

A Theory of Power Law Distributions for the Returns to Capital and of the Credit Spread Puzzle *

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Abstract

I build a model of the cross section of leverage ratios for borrowers - for example, traders or entrepreneurs - assuming heterogeneous beliefs about future asset returns and endogenous collateral constraints à la Geanakoplos (1997). Under minimal assumptions on the underlying distribution of beliefs, the leverage ratios of borrowers follow a Pareto distribution in the upper tail, with an endogenous tail coefficient equal to two, a prediction precisely verified for hedge funds in the TASS database. This Pareto distribution can lead to similar Pareto distributions for the returns of borrowers, be they investment bankers or entrepreneurs, with empirically plausible Pareto-Lorenz coefficients between one and three, providing a new intuition for the skewness of the income distribution. In the model, borrowers and lenders are matched assortatively according to their relative levels of optimism, which can help explain the Over-The-Counter structure of many collateralized asset markets : borrowers with a relatively higher valuation for the asset effectively borrow from more optimistic lenders with a lower collateral requirement. Finally, interest rates earned by lenders are not linked to expected default probabilities, which can shed a new light on the credit spread puzzle.

Keywords: Leverage, Pareto distributions, Over-The-Counter markets.

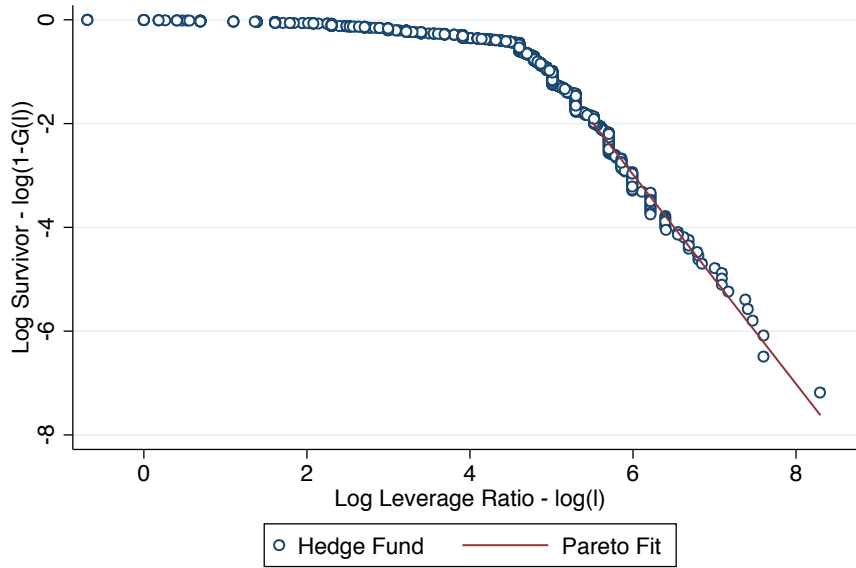
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Introduction

Fat tailed distributions are ubiquitous in economics and have been the subject of much attention. Models generating Pareto distributions usually revolve around a random growth assumption, where skewness is generated through a multiplicative stochastic process with statistical frictions. This paper offers an alternative, microfounded, mechanism for the existence of Pareto distributions based on an investment model with belief disagreement on financial returns. A motivation of this financial mechanism is provided by the figure below. It shows that the distribution of hedge funds' leverage ratios is also characterized by a Pareto distribution in the upper tail. This empirical regularity is arguably harder to connect to random growth models, as there is no a-priori reason to expect shocks to leverage ratios to be multiplicative. The model I develop in this paper, based on belief disagreement and endogenous margins a la Geanakoplos (1997), generates a limiting Pareto distribution for the leverage ratios of borrowers. According to this model, under limited assumptions, leverage ratios should follow a Pareto distribution of exponent two in the upper tail, a prediction I precisely verify in the TASS Hedge Fund Database (with a point estimate of 2.02, and an R^2 in the range of 98%).



The model also offers a new perspective on the driving forces for credit spreads. According to the so-called credit spread puzzle, a well-known anomaly in finance, interest rates on bonds are too high to be explained by average subsequent default probabilities. In my model, spreads are not a reward for risk but are the outcome of a competitive assignment process, where all borrowers would like to borrow from the most optimistic lenders with higher leverage ratios, if prices of loan contracts did not adjust to clear supply and demand. In equilibrium, excess returns are a counterpart for higher leverage, and they make borrowing with less collateral attractive only for the most optimistic borrowers. Hence, the assignment process described in

this paper goes in the way of explaining the credit spread puzzle, even though how much it can explain quantitatively ultimately depends on the distribution of beliefs relative to the objective probability measure for future returns. Moreover, even if contracts are traded anonymously, each lender is effectively lending to one borrower; the market therefore features a high degree of customization and fragmentation, as in real world repo markets which are organized Over-The-Counter.

Finally, the simplicity of the model, as well as the very limited assumptions under which the limiting Pareto distribution is derived, suggest it could be a very good candidate not just for hedge fund's financing, but for entrepreneurs in general, or in any setting where the borrower is more optimistic than his financier. Under certain conditions, the Pareto distribution for borrowers' leverage ratios will translate into Pareto distributions for capital incomes of borrowers also. The model could therefore be a rationalization of the Pareto distribution for top incomes and wealth especially given that in a dynamic setting it will predict empirically plausible Pareto-Lorenz coefficients between one and three.¹

More precisely, I construct a model along the lines of Geanakoplos (1997) and Simsek (2013) where a continuum of agents have heterogeneous beliefs about the payoff of a risky asset. In the simplest environment considered in the paper, agents can invest their cash endowment in three ways: they can buy the risky asset potentially borrowing against it as collateral, lend to investors in the risky asset using the same asset as collateral, and invest in a storage technology. In the equilibrium of this simple model, the agents' population partitions itself into three groups based on their level of optimism: the most optimistic do leveraged investing, agents with intermediate levels of optimism do collateralized lending, and the least optimistic invest in the storage technology. Among borrowers, agents are more or less optimistic; and similarly lenders are more or less optimistic. In Geanakoplos (1997), as well as in subsequent papers such as Simsek (2013), there is however only one leverage ratio for collateralized loans in equilibrium, because all agents agree on the value of the asset conditional upon default. This is a modeling simplification, which allows Geanakoplos (2010) to speak conveniently of a *leverage factor*, and to study its dynamics over the cycle. This merely technical assumption is at odds with empirical evidence and even casual empiricism, as homebuyers purchase houses using collateralized loans with different loan-to-value ratios, repurchase agreements display a dispersion in haircuts, even for the exact same collateral class, and the graph above has shown that hedge funds borrow with very different leverage ratios. In contrast, the general model allows for dissent on the default state also, so that contracts with different endogenous margins are traded in equilibrium.

