing $\delta = 0$ and ζ small.

[8] This property no longer holds if $\bar{\pi}$ is defined as the weighted average of profits across firms. For example, suppose $\pi_{i,t+1} = \pi \;\; \forall i \in H \subset \hat{K}, \pi_{j,t+1} > \pi, q_{j,t} = 0 \;\; \forall j \in \hat{K}/H$. Then $\hat{B}(q_t)$ is empty.

Lemma 1. *Suppose q_t is a vector of proportions and q_{t+1} is determined by recursions Eqs. (6) and (7). Then q_{t+1} is a vector of proportions.*

All proofs are contained in the Appendix available on the JEBO website. This Lemma shows that, when initialized with a vector of proportions, the replicator dynamic, which is summarized by the equations below, produces a vector of proportions.

$$q_{i,t+1} = \begin{cases} q_{i,t} - w_i(q_t) \displaystyle\sum_{j \in \hat{B}(q_t)} r(\pi_{j,t+1} - \bar{\pi}_{t+1})q_{j,t} & \text{if } i \in \hat{G}(q_t) \\ (1 + r(\pi_{i,t+1} - \bar{\pi}_{t+1}))q_{i,t} & \text{if } i \in \hat{B}(q_t) \end{cases} \tag{8}$$

The model is now closed and we can thus define our equilibrium concept.

Definition. The replicator equilibrium dynamics is the system defined by Eqs. (3), (5) and (8).

We note that in the case of two predictors the usual replicator dynamic and our generalization here are qualitatively equivalent. When there are only two predictors, the proportion lost from the one in $\hat{B}$ is automatically adjusted to the other predictor according to its weight (which is one in this case). Thus, the replicator dynamic developed in this paper is a natural generalization of the usual dynamic in a two predictor model.

Our goal is the analysis of this dynamic system, particularly in comparison to the dynamics of the systems defined by Brock and Hommes (1997) and Sethi and Franke (1995).

3. Analytic results

This section collects analytic results on the dynamic system. The next section will present detailed numerical analysis of the non-linear dynamics. In this section, we will first present the results for the RED when $K > 2$ and compare these with the results for the ARED. We will then turn to the special case $K = 2$, which will be the focus of the remainder of the paper.

3.1. Vanishing of dominated predictors

A significant difference between the RED and the ARED of Brock and Hommes is that replicator dynamics yields the intuitively appealing property that the proportion of agents using strictly dominated predictors is always decreasing, and if the rate function is so that $\delta < 0$, this proportion vanishes.

Proposition 2. *Let (p_t, q_t) be an equilibrium path of the RED. If along this path, for all $j' \neq j$ and for all t, $\pi_{j,t} < \pi_{j',t}$, then $q_{j,t} < q_{j,t+1}$. If $\delta < 0$ then $q_{j,t} \to 0$.*[9]

The intuition for this result is as follows. A strictly dominated predictor will always yield a negative net payoff, and thus the proportion of agents using this predictor will be continually diminished.[10] The vanishing of dominated predictors contrasts with Brock and Hommes (1997), for example, in a steady-state. In a steady-state all predictors return the same forecast. Under the RED those predictors with positive costs will lose agents, so in a steady-state, the proportion of

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[9] The requirement that $\delta < 0$ is sufficient but not necessary; one could also place restrictions on the rate at which $r(x) \to 0$ as $x \to 0$ from below.

[10] Notice that this result does not depend on the cobweb model itself.

agents using those predictors must be zero. In the ARED, though, for finite β a positive proportion will always pay a cost to choose a predictor that is strictly dominated.

The difference between the RED and ARED is non-trivial. In Brock and Hommes (1997) instability and complicated non-linear dynamics arise for large but finite β. For such values of β the steady-state will consist of all predictors. A similar instability finding in the RED is not at all obvious because in the RED strictly dominated predictors vanish. In the remaining sections we will show that instability exists in the RED.

3.2. Special case: rational versus naive expectations

To more closely examine the model's dynamics we follow Brock and Hommes (1997) in restricting ourselves to a simple example of the general model. We assume linear supply and demand. The supply equation is derived from profit maximization given a quadratic cost function. We also assume the predictor set is restricted to rational versus naive expectations.[11] Rational expectations in this set-up are equivalent to perfect foresight. Naive expectations are simply last period's realized price.

The model is now given by the equations

$$D(p_{t+1}) = A - Bp_{t+1}, \qquad S(p^{t+1}) = q_t b p_{t+1} + (1 - q_t) b p_t$$

where A, b, $B \in \mathbb{R}_+$, and q_t is the proportion of rational agents in t. Without loss of generality we set $A = 0$. Since agents are divided between the two predictors, $1 - q_t$ is the proportion of naive agents. We focus on the 'unstable' Cobweb case of $b/B > 1$. To emphasize this we note the following:

Remark 3. When the slopes of supply and demand satisfy $b/B > 1$, the 'steady-state' is unstable under naive expectations.

Setting supply equal to demand and solving leads to the equilibrium law of motion,

$$p_{t+1} = \frac{-(1 - q_t) b p_t}{B + q_t b}. \tag{9}$$

As in the general model, predictor choice at time $t + 1$ depends on profits at $t + 1$ and hence price at $t + 1$. In this stylized example the profits associated with the predictors follow the processes

$$\pi_{R,t+1} = \frac{(1/2)(1 - q_t)^2 b^3 p_t^2}{(B + q_t b)^2} - C, \qquad \pi_{N,t+1} = \frac{b}{2} p_t^2 \left(\frac{(-(2 - q_t)b - B)}{B + q_t b} \right), \tag{10}$$

where Eq. (9) has been used to eliminate dependence on p_{t+1}. These profits are computed as the actual realized profits at time $t + 1$ given expectations and predictor proportions in time t.[12] Because agents adapt their predictor choice for p^e_{t+2} in time $t + 1$, this is the relevant metric for assessing forecast accuracy, and thus for determining q_{t+1}.

