ELSEVIER

Journal of Economic Dynamics & Control
27 (2002) 63–85

JOURNAL OF
Economic
Dynamics
& Control

www.elsevier.com/locate/econbase

Local convergence properties of a cobweb model with rationally heterogeneous expectations

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Received 16 February 1999; accepted 26 March 2001

Abstract

This paper gives local stability conditions for convergence of the price dynamics in a cobweb model with rationally heterogeneous expectations, generalizing the example of Brock and Hommes (1997). When agents choose between rational, naive, and adaptive beliefs, the steady state may be locally asymptotically stable if the adaptive predictor places enough weight on past prices and is costless. If adaptive expectations are sufficiently more costly than naive expectations the steady state will be an unstable saddle point. Our results imply that adding a choice can stabilize a system which is unstable under the Brock and Hommes model. These results illustrate how the critical parameter that governs stability is dependent on the array of available predictors.

JEL classification: C62; D83; D84; E30

Keywords: Cobweb model; Heterogeneous beliefs; Adaptive learning; Rational expectations

1. Introduction

The rational expectations hypothesis continues to have a dominant influence on dynamic macroeconomic research. Though many papers such as Bray

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0165-1889/02/$ - see front matter
PII: S0165-1889(01)00027-6

and Savin (1986), Evans and Honkapohja (1999), Guesnerie (1992), Evans and Guesnerie (1993), and Townsend (1978) examine the stability of rational expectations equilibria under various expectation formation schemes, among researchers there is no consensus about how to model the process of structuring beliefs. [1]

Recent work takes a more traditional rational approach to expectation formation by modeling it as an economic decision. Evans and Ramey (1992) assume that expectations are the by-product of a decision to use a costly algorithm to update prior beliefs, or, at no cost leave them fixed. Evans and Ramey (1998) extend this approach by having agents use a mechanism for directly calculating expectations, but they must pay a resource cost for the privilege. In more recent work, Brock and Hommes (1997) consider expectation formation as the rational choice between various costly forecasts.

This seminal approach of Brock and Hommes (1997), called the adaptively rational equilibrium dynamics (ARED), is an intuitively appealing treatment of the expectation formation issue. They consider a cobweb model that has agents choose a predictor from a finite set of expectations functions that are themselves functions of past information, with each predictor assigned a cost for its use.

To illustrate how the general expectation mechanism can generate local instability and complex global dynamics, they devote considerable attention to the case of the model with only rational (which is costly) and naive (which is costless) expectations. In this illustration, which is central to their paper, the system is assumed unstable under naive expectations. When price is far from its steady-state value, the costs of using the rational predictor will be outweighed by the (potential) benefits of forming a more accurate forecast. However, as the system returns to the steady state, both the sophisticated and naive predictors return the same forecast. Because the costs will outweigh the benefits, very few agents will use the sophisticated predictor and the steady-state may become unstable. This approach presents a theoretical underpinning to the more numerical analysis of Arthur et al. (1996).

Brock and Hommes (1997), however, do not fully capture the role played by the predictor set. This paper shows that the set of predictors available to choose from is an important determinant of the local stability criterion. It will be shown that the inclusion of a predictor, even an unsophisticated one, can counteract the influence of the naive predictor. As the size of the set increases, the range of slope values and the intensity of choice under which the steady state is locally stable will also increase.

[1] The papers by Arifovic (1994), Sethi and Franke (1995), Anderson et al. (1997), and Stahl (1993) look at similar issues utilizing genetic algorithms and evolutionary game theory. In these approaches, expectations are not something calculated, but are outcomes of a 'genetic' process.

To illustrate the main point of this paper consider the following thought experiment under the Brock and Hommes assumption of two available predictors. Suppose that in time t price is close, but greater than, its steady-state value and a large portion of agents use the naive predictor. Supply in $t+1$ will be closely determined by the price in t because a disproportionate number of firms use that price as their forecast. Because of the oscillatory nature of the cobweb model the actual price in $t+1$, as determined by the demand curve, will be less than the steady-state price. Consequently, the price in $t+2$ will be greater than its steady-state value. Brock and Hommes (1997) demonstrate that when the relative slopes of supply and demand are greater than one, and the 'intensity of choice' between predictors is high enough, this sequence will diverge.

This paper considers the ARED when there is a factor that dampens this oscillatory behavior of price. Allowing agents to choose another unsophisticated predictor, such as adaptive expectations, provides such a factor. Adaptive expectations are a natural choice to include in the model because it can closely resemble naive expectations or simple-averaging depending upon the size of the adaption parameter. Adaptive expectations were also a mainstay of models during the 1950s and 1960s. However, since adaptive expectations incorporate past information their influence seeks to dampen price oscillations. The results of this paper provide necessary and sufficient conditions for the dampening factor to induce the sequence of prices to converge to the steady state. When the cost of adaptive expectations is zero, these turn out to be equivalent to placing a sufficiently large weight on past prices. What constitutes sufficiently large depends on the slopes of supply and demand, the relative costs of the predictors, and the 'intensity of choice' between predictors. When the cost of adaptive expectations is positive, the range of parameters under which the system is stable increases from the case of rational versus naive expectations.

This paper thus shows that there are important issues not studied in the general results given by Brock and Hommes (1997), such as how the characteristics of the predictor set affect the convergence conditions. This paper also extends the results of the specific case of rational versus naive expectations in Brock and Hommes (1997) by introducing a third important predictor. It will be seen that not all of their stability results carry over to this case. Thus, this paper enhances the results of Brock and Hommes (1997).

This paper proceeds as follows. Section 2 presents the cobweb model with rationally heterogeneous expectations and introduces the case of rational, naive, and adaptive expectations. Section 3 reviews the Brock and Hommes (1997) model of rational versus naive expectations as a special case of the model in Section 2. The main stability results for the model with rational, naive, and adaptive beliefs are presented in Section 4. Finally, Section 5 consists of some concluding remarks.

2. Cobweb model with heterogeneous expectations

This section turns to a specific example of the ARED. It will extend the case of rational versus naive expectations that is the focus of Brock and Hommes (1997) by introducing adaptive expectations. To make the results comparable with that of the Brock and Hommes (1997) specific case, the setup for the cobweb model with rational, naive, and adaptive expectations will closely follow their model. The only change to their framework is the addition of a third choice predictor, adaptive expectations. This change, though small, leads to significant differences in results. The Brock and Hommes (1997) model assumes the simplest possible form of the ARED: agents choose between a perfect-foresight predictor (agents are able to perfectly predict next period's price) and a naive predictor (agents expect last period's price to prevail again). To extend these results, we assume that agents also choose a predictor which is a geometrically weighted average of past prices.