In this paper, I make a different assumption on the structure of beliefs, assuming that agents have point expectations about the future price of the risky asset, and differ in these expectations. With this belief structure, different loan contracts with different implicit leverage ratios are traded

¹Gabaix (2009) notes that "the tail exponent of income seems to vary between 1.5 and 3 [...] Still it is not clear why the exponent for wealth is rather stable across economies. An exponent of 1.5 – 2.5 does not emerge necessarily out of an economic model."

in equilibrium. The new empirical implications of my model can then be traced back to the following key result: in equilibrium, more optimistic lenders lend to more optimistic borrowers. In other words, there is positive assortative matching between borrowers and lenders. The intuition for this result is that more optimistic lenders allow borrowers to achieve a higher leverage ratio, because they agree to lend more for each unit of risky asset in collateral, and that buying more units of the risky asset is relatively worth more to relatively more optimistic borrowers. There is thus complementarity between the beliefs of lenders and the beliefs of borrowers they lend to, which leads to positive assortative matching between lenders and borrowers in the competitive equilibrium.

The positive assortative matching result then allows me to solve for the competitive equilibrium of what now looks very much like an assignment model in the tradition of Sattinger (1975). Lenders choose loans which are fully secured according to their beliefs. The leverage ratios of these loans are higher when they are relatively more optimistic. Those higher leverage loans would be attractive to all borrowers if implicit returns on loan contracts were the same, because borrowers would ideally like to leverage themselves into the asset as much possible. But in equilibrium, interest rates rise when leverage increases, so that only more optimistic borrowers are willing to leverage more. Interest rates therefore play an allocative role, and are higher than the returns to the storage technology, even though they do not compensate lenders for default risk: therefore, interest rates feature *allocative premia*. In a sense, the model is in many respects similar to a hedonic pricing model, as Rosen (1974). More precisely, each lender is effectively lending to one particular borrower using a contract with a different leverage ratio and a different interest rate, and the sorting occurs through the loan contracts that both lenders and borrowers endogenously choose. The market therefore features a high degree of customization and fragmentation, as in real world collateralized debt markets: for example, it is well known that repurchase agreements markets are organized Over-The-Counter (OTC).

An importance difference with assignment models a la Sattinger (1975) however is that the model features only one type of economic entity ex-ante, and that agents split between borrowers and lenders endogenously. In contrast, in previous models of competitive assignment, the mapping is usually from two initially distinct economic entities - skills of workers and complexities of jobs for example. The new predictions on Pareto distributions come from this endogeneity. Indeed, when disagreement tends to zero, the margins of contracts offered by very optimistic lenders tend to zero, and leverage of corresponding very optimistic borrowers go to infinity in a Paretian manner: this is because the most optimistic lenders have very close beliefs to that of the marginal buyers of the asset, hence they would almost be willing to buy the asset at the going price. The most optimistic borrowers, who effectively are borrowing from these lenders, are therefore able to achieve a very high leverage ratio. The fact that agents choose to become borrowers and lenders endogenously is a very important element of the model, and explains its novel implications.

The model also features a discontinuity at the common priors benchmark. When disagreement goes to zero, the leverage of the most optimistic borrowers goes to infinity because lenders

are less and less worried about the collateral. This effect is stronger than the diminished room for speculation allowed by lower disagreement. To the limit, very few borrowers end up borrowing almost all agents' wealth and the fat tailed distribution for leverage ratios obtains even for an epsilon amount of disagreement. This seems to suggest a relative fragility of the common prior assumption, as common prior models do not obtain from disagreement models by continuity when disagreement goes to zero.

The disagreement model with a continuum of types developed in this paper is not just relevant for understanding allocative premia and Pareto distributions for leverage ratios, but also helps uncover new results in line with empirical evidence. First, Zipf Law for the wealth of economic entities (a Power Law with exponent one) is shown to be a fixed point of the problem, and one towards which the economy converges when the static model is repeated. When the distribution of wealth on the support of beliefs is initially fat tailed, the skewness of the distribution of leverage and returns is negative when the distribution is initially more skewed than Zipf, and positive when it is initially less skewed than Zipf. This is due to a congestion effect: when optimistic borrowers are initially too numerous or too wealthy, they cannot leverage themselves from lenders as much. The model could therefore potentially give a microfounded alternative to random growth models with statistical frictions to explain why firm sizes follow Zipf's law in the upper tail (Gabaix (1999)), and shed a new light on firm dynamics more generally.

Second, because the Pareto distribution for leverage ratios can translate into Pareto distributions for capital or entrepreneurial incomes, the model can be extended to a simple dynamic setup to study the dynamics of the tail of the top incomes and of the wealth distributions. In this setting, it is shown to generate long run wealth and top income distributions with empirically observed tail exponents between one and three. The model could therefore potentially be used to understand why both returns to entrepreneurship and to investment banking are very skewed, as well as offer an account of their time series variations. More generally, it suggests a novel mechanism through which a market economy can very naturally generate a large amount of inequality, coming from borrowers' endogenous unequal access to credit. Moreover, because the Pareto distribution for top incomes generated in this model relies on a microfounded mechanism, the model can be potentially used for policy and counterfactual analysis. For example, the model suggests a tight connection between margin requirements, or banking regulation more generally, and top incomes and wealth inequality. It could therefore potentially be applied to study the increase in top income shares witnessed in advanced economies during the last three decades.²

Third, the model has implications for finance. For example, average leverage is increasing when disagreement decreases. When the wealth distribution becomes more skewed, leverage

²The mapping from the model to top income shares is perhaps most direct for investment bankers, who do take extremely large positions on financial markets. Indeed, Bell and Van Reenen (2013) estimate that three fifths of the gains for the top 1% income share between 1997 to 2007 in the United Kingdom went to workers in "Financial Intermediation" (see their Table 2). However, according to the model, margin requirements should impact entrepreneurs, and more generally borrowers, broadly defined.

risers as optimists with the same beliefs become more numerous; and conversely when the fat tail of optimistic investors is wiped out because of a crisis. In other words, the model generates an increase in margins without an assumption of "scary bad news", an important stylized fact that models with a single leverage ratio do not generate (Geanakoplos (2003)). The model also allows to investigate the consequences of introducing margin requirements on the price of risky assets. An interesting result that comes out of the analysis is that margin requirements interact with short-selling regulation: when leverage caps are introduced in a trading environment where short-selling is permitted, the margins both prevent optimists from leveraging and short-sellers from expressing their negative opinions about the asset, so that margin requirements can lead to exacerbate episodes of "irrational exuberance" about asset prices instead of dampening them. Methodologically, addressing this question requires having a model where agents can both short and lend in equilibrium, something that previous disagreement models did not allow, because all pessimists wanted to short rather than lend if they were given the possibility. Finally, the setting is sufficiently tractable that it allows to model "pyramiding lending arrangements", that is economies where loan contracts can also be used as collateral (or collateral can be rehypothecated). These arrangements, which played an important role in the last 2007 – 2009 financial crisis, are shown to allow asset prices to rise even further.