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[11] As shown in Branch (2002) the addition of other stabilizing predictors alters the regions of instability but leaves the central conclusions unchanged.

[12] Given quadratic costs, these formulas are found by plugging (9) directly into the profit function (5) advanced one period.

The numerical analysis section requires a functional form for the rate function. We choose

$$r(x) = \left(\frac{2}{\pi}\right) \tan^{-1}(\alpha x) \tag{11}$$

where $\alpha > 0$. However, the proposition below does not depend on this functional form.[13]

The predictor proportions follow

$$q_{t+1} = \begin{cases} 1 - (1 + r(\pi_{N,t+1} - \bar{\pi}_{t+1}))(1 - q_t) & \text{if } \pi_{R,t+1} > \pi_{N,t+1} \\ q_t & \text{if } \pi_{R,t+1} = \pi_{N,t+1} \\ (1 + r(\pi_{R,t+1} - \bar{\pi}_{t+1}(q_t)))q_t & \text{if } \pi_{R,t+1} < \pi_{N,t+1} \end{cases} \tag{12}$$

These laws of motion follow directly from Eq. (8). To see this, note that if $\pi_{R,t+1} > \bar{\pi}_{t+1}$, then $\pi_{N,t+1} < \bar{\pi}_{t+1}$ so that the weight $w_R(q_t)$ associated to the rational agents is equal to one (see Eq. (8)). Moreover, $\pi_{R,t+1} > \bar{\pi}_{t+1}$ if and only if $\pi_{R,t+1} > \pi_{N,t+1}$. Thus the rational predictor rises precisely by the amount the naive predictor falls:

$$(1 - q_{t+1}) = (1 + r(\pi_{N,t+1} - \bar{\pi}_{t+1}))(1 - q_t).$$

When $\pi_{R,t+1} < \bar{\pi}_{t+1}$ it follows that

$$q_{t+1} = (1 + r(\pi_{R,t+1} - \bar{\pi}_{t+1}))q_t.$$

Finally, when $\pi_{R,t+1} = \bar{\pi}_{t+1}$, it must also be that $\pi_{N,t+1} = \bar{\pi}_{t+1}$. In this case, Eq. (8) would set $q_{t+1} = q_t$ (because $\hat{B}$ is empty); this is imposed directly into Eq. (12).

Notice that for larger values of α, agents respond more quickly to changes in realized profits. The parameter α is the replicator dynamic analog to the 'intensity of choice' parameter β in the ARED. However, in the RED, α directly controls the speed of adaption. This carries a subtly different interpretation from the 'intensity of choice' in the ARED, which is inversely related to the variance of the noise term in the random utility model. The two-equation system (9) and (12) is the RED.

Note again that the timing of the model is crucial. In each time period t, the equilibrium price is determined by expectations formed in time $t - 1$. Agents' expectation formation is, essentially, the choice of whether to pay the cost for rational expectations or use the naive predictor at no cost. Once the new price is observed in time t, agents see realizations of profits and again calculate their expectations of p_{t+1}. Thus, both predictor proportions and equilibrium price follow a system of first order difference equations. We again stress that even though the perfect foresight predictor does not satisfy the restriction placed on the set of forecasting models given in the general model, the timing here and above is identical. See Brock and Hommes (1997) for further details.

3.2.1. Persistence of rational agents

One of the primary results in Sethi and Franke (1995) is that a stochastic model may lead agents to select rational expectations with positive probability even though in a non-stochastic setting rational expectations are strictly dominated by naive expectations. This result follows because the addition of (small) noise makes it possible (probabilistically speaking)

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[13] Also, the numerical results presented in Section 4 appear robust for at least some specifications of the rate function. In particular, we obtained similar results for the rate function $r(x) = e^{\alpha x} - 1$.

that the system will be moved into a region where the benefits to rational expectations out-weighs its costs. Sethi and Franke establish this important result by showing the existence of a unique invariant measure whose support is a subset of the unit interval. The following propo-sition demonstrates that an analogous result holds in the non-stochastic version of the Cobweb model.

Proposition 4. *Assume $p_0 \neq 0$ and $b/B > 1$.*

1. *There exists $q^* \in (0,1)$ so that $q_s < q^*$ implies there exists $t > s$ with $q_t > q^*$.*
2. *There exists $\tilde{q} \in (0,1)$ so that $q_s > \tilde{q}$ implies there exists $t > s$ with $q_t < \tilde{q}$.*

This proposition tells us that under RED, provided $b/B > 1$, coexistence of rational and naive predictors obtains. Importantly, this result holds even though, unlike Sethi–Franke, there is no noise in the model. Moreover, the model is such that in a steady-state, rational expectations are strictly dominated and will vanish asymptotically.

Intuitively, it is precisely the instability of the fixed point under naive expectations that yields the result. The persistent Cobweb oscillations generate orbits that have time varying predictor proportions. The model moves away from the steady-state when there is a large proportion of naive agents. Once the price trajectory is sufficiently far from the steady-state, rational expectations will dominate naive expectations and the dynamics move back towards the steady-state. However, like the $\beta < +\infty$ case of Brock and Hommes (1997), the switch to rational expectations by agents is not complete, so eventually the dynamics will again be repelled from a neighborhood of the steady-state.