2.1. The cobweb model

In the Brock and Hommes (1997) framework, supply decisions are made by choosing that output which maximizes expected profits subject to the one-period production lag. That is,

$$\max_q P^e_{t+1} q - c(q), \tag{2.1}$$

where $c(q)$ is a cost function that is increasing in q. Price expectations are formed by choosing a predictor from a set of predictor functions. Given this heterogeneity in expectation formation, market supply is a weighted sum of the supply decisions of the heterogeneous agents. The weights are simply the proportion of agents using a specific predictor. That is, each agent chooses $H_j \in \{H_1, H_2, \ldots, H_K\}$ where each predictor depends upon a vector of past prices $\vec{P}_t = (P_t, P_{t-1}, \ldots, P_0)$. The fractions of agents using a given predictor, $n_{j,t}(P_t, \mathbf{H}(\vec{P}_{t-1}))$, depends upon price, P_t, and the vector of previous predictors $\mathbf{H}(\vec{P}_{t-1}) = (H_1(\vec{P}_{t-1}), H_2(\vec{P}_{t-1}), \ldots, H_K(\vec{P}_{t-1}))$. Therefore, market equilibrium is given by the equation:

$$D(P_{t+1}) = \sum_{j=1}^{K} n_{j,t}(P_t, \mathbf{H}(\vec{P}_{t-1})) S(H_j(\vec{P}_t)), \tag{2.2}$$

where $D(\cdot)$ is the demand function and $S(\cdot)$ is the supply function.

Because of the nature of the belief dynamics, the equilibrium equations for the cobweb model tend to be complicated. To keep the model analytically tractable, assuming linear demand and supply is important. Therefore, let demand and supply be given by the functions:

$$D(P_t) = A - BP_t,$$

$$S(H_j(\vec{P}_t)) = bH_j(\vec{P}_t), A, B, b \in \mathbb{R}_+. \tag{2.3}$$

Without loss of generalization to the stability properties, set A equal to zero. Market equilibrium when $H_j \in \{H_1, H_2, H_3\}$ is determined by the condition

$$D(P_{t+1}) = n_{1,t}S(H_1(\vec{P}_t)) + n_{2,t}S(H_2(\vec{P}_t)) + n_{3,t}S(H_3(\vec{P}_t)), \tag{2.4}$$

where the predictor functions are defined as [2]

$$H_1(\vec{P}_t) = P_{t+1} \text{ with cost } C \geq 0, \tag{2.5}$$

$$H_2(\vec{P}_t) = P_t \text{ with no cost,}$$

$$H_3(\vec{P}_t) = (1-\gamma)\sum_{k=0}^{t} \gamma^k P_{t-k} \equiv R_t \text{ with cost } \quad D \geq 0,\ 0 < \gamma < 1.$$

To keep the third predictor well-defined, we will always assume that γ is strictly less than one.

Each period, after observing the new price and assessing the accuracy of their forecasts, producers update their prediction of next period's price. The evolution of the proportions of agents using a particular predictor is given by

$$n_{j,t+1} = \frac{\exp[\beta U_{j,t+1}]}{\sum_{j=1}^{K} \exp[\beta U_{j,t+1}]}. \tag{2.6}$$

$U_{j,t+1}$ is a measure of the welfare associated with a certain predictor. The variable β parameterizes preferences over profits. The larger the β, the more likely a producer will switch to an expectation with slightly higher returns. Brock and Hommes call this the 'intensity of choice' parameter. Assume that $U_{j,t+1}$ is net realized profits such that

$$U_{j,t+1} = \pi_j(P_{t+1}, \mathbf{H}(\vec{P}_t)), \tag{2.7}$$

where $\pi_j(P_{t+1}, \mathbf{H}(\vec{P}_t)) = P_{t+1}S(H_j(\vec{P}_t)) - c(S(H_j(\vec{P}_t))) - C_j$. C_j is the fixed cost associated with H_j. [3]

[2] These predictions are equivalent to agents only considering the first moment of the price distribution. Though not explicit in Brock and Hommes (1997), underlying this structure is an assumption of risk neutrality. Assuming risk-aversion could possibly introduce second moments since $E(\pi(P)) \leq \pi(E(P))$ when agents are risk-averse.

[3] In the general setup of this model, Brock and Hommes (1997) assume a more general form of $U_{j,t+1}$. This form is a weighted average of past profits. Their general results, and Theorem 1 below, are based on this more general form of $U_{j,t+1}$.

The cost of production is a simple quadratic cost function $c(q) = q^2/2b$. The profit functions for firms using certain predictor functions are

$$\pi_1(P_{t+1}, P_{t+1}) = \frac{b}{2}P_{t+1}^2 - C, \tag{2.8}$$

$$\pi_2(P_{t+1}, P_t) = \frac{b}{2}P_t(2P_{t+1} - P_t),$$

$$\pi_3(P_{t+1}, R_t) = \frac{b}{2}R_t(2P_{t+1} - R_t) - D.$$

Then plugging (2.8) into (2.6) leads to the laws of motion for the various predictors:

$$n_{1,t+1} = \frac{\exp[\beta((b/2)P_{t+1}^2 - C)]}{Z_{t+1}}, \tag{2.9}$$

$$n_{2,t+1} = \frac{\exp[\beta((b/2)P_t(2P_{t+1} - P_t))]}{Z_{t+1}},$$

$$n_{3,t+1} = \frac{\exp[\beta((b/2)R_t(2P_{t+1} - R_t) - D)]}{Z_{t+1}},$$

where $Z_{t+1} = \sum_{j=1}^{3} \exp[\beta\pi_{j,t+1}]$.

A convenient transformation will lower the system's dimension and simplify the analytic solution. Let $x_{t+1} = n_{1,t+1} - n_{2,t+1} - n_{3,t+1}$ and $y_{t+1} = n_{2,t+1} - n_{3,t+1}$. Then it can be shown that

$$x_{t+1} = \tanh\left[\frac{\beta}{2}\left(\frac{b}{2}((P_{t+1} - P_t)^2 - R_t(2P_{t+1} - R_t)) - (C - D)\right)\right], \tag{2.10}$$

$$y_{t+1} = \tanh\left[\frac{\beta}{2}\left(\frac{b}{2}(R_t - 2(R_t - P_t)P_{t+1} - P_t^2) + D\right)\right].$$

The first equation in (2.10) defines the difference between the proportions of agents choosing rational beliefs and the proportions choosing naive or adaptive beliefs. Similarly, the second equation in (2.10) describes how the difference between the proportions of naive and adaptive evolves over time. If $x_{t+1} = 1$ (-1), then the population has all (no) sophisticated agents.