The rest of the paper proceeds as follows. Section 1 presents the main results in the simplest version of the model, where financial contracts are exogenously restricted to the set of Borrowing Contracts. Section 2 extends this simple model to the case of Short-Sales Contracts in section 2.1 and securitization in section 2.2. Section 3 shows two applications of the model, the effect of margin requirements on asset prices in section 3.1, and the generation of steady-state distributions for wealth with empirically relevant Pareto-Lorenz coefficients in section 3.2. Section 4 concludes.

Literature. Methodologically, this paper borrows from two distinct literatures: the disagreement models literature, and the competitive assignment literature. The basic building block is a disagreement model with endogenous collateral constraints a la Geanakoplos (1997). Geanakoplos and Zame (2002) show the general existence of a collateral equilibrium, and Geanakoplos (2003) investigates liquidity crises and the reason for countercyclical margins, a recurrent feature of the data. Geanakoplos (2010) studies the leverage cycle in the context of a single margin model, Simsek (2013) extends the model to short-sales, and to different types of disagreement (upside and downside optimism). Fostel and Geanakoplos (2012) study the fall in the bubble following the appearance of Credit Default Swaps. But all these papers maintain the assumption of a single leverage ratio. The paper is also part of the larger disagreement literature, starting with Miller (1977) and Harrison and Kreps (1978), and explaining the Internet Bubble and its subsequent crash in 2001, interpreting the crash as a decrease in short-sales constraints as lock-up restrictions were lifted (for example Hong et al. (2006) and Hong and Stein (2007)). There are without doubt institutional constraints which may explain why short-selling is relatively rare, see for example Duffie (1996), Jones and Lamont (2002), D’Avolio (2002), Duffie et al. (2002)

or Lamont and Stein (2004), but an interesting take from my model with short-sales is that short-sales are the exception rather than the norm, even when they are feasible. This paper also uses the methods of the competitive assignment literature, following Roy (1950), Roy (1951), Rosen (1974), Sattinger (1975), Heckman and Sedlacek (1985), Rosen (1981), Teulings (1995) or more recently Gabaix and Landier (2008) which was until now applied to study the assignment of workers to tasks, and which I apply here to the study of financial markets.

Substantially, the paper is part of the literature on Pareto distributions, and therefore speaks to the whole random growth literature, a survey of which is given in Gabaix (2009). Champernowne (1953) is the reference, and Gabaix (1999) and Luttmer (2007) are examples of derivations of Zipf laws for city and firm sizes respectively. It also sheds a new light on the credit spread puzzle: the fact that corporate bond spreads are difficult to reconcile with standard models pricing credit risk, a well-known and well-documented anomaly in finance. For example, Collin-Dufresne et al. (2001) show that variables that should in principle determine credit spread changes have little if any explanatory power. A large literature gives several candidate explanations for the anomaly, but Huang and Huang (2012) calibrate a number of structural models and show that they fall short of explaining the puzzle entirely. Chen et al. (2008) use habit formation in preferences to explain the Baa-Aaa credit spread. Bartolini et al. (2010) show that repurchase agreements' rates on government-sponsored agencies Mortgage-Backed Securities (MBS) as well as private label MBS are not meaningfully different from rates on interbank unsecured loans. Albagli et al. (2013) show that noisy information aggregation models can go in the way of explaining the credit spread puzzle, particularly for high investment grade bonds.

More broadly, this paper is part of a larger literature investigating the effect of borrowing constraints on asset prices and investment. It shows that entrepreneurs' access to credit is endogenously very unequal. The pioneering works on collateral are Bester (1985), Shleifer and Vishny (1992), Hart and Moore (1994) for example. Kiyotaki and Moore (1997) is another important paper emphasizing the feedback from the fall in collateral prices to that in debt capacity. In particular, a key element of my paper is that in order to benefit from one's beliefs, one needs capital as in Shleifer and Vishny (1997).³ This is because other investors in the market do not share the same beliefs, and hence want to be protected if there is default. There are of course many other theories of borrowing limits, that stem from information asymmetries, lack of commitment, or exogenously imposed margin requirements. See, among many others Holmstrom and Tirole (1997), Holmstrom and Tirole (1998), Bernanke et al. (1999), Gromb and Vayanos (2002), Brunnermeier and Pedersen (2009), Gorton and Metrick (2012).

³At an abstract level, the "risky asset" of the paper may indeed very well be Shleifer and Vishny (1997)'s introductory example: the difference between the price of two Bund futures contracts. The belief that two Bund futures contracts, delivering the same exact value at time T , will converge before that, leads some traders to potentially take extreme positions in that direction. But they are able to maintain this position only if they have enough capital until time T , when it will converge for sure. LTCM was actually doing these types of trades, acting as the infinitely leveraged, very optimistic trader of the model.

1 Borrowing Economy \mathcal{E}^B

This section provides the simplest environment to get at the main results of the paper: assortative matching, assignment interest rates, the Pareto distribution for leverage, the "attracting" property of Zipf law, and the generally non-monotonic relationship between leverage and returns. In this section, I assume that agents cannot sell the real or financial assets short and that pyramiding lending arrangements, that is the securitization of loans, or rehypothecation of collateral, are not possible. These assumptions are relaxed in Section 2.

1.1 Setup

There are two periods t and $t + 1$. Subscript t will be omitted for all time t quantities, for conciseness. There is a continuum of agents of measure one born in period t with initial wealth w . Agents care only about period $t + 1$'s consumption, and therefore need assets to store their wealth.