This proposition emphasizes the robustness of the Sethi–Franke result. Their model relied on strategic complementarities and current expectations of future variables; we find similar results in a model dependent on production lags and past expectations of current variables. Further, we find that the introduction of exogenous noise is not necessary to guarantee that the rational predictor will continue to be selected by some agents.

3.2.2. Local stability analysis

The main instability result of Brock and Hommes may be stated as follows:

Theorem 5 (Brock and Hommes, 1997). *Assume that $C_1 > C_2 > \cdots > C_K$. Moreover, assume the steady-state $p^* \equiv (p^*, \ldots, p^*)$ is unstable under homogeneous beliefs H_k. If the intensity of choice β is sufficiently large then the steady-state p^* (under the ARED) is locally unstable.*

An analogous result holds for the replicator dynamic, though some care must be taken in its derivation. We proceed with this exercise now.

Assume throughout this section that $C > 0$. Recall the dynamic system as given by Eqs. (9) and (12) with the profit terms given by Eq. (10). Since both B and b are assumed to be positive, the only possible fixed point to Eq. (9) is $p^* = 0$. In this case, $\pi_R = -C < \pi_N = 0$, so that the evolution of q_t is governed by

$$q_t = (1 + r(\pi_{R,t} - \bar{\pi}_t))q_{t-1}, \tag{13}$$

which, given the properties of the rate function, has a unique fixed point given by $q^* = 0$.

Let $\theta = (p,q)'$, and write the dynamics of θ as $\theta_t = F(\theta_{t-1})$. The previous paragraph demonstrates that F has a unique fixed point at $\theta^* = 0$. Notice that because of the economics behind the derivation of the dynamic system given by F, we think of F as a map from $\mathbb{R} \times [0, 1]$ to itself. Thought

of this way, the fixed point is on the boundary of the relevant space, making stability analysis *a priori* difficult. However, there is nothing in the mathematical definition of F that requires this restriction. So now think of F as a map acting on the plane, and notice that $\mathbb{R} \times [0, 1]$ is simply an invariant space under F.

The profit functions are continuous at the origin, so there is an open set containing θ^* so that the dynamics implied by F are given by Eqs. (9) and (13). This implies that F is differentiable at θ^* and the usual stability analysis may be performed. The derivative of F evaluated at θ^* is easily computed to be

$$DF = \begin{pmatrix} -\dfrac{b}{B} & 0 \\ 0 & 1 + r\left(-\dfrac{C}{2}\right) \end{pmatrix}. \tag{14}$$

Noting that $1 + r \in (0,1)$, it follows that the fixed point is stable if and only if $b/B < 1$, which yields the following proposition analogous to Brock and Hommes:

Proposition 6. *Assume $C > 0$, and that the fixed point $p^* = 0$ is unstable under naive expectations. Then the unique steady-state $(p^*, q^*) = (0,0)$ (under RED) is unstable.*

As is evident from Eq. (14), the ratio b/B acts as a bifurcation parameter. It is worth noting that unlike Brock and Hommes (1997), α does not impact the local stability properties. It does, however, determine how strongly the stable eigenvalue contracts. Thus, in the numerical section below we investigate α's impact on the system's dynamics.

The differentiability of the dynamic system F implies the potential for bifurcation analysis. In particular, as b/B increases past 1, the steady-state under RED is destabilized, and the relevant eigenvalue of the derivative of F (see Eq. (14)) crosses -1, which indicates the possibility of a flip bifurcation. This situation may be analyzed by performing the standard center manifold reduction (see Kuznetsov, 1995) and analyzing the restricted dynamics. Proceeding with this computation, we find the second and third order terms of the restricted system are zero, so that the usual results characterizing the bifurcation fail to hold. Because of this, we turn to numerical analysis to provide evidence of complicated dynamics.

4. Numerical analysis of global dynamics

In this section we turn to numerical analysis to characterize the non-linear dynamics of the RED model. Our aim in this section is to demonstrate that the complicated dynamics found in Brock and Hommes (1997) also occur in the RED. Using standard (though technical) methods in the literature on non-linear dynamics, Brock and Hommes were able to show analytically the presence of stable cycles of arbitrarily high order and demonstrate the existence of chaotic attractors. There are many texts available discussing these techniques: see for example, Guckenheimer and Holmes (1983) and Palis and Takens (1993); also, Brock and Hommes (1997) includes a review of the relevant results. As indicated in the previous section, the standard methods of analysis fail for our model, so we use numerical techniques that demonstrate unequivocally the presence of stable cycling behavior and further that strongly suggest the presence of complex dynamics and chaotic attractors. The relevant numerical analysis is reviewed in Brock and Hommes (1998), and it is to these methods that we appeal when making the arguments below.

We will provide two types of numerical results to demonstrate the existence of complex behavior and to suggest the presence of chaos. First we compute bifurcation diagrams by varying the parameter α and plotting the associated orbits of q. We show there are regions of the parameter space for which stable cycles exist, and as the parameter α is increased, the order of these cycles repeatedly increases. Second, we plot bifurcation diagrams by varying b/B for fixed α and plotting the associated orbits of p. Notice that unlike Brock and Hommes (1997) we treat b/B as a bifurcation parameter in addition to α. This is because b/B induces the primary bifurcation. Given $b/B > 1$, α affects the dynamic behavior qualitatively by changing the magnitude of contractions along the stable manifold. We will also see regions for which the associated orbit of q appears to be a dense subset of a subinterval of $[0,1]$, providing evidence for, though not proving, the existence of aperiodic orbits.