The cobweb model with rational, naive, and adaptive expectations is a system of (non-linear) difference equations that governs the dynamics of price, the geometric weighted average of past prices, the excess proportion of sophisticated agents, and the excess proportion of naive agents. Notice that the weighted average can be re-written as

$$R_{t+1} = (1 - \gamma)P_{t+1} + \gamma R_t. \tag{2.11}$$

The equilibrium for the system can be found by plugging (2.3) and (2.5) into (2.4) and solving for P_{t+1}. Hence, the equilibrium is

$$P_{t+1}(\theta_t) = \frac{-b((2y_t - x_t + 1)P_t + (1 - 2y_t - x_t)R_t)}{4B + 2b(1 + x_t)},$$

$$R_{t+1}(\theta_t) = \frac{-b(1-\gamma)((2y_t - x_t + 1)P_t + (1 - 2y_t - x_t)R_t)}{4B + 2b(1 + x_t)} + \gamma R_t,$$

$$x_{t+1}(\theta_t) = \tanh\left[\frac{\beta}{2}\left(\frac{b}{2}(k(\theta_t) - R_t l(\theta_t)) - (C - D)\right)\right],$$

$$y_{t+1}(\theta_t) = \tanh\left[\frac{\beta}{2}\left(\frac{b}{2}(m(\theta_t) - P_t^2) + D\right)\right],$$

where

$$\theta_t = (P_t, R_t, x_t, y_t),$$

$$k(\theta_t) = \left(\frac{-b((2y_t - x_t + 1)P_t + (1 - 2y_t - x_t)R_t)}{4B + 2b(1 + x_t)} - P_t\right)^2,$$

$$l(\theta_t) = \left(\frac{-b((2y_t - x_t + 1)P_t + (1 - 2y_t - x_t)R_t)}{4B + 2b(1 + x_t)} - R_t\right),$$

$$m(\theta_t) = \left(R_t^2 - \frac{b}{2}(R_t - P_t)\right)\frac{(2y_t - x_t + 1)P_t + (1 - 2y_t - x_t)R_t}{4B + 2b(1 + x_t)}.$$

$\theta_{t+1} = \psi(\theta_t) \equiv (P_{t+1}(\theta_t), R_{t+1}(\theta_t), x_{t+1}(\theta_t), y_{t+1}(\theta_t))$ is a system of (non-linear) difference equations defined by the equilibrium equations for P_t, R_t, x_t, y_t.

3. A digression on the cobweb model with rational versus naive expectations

The model in Section 2 represents the cobweb model of Brock and Hommes (1997) with rational versus naive expectations whenever $n_{3,t}$ and γ are set to zero. Therefore, this model can be seen as a more general case of the ARED than the stylized example of Brock and Hommes (1997).

In addition to the stylized example, Brock and Hommes (1997) also present some general results of the ARED. They show that in a cobweb model that has the steady state locally unstable when all agents use the cheapest predictor, then for sufficiently large β the steady state of the ARED is locally unstable. This result shows that if the preference for higher profits is large enough then the steady state will be unstable. [4] It is important to note that there must exist some ordering to predictor costs that has costs higher for non-naive predictors—a stabilizing and destabilizing predictor cannot cost the same. This distinction will become clear in the cobweb model with rational, naive, and adaptive expectations. This result does not indicate, though, that how large β needs to be for instability depends critically on the set of predictors available. Later results will show that another unsophisticated predictor available to agents will increase the critical β.

The Brock and Hommes (1997) principal results for their stylized example are given in Theorem 1. Note that in the simple two-dimensional case of the model the unique steady state is $E = (0, \bar{x}(\beta)) = (0, \tanh(-\beta C/2)$. The assumption that $b/B > 1$ is equivalent to assuming that the steady state is unstable when $n_{2,t} = 1$.

Theorem 1. Assume that the slopes of supply and demand satisfy $b/B > 1$.

(i) *When the information costs* $C = 0$, *the steady state* $E = (0, 0)$ *and it is always globally stable.*
(ii) *When the information costs* $C > 0$, *then there exists a critical value* β_1 *such that for all* $0 \leq \beta \leq \beta_1$ *the equilibrium is globally stable, while for* $\beta > \beta_1$ *the equilibrium is an unstable saddle point with eigenvalues* 0 *and* $\lambda(\beta) = [-b(1 - \bar{x}(\beta))]/[2B + b(1 + \bar{x}(\beta))] < -1$. *At the critical value* β_1 *the steady-state value is* $\bar{x}(\beta) = -B/b$.
(iii) *When the steady state is unstable, there exists a locally unique period* 2 *orbit* $\{(\tilde{p}, \tilde{x}), (-\tilde{p}, \tilde{x})\}$, *with* $\tilde{x} = -B/b$ *and* $\tilde{p}$ *the unique positive solution of* $\tanh[\beta/2(2b\tilde{p}^2 - C)] = -B/b$. *There exists a* $\beta_2 > \beta_1$ *such that the period* 2 *cycle is stable for* $\beta_2 > \beta > \beta_1$. [5]

Remark 1. Though not explicitly mentioned by Brock and Hommes, the steady state with $C > 0$ is locally and globally stable for all β whenever $b/B < 1$. This result is easily seen in the proof to Theorem 1(ii) and is a direct consequence of the fact that the steady state is locally stable under naive beliefs when $b/B < 1$.

Theorem 1 demonstrates that there are two driving forces behind the dynamics of the steady state in the cobweb model with rational and naive expectations. The first force is the cost of rational expectations. Whenever the perfect foresight predictor is costless ($C = 0$) the steady state is stable,

[4] The formal description of this result, and the necessary regularity conditions, are given in Brock and Hommes (1997, Theorem 2.2).

[5] See Brock and Hommes (1997) for a more detailed discussion and proof of this theorem.

with agents evenly divided between the two predictors. When rational expectations are costly, however, the steady state may no longer be stable. In this case, how quickly agents react to changes in profits govern the convergence of the dynamics. When agents are more responsive to higher profits (corresponding to large values of β), the steady state will be unstable. This paper is concerned with how these instability properties can be dampened by adaptive expectations. The Brock and Hommes results seem to indicate that it is only necessary to have a cheap destabilizing predictor and a costly sophisticated predictor. But, in the next section it will be clear that this is not the case. The array of predictors available, and their cost properties, are also important determinants, and as the number of predictors increases, the stability properties may increase as well.

4. Main results

Before turning to the main results, first note the following definitions.

Definition 1. A steady state of the ARED is a fixed point $\bar{\theta}$ s.t. $\bar{\theta} = \psi(\bar{\theta})$.

Definition 2. A steady state, $\bar{\theta}$, of the ARED is locally asymptotically stable if $\exists \delta > 0$ s.t $\| \theta_t - \bar{\theta} \| \rightarrow 0$ as $t \rightarrow \infty$ whenever $\| \theta_0 - \bar{\theta} \| < \delta$, all $s \geq 0$.

The main stability results of the steady state are given in several theorems, with the proofs contained in the appendix.

4.1. Basic stability results

The basic stability results are given in the following theorem. This theorem considers when adaptive expectations are both costless and costly.