Assets. To transfer wealth into period $t + 1$, agents can invest in *Cash*, with return normalized to $R = 1$, and an Asset in finite supply S , with exogenous price p_{t+1} in period $t + 1$, and endogenous price p in period t , which I will refer to as the *Real Asset* or the *Asset* in the following. In addition, they can agree to collateralized Borrowing Contracts with each other. Formally, I define a Borrowing Contract in Economy \mathcal{E}^B as follows.

Definition 1 (Borrowing Contract, Economy \mathcal{E}^B). *A Borrowing Contract (ϕ) in economy \mathcal{E}^B is a promise of $\phi \geq 0$ units of Cash in period $t + 1$, collateralized by one unit of Real Asset. ϕ is called the face value of the Borrowing Contract.*

Note that without loss of generality, the set of contracts is restricted to a set containing contracts using exactly one unit of Real Asset as collateral, and that this allows to name contract by their face value (ϕ) . Contracts are traded in an anonymous market at competitive price $q(\phi)$, and payment is only enforced by the collateral: agents default as long as the value of the collateral is lower than the face value of the loan they have to repay. The payoff of contract (ϕ) in period $t + 1$ is therefore:

$$\min\{\phi, p_{t+1}\}$$

for a contract with face value ϕ . In period t , this contract is sold by the borrower, who gets $q(\phi)$ units of Cash in exchange for the contract. I will denote by $r(\phi)$ the implicit interest rate on this Borrowing Contract given by:

$$r(\phi) = \frac{\phi}{q(\phi)}.$$

Beliefs. There is disagreement among agents in period t as to what the price of the Real Asset p_{t+1} will be, and agents have point expectations about this future price denoted by p_{t+1}^i for agent i : that is, they are certain of what the price of the Real Asset will be in period $t + 1$ -

therefore, some of them must be wrong.⁴ This assumption of point expectations may seem a bit extreme, however the model generalizes very straightforwardly to a case where agents are risk neutral and agree about a probability distribution for p_{t+1} around this mean.⁵ In other words, I focus on disagreement about means about future asset payoffs rather than about probabilities of certain events, as in the previous heterogenous beliefs and endogenous margins literature. More precisely, the cumulative distribution function representing the wealth of agents with beliefs p_{t+1}^i for future prices is denoted by $F(\cdot)$. The density function representing the wealth of agents on the support of beliefs is $f(\cdot)$, and the upper bound on agents' beliefs is denoted by M and the lower bound m , with $m < M$ (heterogenous beliefs assumption). Moreover, it is assumed that $M \leq w/S$, which is an upper bound for the price of the Real Asset in $t + 1$ if the endowment in $t + 1$ is also w . Therefore, for compactness, this density function represents both how many agents have those beliefs and how wealthy they are. In this static model, the wealth of each agent is equal and normalized to w and therefore it is equivalent to assume that the beliefs p_{t+1}^i of agents i are drawn randomly in the cumulative distribution function $F(\cdot)$.

Equilibrium. All units of the Real Asset are initially endowed to unmodelled agents who sell their asset holdings in period t and then consume. Agent i chooses his positions in the Real Asset n_A^i , a menu of financial Borrowing Contracts (ϕ) denoted by distribution $n_B^i(\cdot)$ and Cash n_C^i , in order to maximize his expected wealth (**W**) in period $t + 1$ according to his subjective beliefs p_{t+1}^i about the Real Asset, subject to his budget constraint (**BC**), the collateral constraint (**CC**):

$$\max_{(n_A^i, n_B^i(\cdot), n_C^i)} n_A^i p_{t+1}^i + \int_{\phi} n_B^i(\phi) \min\{\phi, p_{t+1}^i\} d\phi + n_C^i \quad (\text{W})$$

$$\text{s.t.} \quad n_A^i p + \int_{\phi} n_B^i(\phi) q(\phi) d\phi + n_C^i \leq w \quad (\text{BC})$$

$$\text{s.t.} \quad \int_{\phi} \max\{0, -n_B^i(\phi)\} d\phi \leq n_A^i \quad (\text{CC})$$

$$\text{s.t.} \quad n_A^i \geq 0, \quad n_C^i \geq 0$$

Note this portfolio problem is subject to the additional restriction that $n_A^i \geq 0, n_C^i \geq 0$, that is agents have to choose positive amount of Real Asset and Cash holdings. When $n_B^i(\phi) > 0$, agent

⁴The source of disagreement for prices in period $t + 1$ does not matter here. The model thus includes the particular cases of Harrison and Kreps (1978) (disagreement on dividends), but also of disagreement on resale values themselves. In particular, in the case of a dynamically inefficient economy, the Real Asset can have a rational bubble element to them (Tirole (1985)), so that agents must also entertain independent beliefs about prices. See Geerolf (2013) for an empirical reassessment of dynamic inefficiency in the line of Abel et al. (1989). In that interpretation, agents disagree about which rational expectations equilibrium will prevail next period, and they do disagree about means of asset prices rather than about the probability of different states, unlike in Geanakoplos (1997) and Simsek (2013).

⁵In particular, in that case, all Borrowing Contracts will in equilibrium be indexed by these states, in which the payoff of the asset differs from its expected mean by an amount that everybody agrees on.

i buys Borrowing Contract (ϕ) , and therefore lends. When $n_B^i(\phi) < 0$, agent i sells Borrowing Contract (ϕ) , and therefore borrows.⁶ Each time a borrower sells of unit of Borrowing Contract, he needs to own unit of asset hence equation (CC). The equilibrium concept is that of a *collateral equilibrium*, as defined for example in Geanakoplos (1997), Geanakoplos and Zame (2002) and Simsek (2013), with contracts being treated as commodities. Formally,

Definition 2 (Competitive Equilibrium of \mathcal{E}^B). *A Competitive Equilibrium for Economy \mathcal{E}^B is a price p for the Real Asset and a distribution of prices $q(\cdot)$ for all traded Borrowing Contracts (ϕ) , and portfolios $(n_A^i, n_B^i(\phi), n_C^i)$ for all agents i in the Real Asset, Borrowing Contracts and Cash, such that all agents i maximize expected $t + 1$ period wealth (W) according to their subjective beliefs, subject to budget constraint (BC), collateral constraint (CC), and markets for the Real Asset and Borrowing Contracts clear:*

$$\int_i n_A^i di = S, \quad (MC_A)$$

$$\text{and } \forall \phi, \quad \int_i n_B^i(\phi) di = 0. \quad (MC_B)$$

The assumption of point expectations is crucial, and allows me to solve for the equilibrium of the model. My model thus departs from Geanakoplos (1997) and Simsek (2013) who assumed that agents agreed on the value of collateral conditional upon default, and disagreed either on the value of collateral conditional on other states occurring or on probabilities of different states. Here, in contrast, the value of collateral conditional on default is different for every lender in each possible state, which is what provides me with a distribution of leverage ratios.⁷

Note also that it is assumed that agents cannot impose penalties upon each other other than collateral seizure. In this model, agents do not have any income in period $t + 1$. In a model where they do have such income or other penalties (such as jail) can be imposed, this assumption is tantamount to a no-recourse assumption.