We then analyze the orbits of the vector (p,q) in phase space. We find values of α for which stable periodic orbits obtain. As α is increased the period doubling phenomenon appears, which was already seen in the bifurcation diagram. As α is further increased in order to correspond to observed dense orbits, the presence of what appear to be strange attractors is detected. We conclude, from these numerical results, that our model exhibits complex behavior similar to that exhibited by the model of Brock and Hommes.

4.1. Bifurcation diagrams

We begin with the consideration of the bifurcation plots. In their analysis, Brock and Hommes obtained a primary bifurcation that was a flip or period doubling bifurcation. This indicates that for values of the 'intensity of choice' parameter below a critical value, the steady-state is stable. As the 'intensity of choice' crosses the critical value, the steady-state becomes unstable, and a stable two-cycle emerges. This process repeats ad infinitum. Because the stability properties of the steady-state do not depend on α, we do not see a primary period-doubling bifurcation as α varies. Instead we pick a value of b/B for which the steady-state has already bifurcated (see below). As is evident from the bifurcation diagrams, cycling and/or complex dynamics obtain for all values of α considered: see Figs. 1 and 2.

Figs. 1 and 2 illustrate bifurcations of the predictor proportion q as α is varied for slope values $b/B = 1.1$ and $b/B = 2$. These parameter values both fall in the range of chaotic dynamics analyzed in Brock and Hommes (1997). The bifurcation diagrams were created by running a simulation of 1000 periods (following a transient period of length 10,000) for various values of α in a 1000 point grid of the interval $(0,10)$. Throughout, initial conditions are drawn uniformly from the interval $[0,1] \times [0,1]$. Reading from left to right it is clear that the periodicity or aperiodicity of the orbits depends critically on the parameter α.

We see from these figures that cycling phenomena arise for various values of α. In many regions that correspond to cycling, as the parameter α is increased, the period length increases and results in a cascade to cycles of high order.

For many regions of the parameter space, particularly those regions with high values of α, the associated orbits of q appear to form dense subsets in their containing subintervals. Of course, one can never guarantee numerically that the orbits are in fact dense, as they may correspond to cycles of large length. However even if the orbits are not dense the associated dynamics of q are quite complicated. This will be further demonstrated in the subsection below.

Fig. 3 demonstrates the primary bifurcation as the ratio b/B crosses one. The bifurcation diagram plots values of p_t against b/B, where α is set to be 1. The ratio b/B was not treated as a bifurcation parameter in Brock and Hommes (1997) since the 'intensity of choice' parameter
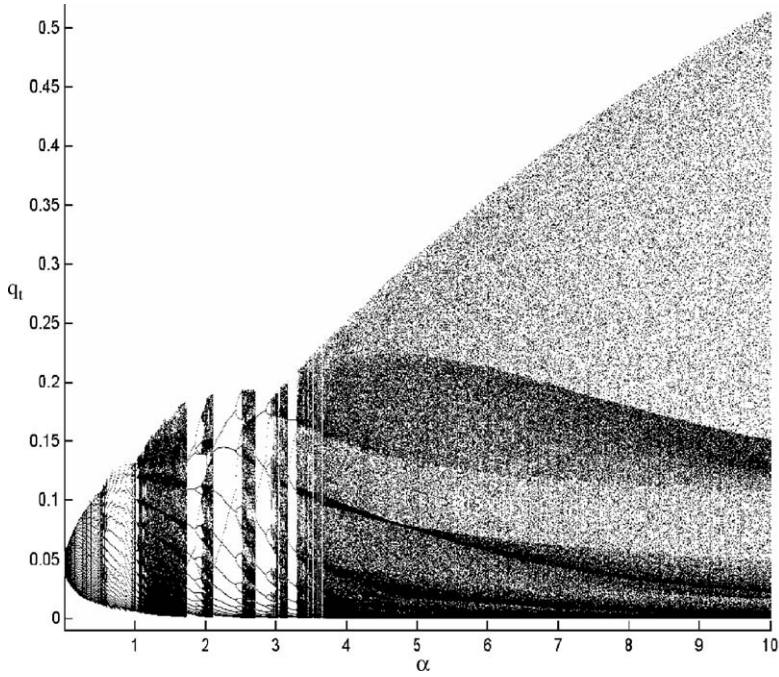
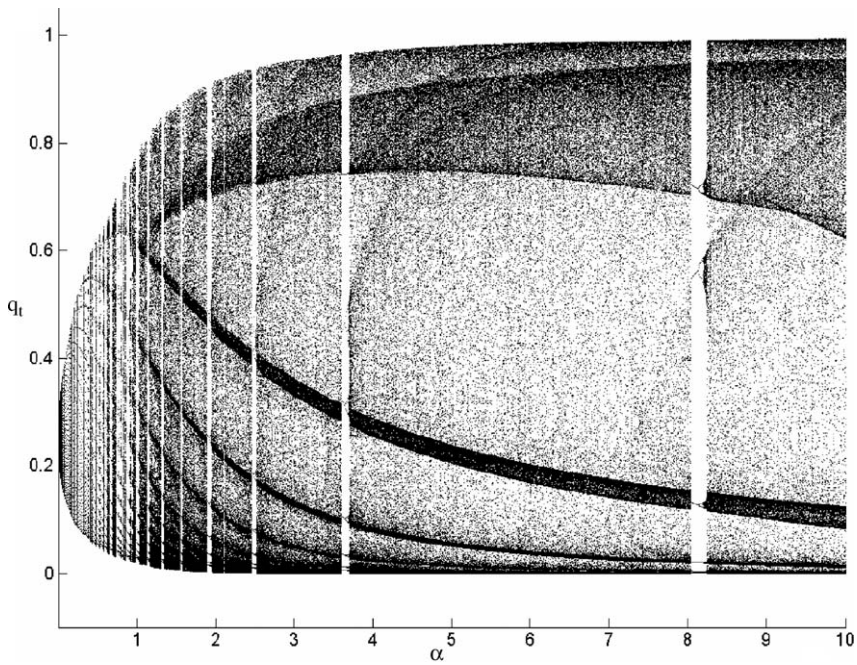Fig. 1. Bifurcation diagram for q_t when $b/B = 1.1$.
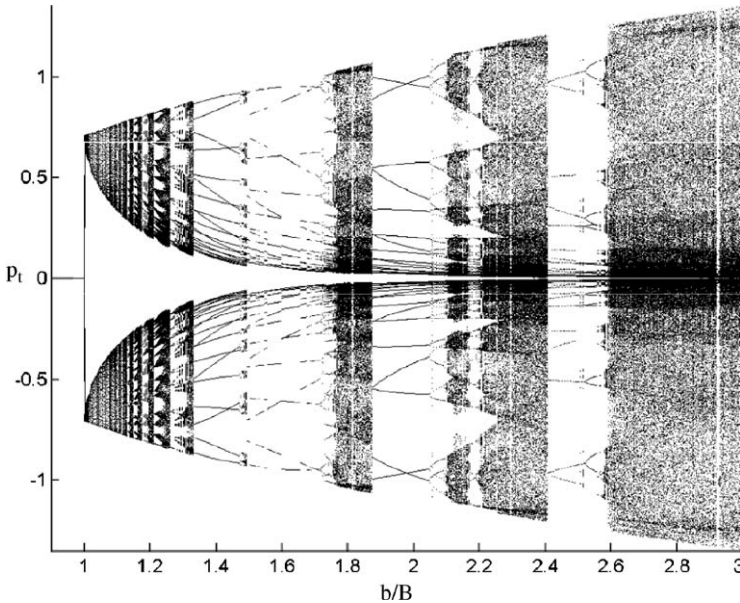