Theorem 2. Assume $b/B > 1$ and $C \geq D \geq 0$. In the cobweb model with rational, naive, and adaptive expectations, there is a unique steady state defined by $E = (0, 0, \bar{x}(\beta), \bar{y}(\beta))$ with $\bar{x}(\beta) = \tanh(-\beta(C - D)/2)$ and $\bar{y}(\beta) = \tanh(\beta D/2)$. Further, the steady state has the following properties.

(a) *When $C=0, D=0$, $\bar{x}=\bar{y}=0$ and the steady state is locally asymptotically stable.*
(b) *When $C > D > 0 \exists \beta'$ such that the steady state is an unstable saddle point $\forall \beta > \beta'$.*

Theorem 2 gives the conditions for local stability of the unique steady state for various orderings of the cost structure. Part (a) shows that the steady state is always locally stable whenever the cost to the perfect foresight and adaptive predictor is zero. Part (b) shows that if the cost ordering is consistent with Theorem 1(ii) then the steady state is an unstable saddle point for large enough β. Theorem 2(b) demonstrates that with positive costs to all predictors other than the naive predictor, the steady state may be locally unstable when it is unstable under the cheapest predictor. Such a result is similar to that of the Brock and Hommes (1997) general result. Both Theorem 1(ii) and Theorem 2(b) suggest that as long as there is some monotonic ordering of costs, the steady state may be an unstable saddle point.

These are essentially the same stability results in the simple example considered by Brock and Hommes (1997). When all predictors are costless the steady state is locally asymptotically stable. Once there is a strict ordering to costs, this local stability may no longer hold for a large enough intensity of choice parameter β. This result does not indicate, though, how β depends on the array of available predictors. To see how the addition of adaptive expectations affects the instability conditions the following corollary compares the critical β in this case to the critical β in the Brock and Hommes model.

Corollary 3. Let β_1 *be the critical value in Theorem* 1. *Assume* $C > D > 0$. *Consider the cobweb model with rational, naive, and adaptive expectations when* $b/B > 1$. *For D sufficiently small there exists* $\tilde{\gamma}(b, B, \beta, C, D)$ *and* $\beta_2 > \beta_1$ *so that whenever* $\beta < \beta_2$ *and* $\gamma > \tilde{\gamma}$ *the steady state is locally stable. If* $\beta > \beta_2$ *the steady state is locally unstable for all* $0 \leq \gamma \leq 1$.

Corollary 3 states that the Brock and Hommes critical value, β_1, leads to a stable steady state whenever γ is sufficiently close to one. How 'sufficiently' is defined depends on the exact values of b, B, and β. However, for a larger critical value, β_2, the additions of adaptive expectations does not induce stability for any value of γ. This is because when $\beta > \beta_2$, $\tilde{\gamma} > 1$. Corollary 3 shows that it is not β alone that determines the stability of the steady state. While it is true for some large enough β that the steady state becomes unstable, the Brock and Hommes (1997) general result does not give any indication as to how this critical β changes as the predictor set changes. Their results (Theorem 1) show that when β is large enough the steady state will become unstable for any arbitrary positive cost to the sophisticated predictor. However, for that β which induced instability in the rational versus naive case, if D is small enough, and γ is sufficiently close to unity, then the case with rational, naive, and adaptive beliefs will be stable. The cost D should be small since adaptive expectations are fairly unsophisticated, and could even be set arbitrarily close to zero, the cost of naive expectations. These results show that there is an interplay between costs, the 'intensity of choice' parameter, the adaption parameter on past prices, and the relative slopes of supply and demand which, in the end, determine the convergence properties of the dynamical system.

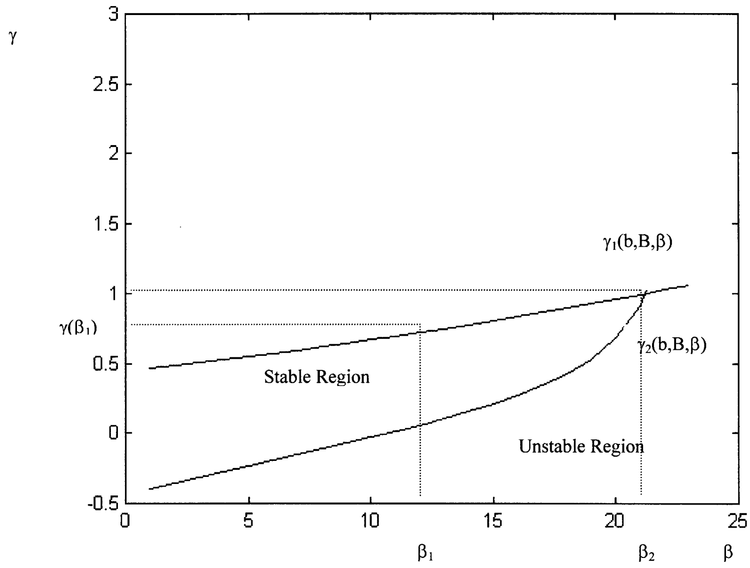

Fig. 1.

Fig. 1 illustrates the results of Corollary 3. It shows in (β, γ) space, the stable and unstable regions described in Corollary 3. The graph is drawn for $C = 1, D = 0.5$, and $b/B = 3$. Here $\tilde{\gamma}(b, B, \beta, C, D) = \min\{\gamma_1, \gamma_2\}$ where the functions γ_1, γ_2 are defined in the proof to Theorem 2. The value of β_2 is given by $\tilde{\gamma}(b, B, \beta_2, C, D) = 1$.[6] The area above the curve γ_1 represents all of the (β, γ) combinations in which the steady state is locally asymptotically stable when $\gamma_1 = \tilde{\gamma}(b, B, \beta, C, D)$, i.e. $\gamma_1 < \gamma_2$. Similarly, the curve γ_2 represents the (β, γ) combinations for which the steady state is locally stable when $\gamma_2 = \tilde{\gamma}(b, B, \beta)$, that is, when $\gamma_2 < \gamma_1$. Notice that some of the stability region lies above $\gamma = 1$, an area of infeasibility. The area below both lines are all of the (β, γ) pairs in which the steady state is locally unstable. The stability region for Brock and Hommes is $\beta \leq \beta_1$. In the model with rational, naive, and adaptive expectations, a sufficient amount of weight must be placed on past prices to induce stability. This graph clearly illustrates that a larger γ is necessary to preserve stability under larger values of β.