1.2 Equilibrium: Positive Assortative Matching and Allocative Premia

The following proposition shows that at a competitive equilibrium of Economy \mathcal{E}^B , more optimistic agents buy the asset borrowing from moderately optimistic agents, using the asset as collateral, while pessimistic agents invest in cash. Moreover, the equilibrium is characterized by assortative matching between borrowers and lenders, with more optimistic investors borrowing from more optimistic lenders; and this matching occurs through allocative premia on bonds that

⁶This convention follows those used in repurchase agreements. The seller of the repurchase agreement is the borrower and sells the security used in collateral, agreeing to buy it back at a later date. In contrast, the lender buys the repurchase agreement as well as the security. (reverse-repo)

⁷Moreover, relative to Simsek (2013), I work with a continuum of types instead of two, which allows me to potentially have short-selling, lending, and investing in equilibrium. Geanakoplos (1997) also considered a continuum of agents, but he then always assumes that they all agree on the asset payoff conditional on there being a bad state, which effectively pins down the face values of the loans for one unit of collateral. In contrast, in the equilibrium of my model, many contracts with different face values will be traded in equilibrium, and hence margins and haircuts are heterogeneous.

optimistic borrowers sell. For expositional convenience, this Proposition 1 however only highlights the most important elements in the competitive equilibrium of Economy \mathcal{E}^B . The reader is referred to the end of this section for a complete characterization of equilibrium prices and agents' portfolios.

Proposition 1 (Equilibrium of Economy \mathcal{E}^B). *At a competitive equilibrium of Economy \mathcal{E}^B ,*

1. *The space of agents' beliefs $[m, M]$ is partitioned through π'' and π into three intervals:*
 - *Agents i with beliefs $p_{t+1}^i \in [m, \pi'']$ (cash investors) invest in Cash.*
 - *Agents i with $p_{t+1}^i \in [\pi'', \pi]$ (lenders) buy Borrowing Contracts.*
 - *Agents i with $p_{t+1}^i \in [\pi, M]$ (borrowers) buy the Asset and sell Borrowing Contracts.*
2. *Lenders with beliefs x buy Borrowing Contracts (x) with face value x , sold by borrowers of type $y = \Gamma(x)$. $\Gamma(\cdot)$ is a strictly increasing function from beliefs of lenders in $[\pi'', \pi]$ to beliefs of borrowers in $[\pi, M]$ (positive assortative matching).*
3. *The implicit interest rate of Borrowing Contract (x) , $r(x)$, and the assignment function $\Gamma(x)$ are solutions of a system of two first-order Ordinary Differential Equations:*

$$x(\Gamma(x) - x)r'(x) - \Gamma(x)r(x) + pr(x)^2 = 0 \quad \text{with} \quad r(\pi'') = 1 \quad (\text{A})$$

$$f(\Gamma(x))\Gamma'(x) = \frac{pr(x) - x}{x}f(x) \quad \text{with} \quad \Gamma(\pi'') = \pi. \quad (\text{C})$$

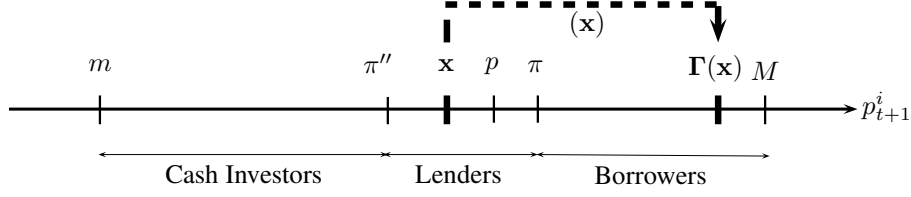
4. *π'' , π and p (the asset price) are such that $\pi'' \leq p \leq \pi$ and solution to:*

$$(a) \quad \frac{1 - F(\pi'')}{p} = \frac{S}{w}, \quad (b) \quad r(\pi) = \frac{\pi - \pi''}{p - \pi''}, \quad \text{and} \quad (c) \quad M = \Gamma(\pi). \quad (1)$$

Discussion. The first point of proposition 1 states that there are Cash Investors, Lenders, and Borrowers / Asset Buyers in this economy, as in Geanakoplos (1997) and Simsek (2013). An important difference though with these papers is that the marginal buyer of the Real Asset does not have beliefs equal to the price of the Real Asset, because his outside option is to lend and not to invest in Cash, and that lending brings an excess return relative to Cash, a point I shall come back to later.

The second point states that lenders with beliefs $p_{t+1}^i = x$ buy Borrowing Contracts with face value x for one unit of collateral (contract (x) of Definition 1), which means that more optimistic lenders lend buying contracts with a higher face value or a higher leverage ratio (equivalently, less collateral). This is a direct consequence of the assumed structure of beliefs. Lenders believe that the collateral will be worth x in all states of nature, they are therefore willing to lend more against each unit of collateral if they have a relatively higher p_{t+1}^i . This effect is absent in Geanakoplos (1997) and Simsek (2013), where lenders agree on the value of

ECONOMY \mathcal{E}^B : BORROWING, NO PYRAMIDING, NO SHORT-SALES



Note: Agents have heterogenous beliefs $p_{t+1}^i \in [m, M]$. Agents' population partitions itself endogenously into Cash Investors, Lenders and Borrowers, according to cutoffs π'' and π . p is the endogenous price of the Real Asset. Lenders with beliefs $p_{t+1}^i = x$ buy Borrowing Contracts (x) with face value x and one unit of collateral, sold by borrowers with beliefs $p_{t+1}^i = \Gamma(x)$. Lenders with beliefs x and borrowers with beliefs $\Gamma(x)$ are thus *effectively matched* through Borrowing Contracts (x) , even though Borrowing Contracts are traded anonymously.

collateral conditional on the default state, which is the one relevant for pricing the corresponding Borrowing Contract. This is why they have only one Borrowing Contract and hence one leverage ratio while this model has a distribution of them - and, for low disagreement, this distribution of these Borrowing Contracts have a corresponding distribution of leverage ratios which is close to a Pareto of shape two, as I will show later.