Fig. 2. Bifurcation diagram for q_t when $b/B = 2$.

Fig. 3. Bifurcation diagram for various values of b/B.

acted as the critical bifurcation parameter. In the RED α has no bearing on the instability of the steady-state, so we require, as Fig. 3 demonstrates, that b/B first bifurcate the system.[14]

Each bifurcation can be examined more closely in time-series plots. Figs. 4–7 show typical trajectories when the relative slopes are $b/B = 2$. Each figure corresponds to a different value of α and each makes the normalization $C = 1$. Each plot shows various orbits from the bifurcation diagram in Fig. 2. In Fig. 4 ($\alpha = 1.9$), price follows a regular oscillatory path around the unstable steady-state. Fig. 4 shows the arrival of a stable period 18 orbit. Figs. 5–7 illustrate the dynamic effects of increasing the adaption parameter α. Fig. 5 illustrates that for a larger value of α (1.935) a stable period 36 orbit arises. Eventually, as α increases further, the periodic behavior disappears and the chaotic, aperiodic behavior seen in Figs. 6 ($\alpha = 2.3$) and 7 ($\alpha = 9$) prevail. These more complicated dynamic paths correspond to the regions in the bifurcation diagram that suggest the existence of dense orbits.

4.2. Attractors

In this subsection we take a closer look at the stability properties of the model. We show evidence of attractors that are stable cycles. We then find that by altering the value of α, these attracting cycles bifurcate into cycles of twice the order. This process continues, and eventually, as α continues to increase, these attractors become increasingly complex and appear to evolve into chaotic attractors.

Figs. 8–11 illustrate the bifurcations, periodic attractors, and evidence for the existence of strange attractors. In each figure we set $b/B = 2$, $C = 1$, and ran simulations of 50,000 periods.

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[14] Note that as b/B crosses 1, while the steady-state becomes unstable and the system clearly bifurcates, the subsequent behavior of p, and thus the nature of the bifurcation, is difficult to describe. This is consistent with our findings that the primary bifurcation defied the standard center manifold reduction technique.
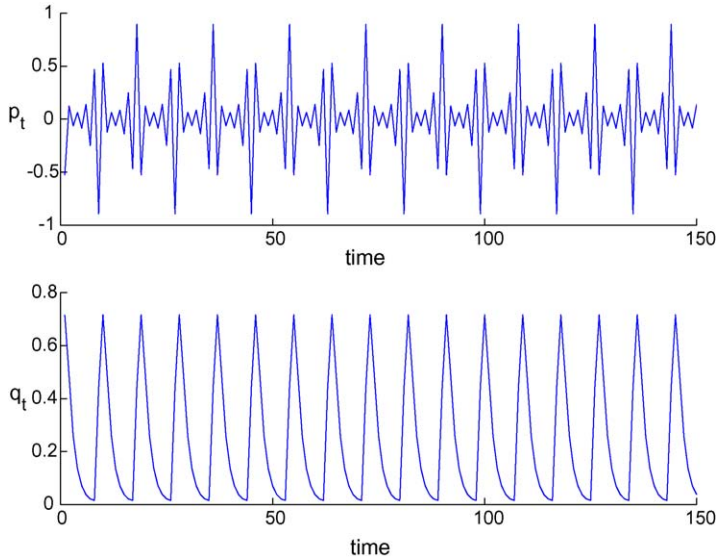
Fig. 4. Trajectory for (p_t, q_t) when $b/B = 2$ and $\alpha = 1.9$.