Again, the intuitive justification for this result is as follows. The cobweb model with naive beliefs when the equilibrium is away from the steady state has price switching from positive to negative every other period. Larger values of β make this oscillation more acute. It is this dynamic process that

[6] The graph in Fig. 1 shows $\gamma_1(b, B, \beta_2, C, D) > \gamma_2(b, B, \beta_2, C, D)$. Depending on the parameter values, $\gamma_1(b, B, \beta_2, C, D) < \gamma_2(b, B, \beta_2, C, D)$ could also hold.

leads to explosive oscillation under the conditions of Brock and Hommes. With rational expectations, irregular switching takes place as the net benefits of sophistication oscillate as well. The addition of adaptive expectations introduces a dampening effect that helps reduce these oscillations and under conditions mentioned above, guides the system back towards the steady state.

4.2. Stability results under costless adaptive expectations

This subsection examines the stability conditions when $D=0$, that is, when adaptive expectations are costless. This is an interesting case because without a natural metric on sophistication one could argue that adaptive expectations are equally costless as naive expectations.

Before stating Theorem 4 we introduce the following notations:

$$\bar{x}(\beta) = \tanh\left(\frac{-\beta(C-D)}{2}\right),$$

$$\bar{y}(\beta) = \tanh\left(\frac{\beta D}{2}\right),$$

$$\hat{\gamma}(b,B,\beta,C,D) = \min(S(b,B,\beta,C,D), V(b,B,\beta,C,D)),$$

where

$$S(b,B,\beta,C,D) = \frac{2b(1-\bar{x}(\beta))}{4B + b(3+\bar{x}(\beta))},$$

$$V(b,B,\beta,C,D) = \frac{a_1(b,B,\beta)a_2(b,B,\beta) - a_3(b,B,\beta) - a_4(b,B,\beta)}{a_3(b,B,\beta) + a_4(b,B,\beta) - a_1(b,B,\beta)a_5(b,B,\beta)},$$

$$a_1(b,B,\beta) = (4B + 2b(1+\bar{x}(\beta))),\ a_2(b,B,\beta) = (4B + b(1+3\bar{x}(\beta))),$$

$$a_3(b,B,\beta) = 4bB(1-\bar{x}(\beta)),\ a_4(b,B,\beta) = 2b^2(1-\bar{x}(\beta))(1+\bar{x}(\beta)),$$

$$a_5(b,B,\beta) = (4B + b(3+\bar{x}(\beta))).$$

Theorem 4. Assume that $b/B > 1$ and $C > D=0$. In the cobweb model with rational, naive, and adaptive expectations there is a unique steady state defined by $E=(0,0,\bar{x},0)$ and it is locally asymptotically stable if and only if $\gamma > \hat{\gamma}(b,B,\beta,C,D)$.

Theorem 4 gives the necessary and sufficient conditions for the steady state to be stable when the perfect foresight predictor has a positive cost of C, but the adaptive predictor is still costless. For certain values of β, the condition for stability may require the impossible; that is, stability cannot occur since the condition requires $\gamma > 1$.

The following lemma gives a more intuitive description of how $\hat{\gamma}$ depends on b/B and β, and it will be utilized in the proofs of the subsequent corollaries.

Lemma 5. If $b/B < 2$, then $S > V$ for sufficiently large β. If $b/B > 2$, then $V > S$ for sufficiently large β.

Proof. Let $\beta \to \infty$. Since

$$\bar{x} \to -1, \quad S \to \frac{4b}{4B + 2b}, \quad V \to \frac{16B^2 - 16bB}{-16B^2}.$$

$$S > V \iff \frac{b}{B}\left(\frac{b}{B} - 1\right) < 2.$$

The last condition is satisfied whenever $b/B < 2$. □

The necessary and sufficient condition given in Theorem 4 depends on the relative slopes and the 'intensity of choice' parameter. Whenever β is large, and b/B is not too far into the unstable cobweb region (i.e. $(1, \infty)$), then $\gamma > S$ is the necessary and sufficient condition for local asymptotic stability. Conversely, when b/B is further into the unstable region, $\gamma > V$ is the condition. The parameter β is important because it, along with b/B, affects the size of S and V.

The following corollary gives greater insight into when the stability conditions are satisfied.

Corollary 6. Assume that $C > 0$ and $D=0$. Consider the cobweb model with rational, naive, and adaptive expectations. The steady state $E = (0, 0, \bar{x}, 0)$ is locally asymptotically stable for all sufficiently large β if $b/B < 2$ and $\gamma > b/B - 1$.

Proof. Suppose $\beta \to \infty$, then $\bar{x} \to -1$. By Lemma 5 if $b/B < 2$ then $S > V$. Hence, by Theorem 4 local asymptotic stability of the steady state is given by the condition $\gamma > b/B - 1$. When $b/B < 2$ there exists a $0 < \gamma < 1$ that will satisfy this condition. Hence, the steady state is locally stable for $b/B < 2$ and $\gamma > b/B - 1$ for all β sufficiently large. □

Corollary 6 gives the conditions under which the steady state with $C > 0$, $D = 0$ is locally stable even when β is large; it shows that for large β, large enough γ, and small enough relative slopes of supply and demand the steady state will be locally stable. This result is a departure from Brock and Hommes.

Corollary 6 addresses the stability issue when $b/B < 2$. A natural question is what happens when $b/B \geq 2$. Because of the analytical complexity of this case, we present a numerical analysis. Fig. 2 considers a particular numerical parameterization to examine the stability conditions for a variety of relative slope values; that is, Fig. 2 illustrates the results of Theorem 4. It plots the stability condition $\hat{\gamma}(b, B, \beta, C, D)$ as a function of b/B, with $\beta = 20$, $C = 1$,

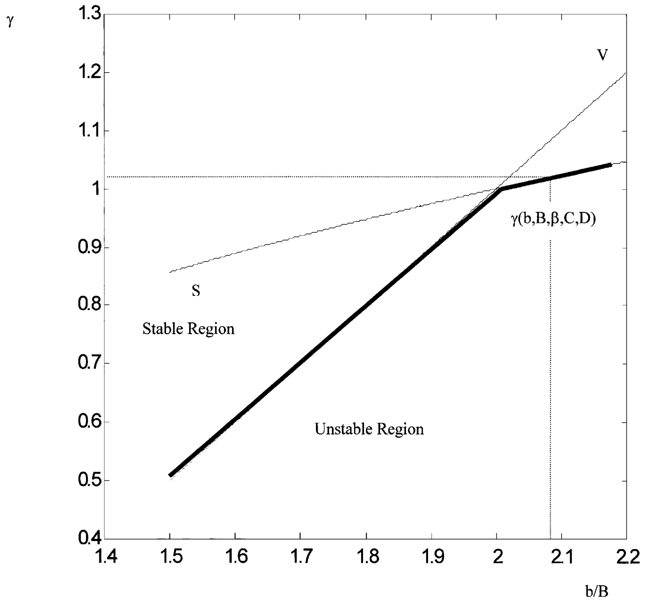

Fig. 2.