Moreover, these more optimistic lenders are *de facto*, at the competitive equilibrium of this economy, lending to more optimistic borrowers. There is positive assortative matching between borrowers and lenders, expressed by the means of a strictly increasing assignment function $\Gamma(\cdot)$ mapping the beliefs of a lender with that of the corresponding borrower on the other side of the Borrowing Contract. Lenders with beliefs x buy Borrowing Contracts (x) with face value x , and borrowers with beliefs $\Gamma(x)$ sell these Borrowing Contracts. This arrangement is the one which maximizes the aggregate surplus: a higher leverage ratio for a collateralized loan is worth more to a more optimistic borrower, because the extra leverage he can achieve gives him a higher expected wealth. The way in which more pessimistic borrowers do not want to choose these contracts at the competitive equilibrium is that implicit interest rates on higher leverage contracts rise to ensure that only more optimistic borrowers want to sell them. Equation (A), in the third part of proposition 1 is an assignment equation expressing the fact that interest rates rise just enough so that more pessimistic borrowers are excluded from contracts with a higher leverage ratio. Equation (C) is a market clearing condition on financial Borrowing Contracts expressing how many lenders there must be in an interval around x to be lending to corresponding borrowers in a interval around $\Gamma(x)$. The mathematical details of the proof of this proposition 1 are in Appendix B.1.1, but it is now useful to develop some intuition for all of these results.

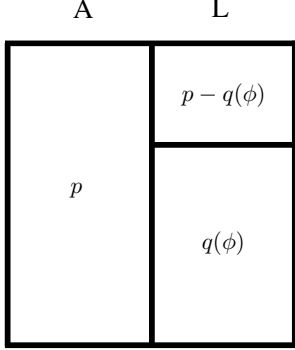
Agents' Behavior. The first point of the Proposition 1 states that at the equilibrium of the Borrowing Economy \mathcal{E}^B , traders split into three sets with respect to the type of investments

they make: there are Cash investors, lenders and leveraged investors, as in Simsek (2013) and Geanakoplos (1997). I refer to Appendix B.1.1 for a proof. Intuitively, the reason why there are lenders is that investors always have an incentive to borrow from moderately optimistic agents, using assets they buy as collateral, because they want to lever up into the asset which has a positive excess return. But this logic has its limit, as there are a limited number of Real Assets which can potentially be used as collateral, and therefore the most pessimistic investors cannot be offered collateralized Borrowing Contracts and invest in cash. Ultimately, this is coming from the previously imposed discipline on beliefs, the fact that $M \leq w/S$. If all agents' endowments were invested in the asset, then the price of the real asset would be w/S by market clearing, and therefore the measure of buyers would be zero as long as the density of beliefs does not have a mass point for beliefs equal to M , a contradiction. Note that here, even though the return on cash is $R = 1$, the marginal buyer does not have beliefs equal to the price of the asset, but has beliefs π strictly greater than the price of the asset p . The reason is that in this model, because of the assignment process I shall talk about later, lenders earn an excess return relative to cash investors. For agents with beliefs just above the price of the asset p , it is therefore more profitable to lend than to invest in the asset. Let us turn now into more detail to points two to four of proposition 1: the determination traded contracts (point two), interest rates and assignment function (point three), and cutoffs (point four).

Traded Contracts. Denote lenders by their beliefs about the future price of the Real Asset $x = p_{t+1}^i \in [\pi'', \pi]$, and borrowers by $y = p_{t+1}^i \in [\pi, M]$. In contrast to Geanakoplos (1997) and Simsek (2013), the competitive equilibrium in Definition 2 is a priori a high dimensional object. The set of contracts that borrowers and lenders can potentially sign is indexed by all possible face values on the real line $\phi \in \mathbb{R}_+$, and the double continuum of agents (x, y) a priori needs to choose its position in all these traded contracts. However, it is possible to show that a lender of type x will choose contract (x) , that is will never choose an over or an under-collateralized contract relative to his beliefs. Second, the assortative matching property pins down the type of contracts that borrowers with beliefs y will choose, and that come from the limited supply of contracts of type (x) , coming from the limited supply of lenders' equity (see Appendix B.1.1 for a rigorous derivation of these steps).

To show that a lender of type with beliefs $p_{t+1}^i = x$ will buy contract (x) in equilibrium, note that the payoff for a lender of type x of buying Borrowing Contract (ϕ) is given by $\min\{\phi, x\}$. In a competitive equilibrium, a lender with beliefs x will never buy a Borrowing Contract (ϕ) with $\phi < x$, which is an over-collateralized contract for him. This is because a borrower always strictly prefers to borrow through contracts with lower collateral, but increasing collateral does not give anything to the lender. Therefore $\phi \geq x$. Moreover, all contracts with $\phi > x$ are undercollateralized contracts, and they lead to default for sure for lenders, who expect to recoup one unit of the collateral, and are thus all equivalent for them. For borrowers on the other hand, who expect to repay their loans, they are strictly worse, so they will never be traded in equilibrium. Consequently $\phi \leq x$. All in all, contracts will neither be under nor over collateralized

with respect to lender's beliefs, and $\phi = x$: a lender with beliefs x will only trade a Borrowing Contract of type (x) .



E

D

To understand which contracts borrowers choose to sell in equilibrium, note as a preliminary that $\pi \geq p$ because otherwise borrowers with valuation in $[\pi, p]$ would prefer to invest in Cash. Optimists' unlevered expected returns are therefore strictly greater than the returns to Cash $R = 1$ for $y > \pi \geq p$, and equal to $\frac{y}{p}$, so that all borrowers would ideally like to leverage themselves into the asset as much as possible, or use loans with the lowest possible margins. However, expected returns for highly optimistic borrowers are larger than that for less optimistic borrowers, and they thus benefit more from borrowing at the margin. In other words, there is *complementarity* between beliefs of borrowers and leverage, and therefore complementarity between the

beliefs of borrowers and that of lenders. Indeed, more optimistic lenders allow borrowers to achieve a higher leverage ratio, or to buy a greater number of real assets, because loans with higher face values have a higher implicit leverage ratio. (this intuitive property is actually an equilibrium statement, but will be proved later) The complementarity between the beliefs of lenders and that of borrowers is the key intuition behind the assortative matching result. In equilibrium, this positive assortative matching will arise in the competitive equilibrium through rising interest rates on high leverage ratio loans so that lenders' wealth go more predominantly to borrowers who want it most.