The figures contain plots in (p,q)-space of trajectories of length 40,000, after an initial 10,000 transient period. Fig. 8 corresponds to an α value of 1.9 and clearly shows the existence of a period 18 attractor. When α is increased slightly to $\alpha = 1.935$ (Fig. 9) a new attractor arises which has double the period of $\alpha = 1.9$. As α is increased further to, say, $\alpha = 1.94$ (Fig. 10) the increasing length of the cycles leads to an attractor that appears chaotic. It is this cascading of bifurcations
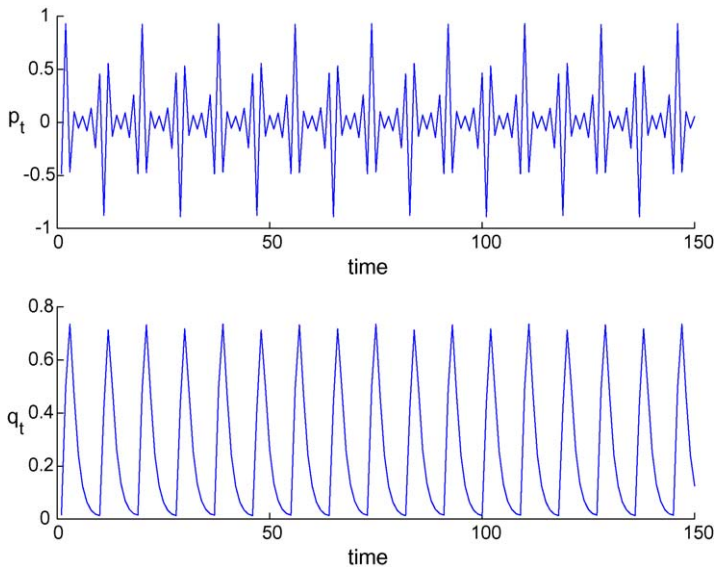
Fig. 5. Trajectory for (p_t, q_t) when $b/B = 2$ and $\alpha = 1.935$.
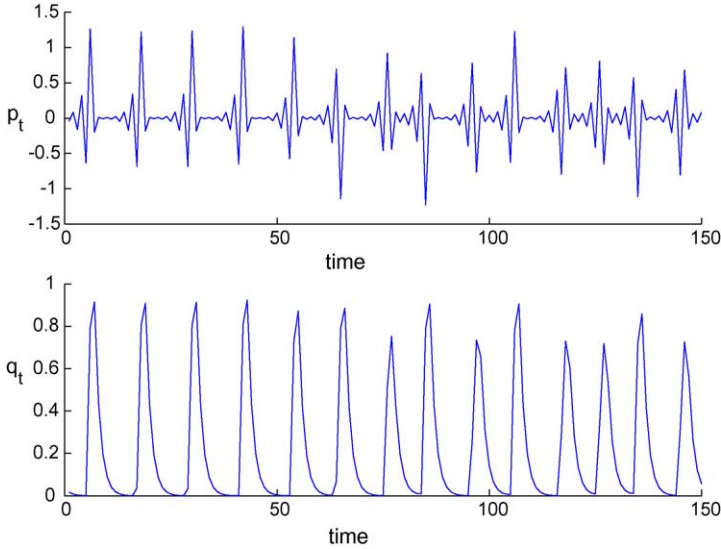
Fig. 6. Trajectory for (p_t, q_t) when $b/B = 2$ and $\alpha = 2.3$.

that suggests chaos. As α is further increased to $\alpha = 2.3$ the phase plot in Fig. 11 now suggests a strange attractor.