$D = 0$. The value for β was chosen so that $b/B = 2$ is the switching point for whether S or V is the minimum of $\hat{\gamma}(b, B, \beta, C, D)$. Corollary 6 shows, and Fig. 2 illustrates, that when $b/B < 2$ and if $\gamma > V = \hat{\gamma}(b, B, \beta, C, D)$ then the steady state will be locally stable. When $b/B \geq 2$ we have $\hat{\gamma} = S$, and Theorem 4 shows that for $\gamma > S = \hat{\gamma}(b, B, \beta, C, D)$ the steady state will be locally stable. The figure also shows that for a sufficiently large, fixed β, the steady state will become unstable for all γ as b/B increases beyond 2. This demonstrates that the addition of adaptive expectations increases the range of slope values for which the steady state is locally stable. However, for large β, as the slope values increase, the steady state will become unstable regardless of how much adaptive expectations dampen past prices..

Our result is surprising given that Carlson (1968) and Auster (1971) who show that the cobweb model is invariably stable under simply-averaged expectations. Simply-averaged expectations is equivalent to the case of large γ in this model. Unlike Carlson (1968) or Auster (1971), in our framework the steady state is not invariably stable even when γ is large. The presence of naive beliefs as a choice possibility can lead to divergent dynamics. For larger values of b/B and β, the steady state may be locally unstable for large γ. For a range of 'unstable' slope values, if the adaption parameter is large enough the steady state will be locally stable. Predictor choice can stabilize or destabilize depending on the set of predictors and the market conditions.

That the slope condition for instability becomes larger with adaptive expectations is one of the principal implications of Theorem 4. The local stability conditions depend on the selection of predictors. Intuitively, the addition of another predictor lowers the weight, in a steady state, placed on the other predictors. In the neighborhood of a steady state, naive expectations will have less effect on equilibrium price than even before agents could choose adaptive expectations. Since they have less influence the critical 'intensity of choice' must increase.

Note that the Brock and Hommes model of rational versus naive expectations is a special case of this model. For this reason adaptive expectations are a natural choice for inclusion in a model of rationally heterogeneous expectations. Depending on the size of γ, this model could be the case of rational versus naive expectations, rational versus simply-averaged expectations, or something in between. This paper shows that the latter two cases have interesting properties. The cobweb model with rationally heterogeneous expectations tends to be stable under more cases than Brock and Hommes' (1997) implies, but when these cases arise depends on a complex set of conditions.

4.3. *Rational versus adaptive expectations*

The previous results indicate that the mix of predictors available to agents is important for system stability. This subsection investigates what happens when there are two choices of predictors available to agents and both tend to be stabilizing.

First, however, the following theorem examines steady-state stability when $n_{3,t} = 1 \forall t$, in order to see that adaptive expectations are stabilizing. That is, when agents can only use adaptive expectations, the steady state will be locally asymptotically stable when sufficient weight is placed on past prices.

Theorem 7. Assume $b/B > 1$. The steady state $E = (0,0)$ of the cobweb model with adaptive expectations is locally asymptotically stable $\forall \beta > 0$ whenever

$$\gamma > \frac{(b/B) - 1}{1 + (b/B)}.$$

The following result gives the stability conditions when agents choose between a costly rational predictor and a costless adaptive predictor. For the case of rational and adaptive beliefs, the model is modified by setting $n_{2,t} = 0$ and $Z_t = \exp \beta\pi_{1,t} + \exp \beta\pi_{3,t}$.

Theorem 8. Assume $b/B > 1$ and $C > D = 0$. In the cobweb model with rational and adaptive expectations and a sufficiently large intensity of choice, there is a unique steady state defined by $E = (0, 0, \bar{x})$ and it is locally asymptotically stable if and only if $\gamma > 1 - B/b$.

In both results, γ must dampen sufficiently for local stability to hold. Note that

$$1-\frac{B}{b}>\frac{(b/B)-1}{1+(b/B)}.$$

So those γ for which the steady state is stable under homogeneous expectations, will also lead to stability under rationally heterogeneous expectations when agents choose between rational and adaptive expectations. [7] As has been seen before, adaptive expectations must weight the past strongly enough for the stability conditions to be satisfied.

5. Concluding remarks

This paper has investigated a cobweb model with rationally heterogeneous beliefs. The results extend and enhance those from the ARED of Brock and Hommes (1997). The crucial difference between their model and the one presented here is the addition of adaptive expectations. Adaptive expectations arguably should be included in a rationally heterogeneous model of beliefs since they do require some memory on the part of agents and are a method traditionally used in macroeconomics. By allowing for two types of unsophisticated beliefs we have achieved two aims. First, an investigation into the specifics of the belief switching mechanism, and how the interaction between unsophisticated predictors affects the local convergence properties. Second, to create a more general example of the Brock and Hommes model that, depending upon the strength of agents' beliefs, could have naive or simply-averaged expectations as special cases, and still allow for analysis of the system's dynamics.

The results of this paper can be summarized as follows:

1. When adaptive expectations are a second unsophisticated choice that is costless, the steady state may be locally asymptotically stable, for fixed values of the 'intensity of choice' parameter, when agents' memories are sufficiently strong.
2. When adaptive expectations are a second unsophisticated choice that is costly, the steady state will be an unstable saddle point for a large 'intensity of choice' between predictors. The adaptive predictor, though, can increase the 'intensity of choice' at which the steady state switches from a stable steady state to an unstable saddle point.

[7] Note that the opposite results are also possible by adding naive expectations to the model of rational and adaptive beliefs. In this case, the presence of naive expectations will tend to destabilize the system. This underscores the main message of this paper: the addition of stabilizing predictors will tend to make the system 'more stable'.

3. When agents choose between two predictors, an expensive sophisticated predictor (perfect foresight) and a cheap unsophisticated predictor (adaptive expectations), the steady state is locally asymptotically stable when the adaption parameter is large enough. The 'intensity of choice' has no effect on the stability conditions in this case.

Adaptive expectations may create a more stable environment if the adaption parameter is large enough. The results in this paper clearly indicate that the cobweb cycling that has price oscillating around its steady state is dampened when agents may also choose adaptive expectations with strong dampening. Section 4 gave the conditions under which the sequence of prices will converge to the steady state. Thus, this paper shows that predictor choice is important for the exact form of the stability properties in a way not covered by the general result of Brock and Hommes. However, the results also lend support to the Brock and Hommes (1997) assertion that predictor choice leads to market instability for some range of slope values, and parameter values. What these ranges are depends upon the array of predictors available to agents.

When agents choose between an expensive perfect foresight predictor and a cheap adaptive predictor, the conditions for convergence are more general than considered by Carlson (1968) and Auster (1971). Thus, the choice, and subsequent predictor switching, can have a stabilizing influence. It is possible to add a choice, which is not the cheapest one, but which (for some β) will stabilize a system which is unstable under Brock and Hommes (1997) example.