Formally, denote by $ROE^B(p_{t+1}^i, \phi)$ the expected wealth of an optimist with beliefs p_{t+1}^i in period $t + 1$, who borrows with a contract of face value ϕ . Denote the leverage ratio on the loan with face value ϕ by $l(\phi)$, and given by:

$$l(\phi) = \frac{q(\phi)}{p - q(\phi)} = \frac{\phi}{pr(\phi) - \phi}$$

Selling one loan contract of type ϕ requires buying one unit of asset at price p first which then allows to raise $q(\phi)$ from this sale, using the asset as collateral, and complementing with one's own funds (see the Figure above for an illustration of this trade). Note that borrowers will bind on their collateral constraint (CC), because they want to leverage themselves as much as possible, and so they always sell one borrowing contract when they buy one unit of asset. With one unit of wealth (equity), a borrower can therefore sell $1/(p - q(\phi))$ loan contracts, raising an amount $l(\phi)$ given by:

$$l(\phi) = \frac{q(\phi)}{p - q(\phi)}.$$

Because $l(\phi)$ is the leverage that a loan contract (ϕ) allows to achieve⁸, an optimist with

⁸Note that leverage here is equal to the amount of the loan divided by the equity of the borrower (D/E on the figure on the left). Sometimes, it is defined as the total amount of assets divided by equity (A/E above), or $1 + l(x)$ using the notations of this model, but it does not change anything apart from a constant.

beliefs p_{t+1}^i has an expected return $ROE^B(p_{t+1}^i, \phi)$ conditional on selling a contract of type ϕ given as the mean of the expected return on the asset according to his beliefs p_{t+1}^i/p with weight the size of the purchase for each unit of equity $1 + l(\phi)$ and the return given to lenders $r(\phi) = \phi/q(\phi)$ with weight the size of the borrowed funds $-l(\phi)$ ⁹:

$$ROE^B(p_{t+1}^i, \phi) = \frac{p_{t+1}^i}{p}(1 + l(\phi)) - r(\phi)l(\phi).$$

This expected return is strictly supermodular in the beliefs of borrowers p_{t+1}^i and the face value of the loan ϕ . This is because the price of loan contracts $q(\phi) = \phi/r(\phi)$ is increasing in ϕ if the contract is traded: no borrower would borrow selling a contract with a lower price and a higher repayment in $t + 1$. From $q'(\phi) > 0$ it follows that the leverage ratio $l(\phi)$ is strictly increasing in the face value of the loan ϕ . Therefore so is the cross derivative of the expected return of borrowers buying a contract of type ϕ , which gives strict supermodularity:

$$q'(\phi) > 0 \quad \Rightarrow \quad l'(\phi) = \left(\frac{q(\phi)}{p - q(\phi)} \right)' > 0 \quad \Rightarrow \quad \frac{\partial^2 ROE^B(p_{t+1}^i, \phi)}{\partial p_{t+1}^i \partial \phi} = \frac{1}{p} l'(\phi) > 0.$$

Because borrowers choose the face value of the contract ϕ to maximize $ROE^B(p_{t+1}^i, \phi)$, the first order condition gives by total differentiation:

$$\begin{aligned} \frac{\partial ROE^B(p_{t+1}^i, \phi)}{\partial \phi} = 0 & \Rightarrow \frac{\partial^2 ROE^B(p_{t+1}^i, \phi)}{\partial^2 \phi} d\phi + \frac{\partial^2 ROE^B(p_{t+1}^i, \phi)}{\partial p_{t+1}^i \partial \phi} dp_{t+1}^i = 0 \\ & \Rightarrow \frac{d\phi}{dp_{t+1}^i} = - \frac{\frac{\partial^2 ROE^B(p_{t+1}^i, \phi)}{\partial p_{t+1}^i \partial \phi}}{\frac{\partial^2 ROE^B(p_{t+1}^i, \phi)}{\partial^2 \phi}} > 0. \end{aligned}$$

The last statement results from the fact that ϕ is chosen so as to maximize $ROE^B(p_{t+1}^i, \phi)$ given p_{t+1}^i , so the second derivative in the denominator is negative, together with the supermodularity condition in the numerator derived earlier. Using the fact that lenders choose to lend with Borrowing Contracts which have face values equal to their beliefs about the Real Asset, one concludes that more optimistic borrowers borrow from relatively more optimistic lenders.

At the competitive equilibrium of this economy, there is consequently assortative matching between borrowers and lenders, as in Sattinger (1975) or more recently Teulings (1995). An interesting feature of the model is therefore that starting from anonymous markets as in Geanakoplos (1997), the market reveals to have a high degree of customization and fragmentation, as in real world repos markets which are organized Over-The-Counter (OTC)¹⁰: in this model, each borrower is effectively borrowing from a different lender with a different interest rate, and a different leverage ratio, both of which are all the higher that borrowers and lenders are relatively more optimistic. The double dimensionality with potential contracts signed between any lender-borrower

⁹This is heuristic. In Appendix B.1.1, this expected return $ROE^B(p_{t+1}^i, \phi)$ is derived rigorously from the agents' problem, and in particular the Collateral Constraint (CC) and the Budget Constraint (BC).

¹⁰Rigorously, a more detailed model would be needed to understand how very small costs of setting up walrasian markets, and there are a double continuum of them there, could justify setting up an OTC structure.

pair $(x, y) \in [\pi'', \pi] \times [\pi, M]$, is thus reduced to a one dimensional problem, with the set of traded contracts given by:

$$\left\{ (x)_x^{\Gamma(x)}, x \in [\pi'', \pi] \right\},$$

where $\Gamma(\cdot)$ is the strictly increasing assignment function mapping lenders to borrowers (so that $y = \Gamma(x)$). This strictly increasing assignment function comes from the observation that lenders with beliefs x have limited wealth, and therefore borrowing contracts with a given level of leverage also are in limited supply. Importantly, this equilibrium arises here as in assignment models as a result of competitive forces: it is not that borrowers will be looking for particular lenders to borrow, but given prices, they will only be looking for a certain type of contract. But in the competitive equilibrium, it so happens that there is a one-to-one mapping between contracts and lenders because lenders only buy contracts with a face value equal to their beliefs $x = \phi$. We are therefore left to determine the assignment function as well as the prices of the contracts so that this mapping arises at the competitive equilibrium.