That the steady-state acts as a repellor under naive expectations and an attractor under rational expectations is the source for the complicated dynamics seen in Brock and Hommes' ARED and of the RED here. It has been emphasized by many in the chaotic dynamics literature (e.g. Guckenheimer and Holmes) that stability properties that depend on the 'regime' are an important
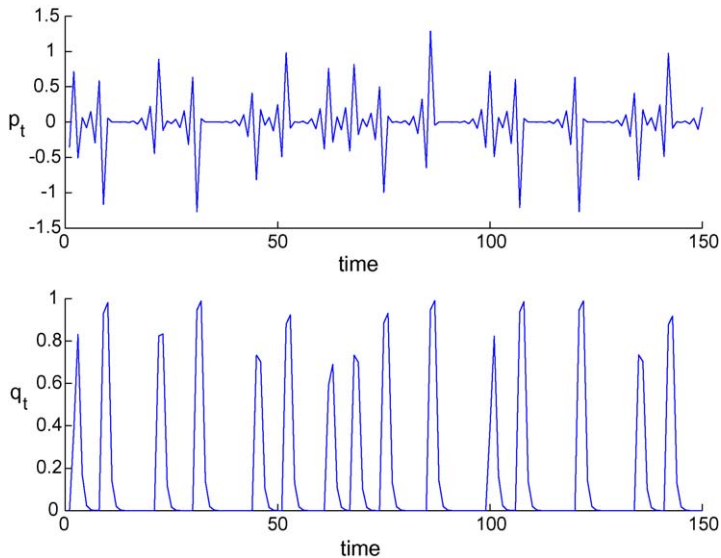
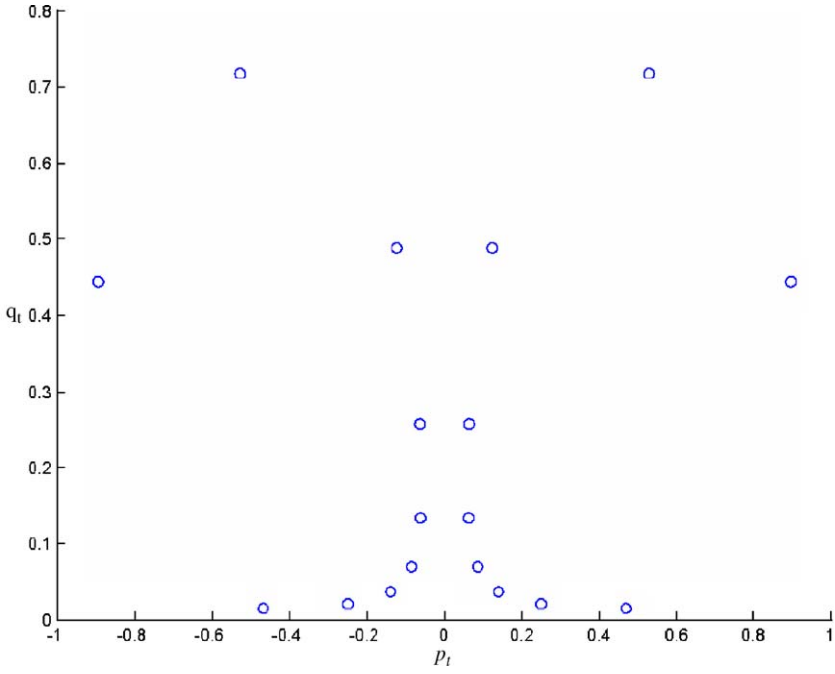Fig. 7. Trajectory for (p_t, q_t) when $b/B = 2$ and $\alpha = 9$.
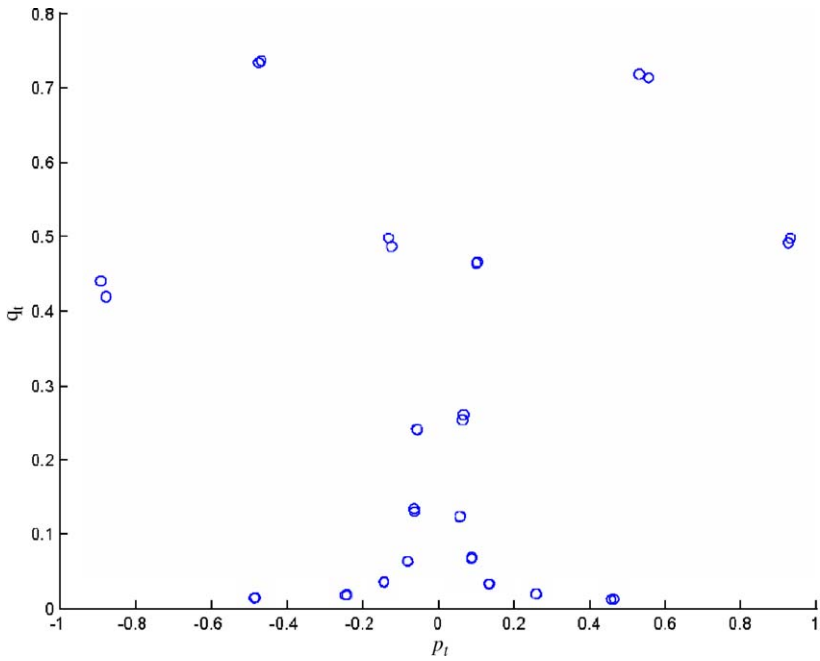
Fig. 8. Phase plot when $b/B = 2$ and $\alpha = 1.9$.