Finally, the results presented in Section 4 allow us to conjecture as to what happens when the choices available to agents increase. It was seen that increasing the predictor choices from two to three tended to stabilize the steady state so long as the added choice is stabilizing. As the choice set increases further, and the weight placed on the de-stabilizing predictor decreases, the bounds on which the steady state is stable may also increase. But, as mentioned above, how these bounds increase depends upon the make up of the set of predictors and their cost characteristics. Future research should investigate exactly how these stability properties will change as the predictor choice set expands even further. This paper suggests that the parameter range over which the system is stable will increase as the predictor set expands.

Acknowledgements

This paper has benefited enormously from comments and suggestions by George W. Evans. I would also like to thank Van Kolpin and Bruce McGough for stimulating remarks and comments. In addition I am grateful for helpful suggestions provided by two anonymous referees.

Appendix A.

In this appendix the proof of the main theorems are presented.

Proof of Theorem 2. First consider the uniqueness of the steady state. The steady state is found by setting for all time periods $P_t = \bar{P}$. Then solving for the steady-state price $\bar{P}$ in the equilibrium equations leads to the unique steady-state price $\bar{P}=0$. By plugging this steady-state value into the remaining dynamic equilibrium equations, it is easily seen that the steady state is unique.

To prove local stability it will be useful to linearize the system around the corresponding steady state and show that the eigenvalues lie inside the unit circle (see Theorem 6.5 in Stokey et al. (1989) or Theorem 6.2 in Azariadis, 1993).[8] Denoting $f(\theta) \equiv P_{t+1}(\theta)$, $i(\theta) \equiv R_{t+1}(\theta)$, $g(\theta) \equiv x_{t+1}(\theta)$, $h(\theta) \equiv y_{t+1}(\theta)$, the eigenvalues are

$$\lambda_1 = \frac{1}{2}\{(f_p + i_R) + \sqrt{(f_p + i_R)^2 - 4f_p(i_R - (1-\gamma)f_p)}\}, \tag{A.1}$$

$$\lambda_2 = \frac{1}{2}\{(f_p + i_R) - \sqrt{(f_p + i_R)^2 - 4f_p(i_R - (1-\gamma)f_p)}\},$$

$$\lambda_3 = \lambda_4 = 0.$$

If all the eigenvalues in (A.1) lie inside the unit circle then local asymptotic stability holds. This condition is clearly satisfied for λ_3, λ_4, so it only remains to check that the condition is satisfied for λ_1, λ_2. Note first that $f_p = -b/(4B + 2b)$ and $i_R = [-(1-\gamma)b]/(4B + 2b) + \gamma$, and that the eigenvalues are real whenever $\gamma > 0$.

We want to show that $|\lambda_1|, |\lambda_2| < 1$, which, since $|a + b| \leq |a| + |b|$, is equivalent to

$$\sqrt{(f_p + i_R)^2 - 4f_p(i_R - (1-\gamma)f_p)} < (2 - |f_p + i_R|). \tag{A.2}$$

There are two cases to consider: (i) $f_p + i_R > 0$; (ii) $f_p + i_R < 0$.

Consider case (i). $|\lambda_1|$, $|\lambda_2| < 1$ is equivalent to (rewriting (A.2) and assuming (i))

$$f_P(1 - i_R) + f_P^2(1-\gamma) < 1 - i_R.$$

Since $|i_R| < 1$, $|\lambda_1|$,$|\lambda_2| < 1$ can be established if

$$f_P^2(1-\gamma) < 1 - i_R \quad \Leftrightarrow \quad \frac{-b(1-\gamma)}{4B + 2b} < 0.$$

Therefore, $|\lambda_1|$, $|\lambda_2| < 1$ for $f_p + i_R > 0$.

[8] See LaSalle (1986) for an excellent mathematical treatment of discrete dynamical systems.

Now consider case (ii): $f_p + i_R < 0$. $|\lambda_1|$, $|\lambda_2| < 1$ is equivalent to

$$\gamma > \frac{-8bB - 16B^2}{(4B+3b)(4B+2b) - (4bB+2b)^2}. \tag{A.3}$$

This last condition is always satisfied since $0 < \gamma < 1$. The steady state, then, is locally asymptotically stable (see Theorem 6.2 in Stokey et al. 1989).

The proof of part (b) follows the same as for part (a). Note first that when $D > 0$ then $\bar{n}_2 > 0$. Linearizing the system around $E = (0, 0, \bar{x}(\beta), \bar{y}(\beta))$ leads to the eigenvalues,

$$\lambda_1 = \frac{1}{2}\{(f_p + i_R) + \sqrt{(f_p + i_R)^2 - 4f_p(i_R - (1-\gamma)f_R)}\}, \tag{A.4}$$

$$\lambda_2 = \frac{1}{2}\{(f_p + i_R) - \sqrt{(f_p + i_R)^2 - 4f_p(i_R - (1-\gamma)f_R)}\},$$

$$\lambda_3 = \lambda_4 = 0.$$

It can be easily verified that all of the eigenvalues are real numbers, and that $(f_p + i_R)^2 > 4f_p(i_R - (1-\gamma)f_R)$. Again, there are the two cases: (i) $f_p + i_R > 0$; (ii) $f_p + i_R < 0$.

Begin with case (i). $|\lambda_1|$, $|\lambda_2| < 1$ is equivalent to

$$(1-\gamma)f_P f_R < (1 - f_P)(1 - i_R). \tag{A.5}$$

It can be shown (A.5) holds everywhere. However, this condition will only be valid for

$$f_p + i_R > 0 \quad \Leftrightarrow \quad \gamma > \frac{2b(1 - \bar{x}(\beta))}{4B + b(3 + \bar{x}(\beta) - 2\bar{y}(\beta))} \equiv \gamma_1(b, B, \beta, C, D). \tag{A.6}$$

But, as $\beta \to \infty$ (A.6) is only feasible (i.e. $0 < \gamma < 1$) for $\frac{b}{B} < 1$. $|\lambda_1|$, $|\lambda_2| < 1$ does not hold when $f_p + i_R > 0$. We remark that γ_1 depends on C and D via $\bar{x}(\beta)$ and $\bar{y}(\beta)$.

Now consider case (ii).

$$|\lambda_1|, |\lambda_2| < 1 \Leftrightarrow -f_P(1 + i_R) + (1-\gamma)f_P f_R < 1 + i_R \Leftrightarrow \tag{A.7}$$

$$\gamma > \frac{1 - f_P f_R + f_P^2 + 2f_P}{(f_P + 1)(f_P - 1) - f_P f_R} \equiv \gamma_2.$$

It can be easily verified that as $\beta \to \infty$, $\gamma_2 > 1$. Hence, for sufficiently large β, there is an unstable saddle point with real eigenvalues. □

Proof of Corollary 3. We will prove this result by first examining stability when β is fixed at the critical value that induces instability in Brock and Hommes (1997). Then we will show that for some larger value of β the model with rational, adaptive, and naive expectations will turn unstable.

Fix $\beta=\beta_1$ s.t. $\bar{x}(\beta_1)=\tanh(-\beta_1(C-D)/2)=-(B/b)+\varepsilon > (-B/b)$. Reexamine the stability argument from the above theorem. From the above proof, the relevant stability condition is $\gamma > \tilde{\gamma}(b,B,\beta,C,D) = \min\{\gamma_1,\gamma_2\}$. We want to show that when $\beta=\beta_1$ there exists a $\gamma \in (0,1)$ that satisfies this condition. Without loss of generality, plug in $\bar{x}(\beta_1) = (-B/b)+\varepsilon$ into γ_1

$$\gamma > \frac{2b(1+(B/b)-\varepsilon)}{4B+b(3-(B/b)+\varepsilon-2\bar{y}(\beta_1))}. \tag{A.8}$$

There exists such a $\gamma \in (0,1)$ whenever,

$$\bar{y}(\beta_1) < \frac{(b+B)}{2b} + \frac{3}{2}\varepsilon. \tag{A.9}$$

Note that as $D \to 0$, $\bar{y}(\beta_1) \to 0$, and since $\bar{y}(\beta)$ is continuous in D, $\exists D > 0$ s.t. condition (A.9) is satisfied. Hence, for $\beta \le \beta_1$ and $\gamma > \tilde{\gamma}(b,B,\beta,C,D)$ the steady state is locally asymptotically stable.

The second step is to show that $\exists \beta_2 > \beta_1$ s.t. the steady state is locally stable (unstable) $\forall \beta < \beta_2 (> \beta_2)$. Suppose $\beta_2 > \beta_1$ and β_2 is defined such that $\bar{x}(\beta_2) = \tanh(-\beta_2(C-D)/2) = -B/b$. The stability condition will be satisfied for some γ when

$$2y(\beta_2) < \frac{1+B/b}{2}. \tag{A.10}$$

Again $y(\beta) \to 0$ as $D \to 0$. So $\exists \beta_2 > \beta_1$ such that the steady state is locally asymptotically stable. By Theorem 2, if we increase β sufficiently the steady state will become unstable. □

Proof of Theorem 4. To prove local stability in this case simply reevaluate the Jacobian at the steady state $E = (0,0,\bar{x}(\beta),0)$, and check that the eigenvalues lie inside the unit circle. The only difference in the Jacobian from part (a) to Theorem 2 is that

$$f_p = \frac{-b(1-\bar{x}(\beta))}{4B+2b(1+\bar{x}(\beta))} \quad \text{and} \quad i_R = \frac{-(1-\gamma)b(1-\bar{x}(\beta))}{4B+2b(1+\bar{x}(\beta))} + \gamma.$$

Again, from above,

$$|\lambda_1|,|\lambda_2| < 1 \quad \Leftrightarrow \quad (f_p+i_R)^2 - 4f_p(i_R-(1-\gamma)f_p) < (2-|f_p+i_R|)^2. \tag{A.11}$$

which in turn has two cases: (i) $f_p + i_R > 0$; (ii) $f_p + i_R < 0$.

Consider case (i).

$$|\lambda_1|, |\lambda_2| < 1 \Leftrightarrow -4bB - 2b^2 - 2b^2\bar{x}(\beta) < 16B^2 + 4bB(5 + 3\bar{x}(\beta)) + 2b^2(3 + 4\bar{x}(\beta) + \bar{x}(\beta)^2).$$

The left-hand side of the last condition is negative, and the right-hand side is positive. Therefore, $|\lambda_1|$, $|\lambda_2| < 1$ whenever $f_p + i_R > 0$, or $\gamma > [2b(1-\bar{x}(\beta))]/4B + b(3 + \bar{x}(\beta))$. Hence, the condition $\gamma > S$.

Now consider case (ii).

$$|\lambda_1|, |\lambda_2| < 1 \quad \Leftrightarrow \quad -f_P(1 + i_R) + (1 - \gamma)f_P^2 < 1 + i_R. \tag{A.12}$$

$$\Leftrightarrow \gamma > \frac{(4B+2b(1+\bar{x}(\beta)))(4B+b(1+3\bar{x}(\beta))) - 4bB(1-\bar{x}(\beta)) - 2b^2((1-\bar{x}(\beta))(1+\bar{x}(\beta)))}{4bB(1-\bar{x}(\beta)) + 2b^2((1-\bar{x}(\beta))(1+\bar{x}(\beta))) - (4B+2b(1+\bar{x}(\beta)))(4B+b(3+\bar{x}(\beta)))}.$$

Hence, the condition $\gamma > V$. Therefore, for a sufficiently large adaption parameter the steady state is locally asymptotically stable. □

Proof of Theorem 7. Under the reduced system, the two eigenvalues are

$$\lambda_1 = 0, \tag{A.13}$$

$$\lambda_2 = \gamma - \frac{b}{B}(1 - \gamma).$$

The condition for local stability is equivalent to

$$\left|\gamma - \frac{b}{B}(1 - \gamma)\right| < 1. \tag{A.14}$$

There are two cases to consider: (a) $\gamma < b/B(1-\gamma)$; and, (b) $\gamma > b/B(1-\gamma)$. Note that case (b) is relevant when $\gamma > b/B/(1 + (b/B))$.

Now consider case (a). Local stability holds when

$$\gamma > \frac{b/B - 1}{1 + (b/B)} \tag{A.15}$$

Now consider case (b). Here stability of the steady state holds for all $\gamma < 1$. Since

$$\frac{b/B - 1}{1 + (b/B)} > \frac{b/B}{1 + (b/B)},$$

then whenever

$$\gamma > \frac{b/B - 1}{1 + (b/B)}$$

local asymptotic stability holds. □

Proof of Theorem 8. Fix $n_{2,t}=0$, $\forall t$, and without a loss of generality assume $D=0$. (It is only necessary that $C>D$ so that $\bar{x}(\beta)<0$). Under the new steady state $E=(P,R,\bar{x}(\beta))=(0,0,\bar{x}(\beta))$ the eigenvalues are

$$\lambda_1 = \frac{-b(1-\gamma)(1-\bar{x}(\beta))}{2B+b(1+\bar{x}(\beta))}, \tag{A.16}$$

$$\lambda_2=\lambda_3=0.$$

The steady state will be locally asymptotically stable whenever $|\lambda_1|<1$, or

$$-(1-\gamma)b\bar{x}(\beta)-\gamma b < 2B + b\bar{x}(\beta). \tag{A.17}$$

Let $\beta\to\infty$. Then $|\lambda_1|<1$ if and only if $\gamma>(1-B/b)$. Hence, the steady state is locally asymptotically stable for all values of β if the adaption parameter is large enough. □

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