Fig. 9. Phase plot when $b/B = 2$ and $\alpha = 1.935$.
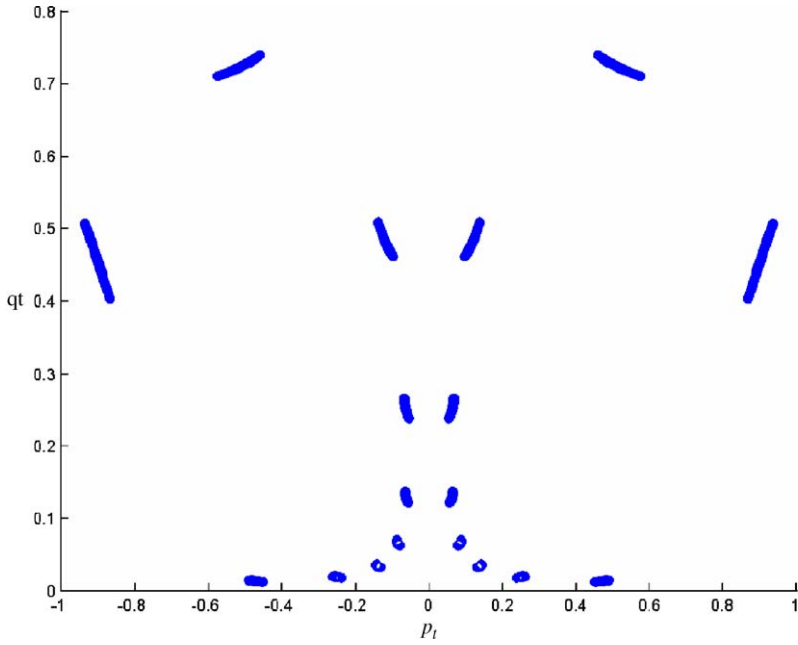
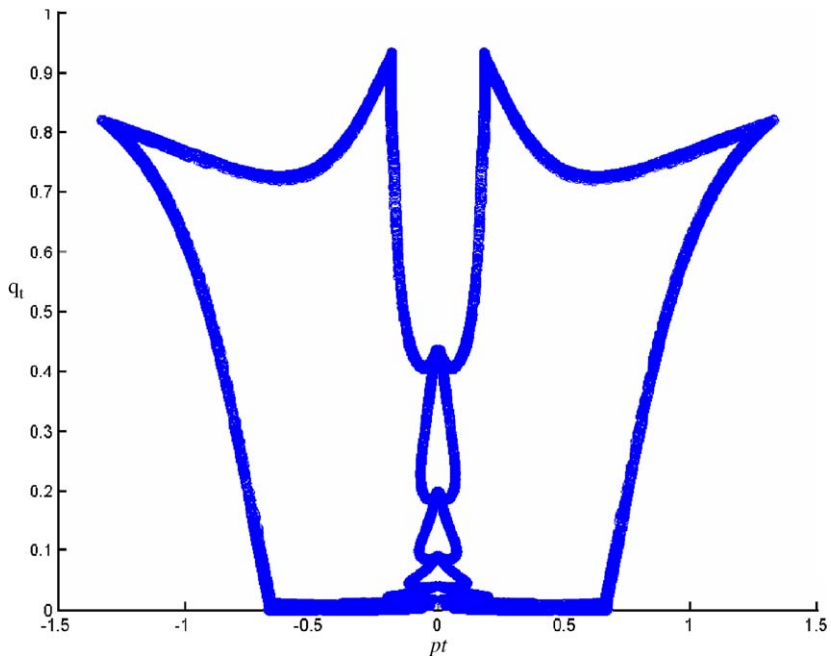Fig. 10. Phase plot when $b/B = 2$ and $\alpha = 1.94$.

Fig. 11. Phase plot when $b/B = 2$ and $\alpha = 2.30$.

source of chaotic behavior in multidimensional non-linear systems. This point has been made clearly in the applications of the ARED in other settings by Brock and Hommes (1998), Brock et al. (2005) and Droste et al. (2002), and we see its importance here in the case of a replicator dynamic.[15]

5. Conclusion

This paper introduces the replicator dynamic into a Cobweb model with rationally heterogeneous expectations. We define the coupled non-linear system, which consists of the equilibrium law of motion for price and the replicator dynamic that governs predictor proportions, as the replicator equilibrium dynamics (RED). The replicator dynamic used here is inspired by Sethi and Franke; however, we generalize their replicator to govern a vector of predictor proportions of arbitrary finite length. We compare the resulting dynamics with the ARED of Brock and Hommes and the model of strategic complementarity of Sethi and Franke.

We find support for both the main results of Brock and Hommes (1997) and Sethi and Franke (1995). Our analytic results show that it is possible to generalize the Sethi–Franke replicator dynamic to a model with an arbitrarily large finite number of predictors. We show that under the replicator dynamic, unlike in Brock and Hommes (1997), strictly dominated predictors vanish asymptotically. However, the instability result of Brock and Hommes obtains in case of naive versus rational predictors: the steady-state of a Cobweb model under RED is unstable if it is unstable when all agents use the naive predictor. Further, this instability result follows from a surprising deterministic analog to the main result of Sethi and Franke: costly, sophisticated predictors do not necessarily vanish asymptotically.

This latter result is interesting and surprising. In Sethi and Franke's model, agents' decisions strategically complement each other. Thus, the stochastic nature of their model implies that with positive probability some agents will be rational. This in turn implies that other agents will also want to be rational since agents' actions are complementary. In the RED, there is a negative feedback from expectations, and agents have an incentive to deviate from consensus actions.[16] We find that the instability of the steady-state under naive expectations and the oscillatory nature of the dynamics of the model add the necessary volatility for a positive proportion of agents to select rational expectations. We are also able to demonstrate numerically that the complicated dynamics of Brock and Hommes (1997) arise in the RED. This too is interesting as it indicates that chaos is not a feature particular to the set up of Brock and Hommes, but may in fact be generic behavior.

The results of this paper suggest that complicated dynamics in models with dynamic predictor selection such as Brock and Hommes are an important source of volatility. We show that these endogenous dynamics may generate the necessary volatility exogenously imposed by Sethi and Franke (1995) to insure the survival of rational agents. The results also imply that the exact evolutionary nature of the dynamics is not important for generating complex dynamics. It is the dual repellor/attractor property of expectations around a steady-state that yield the complex dynamics.

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[15] The CeNDEF has many researchers devoted to studying non-linear systems with dynamic predictor selection. Many of the most promising applications are in asset pricing models.

[16] This point is made in Branch and Evans (2006) who construct a stochastic version of the cobweb model where agents may choose between underparameterized models.

Acknowledgements

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jebo.2005.09.003.

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