Interest Rates and Assignment Function. Two first-order differential equations and two initial conditions then determine functions $\Gamma(\cdot)$ and $r(\cdot)$. The first initial condition is $\Gamma(\pi'') = \pi$, which expresses that the most pessimistic lender with beliefs π'' must be matched with the most pessimistic borrower and beliefs π . The second is $r(\pi'') = 1$, given by contradiction. If $r(\pi'') > 1$, then Cash investors with beliefs just below π'' find it optimal to lend too, which is not an equilibrium. If $r(\pi'') < 1$, then lenders just above beliefs π'' are better off investing in Cash.

As for the differential equations, the first equation is a mechanical, accounting equation expressing that optimists need to use their equity to fund asset purchases that they cannot fund through borrowing. Consider lenders in a small interval $[x, x + dx]$, who buy contracts in set $\{(u, \Gamma(u)), u \in [x, x + dx]\}$. The optimists selling those contracts on the other side then belong to the set $[y, y + dy] = [\Gamma(x), \Gamma(x) + \Gamma'(x)dx]$, since $dy = \Gamma'(x)dx$. Their endowment is given by $wf(y)dy$, and this must complement lenders' providing of funds to purchase the Real Asset. Now lenders of type x will contribute their endowment $wf(x)dx$ to the buying of assets. Using this endowment, they will buy $\frac{wf(x)dx}{q(x)}$ loan contracts of type (x) . Optimists will need to complement the purchase of the corresponding collateral to the amount of $p - q(x)$. This gives:

$$w \frac{p - \frac{x}{r(x)}}{\frac{x}{r(x)}} f(x) dx = wf(y) dy.$$

From this equation, one arrives at the accounting equation:

$$\Gamma'(x) f(\Gamma(x)) = \frac{pr(x) - x}{x} f(x). \quad (C)$$

A second equation is obtained by noting that for the above derived assortative matching to be an equilibrium, prices of contracts must adjust so that when choosing their favorite contracts, optimists are effectively implementing the assortative matching equilibrium. The way this happens is that more optimistic borrowers borrow with higher leverage ratios but also higher interest

rates, so that less optimistic borrowers will not want to use these high leverage ratio loans. This intuition of prices playing an allocative role is known in the assignment literature since Sattinger (1975), Rosen (1981) and used in Gabaix and Landier (2008) among many others, but the key contribution of this paper is to apply it to financial markets. The assignment equation is obtained by noting that a choice of a face value of the Borrowing Contract ϕ for borrower y , when maximizing his expected return on equity $ROE^B(y, \phi)$ must lead him to choose the contract also chosen by the lender to whom is assigned to by the assignment function, so that $y = \Gamma(x) = \Gamma(\phi)$:

$$\Gamma^{-1}(y) = \arg \max_{\phi} \frac{y - \phi}{p - q(\phi)} = \arg \max_{\phi} \frac{y - \phi}{pr(\phi) - \phi} r(\phi).$$

Because a lender with beliefs x chooses a contract of face value $\phi = x$ to lend, it must be that the choice of the face value by the borrower is such that $\phi = \Gamma^{-1}(y)$. A first order condition for this maximization program and the use of the implicit interest rate $r(\phi) = \phi/q(\phi)$ instead of the price of the Collateralized Bond allows to write a second first-order differential equation, the *assignment equation*:

$$x(\Gamma(x) - x)r'(x) - \Gamma(x)r(x) + pr(x)^2 = 0 \quad (\text{A})$$

Importantly, because of the competition of Cash Investors for lending, the interest rates will be minimal such that this constraint is met, which shows up in the initial condition $r(\pi'') = 1$.

To build intuition, one can refer to the [Online Appendix](#), where the balance sheet of a borrower of type y matched with a lender of type x are shown, and where aggregate leverage, margins, and assets are also shown and derived.

Cutoffs. The equations defining the aggregate cutoffs can finally be derived. Equation (3a) is the market clearing condition, expressing that both lenders and buyers' equity are used to finance the purchase of the Real Asset. Sp is asset supply, while $(1 - F(\pi''))w$, equal to borrowers' and lenders' funds that is $(1 - F(\pi))w + (F(\pi) - F(\pi''))w$, is asset demand, and therefore:

$$\frac{1 - F(\pi'')}{p} = \frac{S}{w} \quad (\text{3a})$$

Equation (3b) expresses that agents with beliefs equal to π must be indifferent between lending to optimists with beliefs M and investing in the asset, borrowing from agents with beliefs π'' . This allows to arrive at a "pasting" equation:

$$r(\pi) = \frac{\pi - \pi''}{p - \pi''} \quad (\text{3b})$$

Finally, equation (3c) results from the fact that the most optimistic borrower is matched at the competitive equilibrium with the most optimistic lender $M = \Gamma(\pi)$.

Formal Expressions for Agents' Portfolios. Proposition 1 fully characterizes prices at the competitive equilibrium in Definition 2: the price for the real asset p , and prices $q(\cdot)$. Points

1 and 2 of Proposition 1 however only characterize agents' portfolios partially. It is possible to characterize them completely using formal notations in Definition 2 (again, see Appendix B.1.1 for a full proof). $\delta_x(\cdot)$ denotes the Dirac measure with mass point at x .

- Portfolios of cash investors with beliefs such that $p_{t+1}^i \in [m, \pi'']$ are:

$$n_A^i = 0, \quad n_B^i(\cdot) = 0, \quad n_C^i = w.$$

- Portfolios of lenders with beliefs such that $p_{t+1}^i \in [\pi'', \pi]$ are:

$$n_A^i = 0, \quad n_B^i(\cdot) = \frac{w}{q(p_{t+1}^i)} \delta_{p_{t+1}^i}(\cdot), \quad n_C^i = 0.$$

- Portfolios of borrowers with beliefs such that $p_{t+1}^i \in [\pi, M]$ are:

$$n_A^i = \frac{w}{p - q(\phi')}, \quad n_B^i(\cdot) = -\frac{w}{p - q(\phi')} \delta_{\phi'}(\cdot), \quad n_C^i = 0,$$

with ϕ' given by: $\phi' = \arg \max_{\phi} \frac{p_{t+1}^i - \phi}{p - q(\phi)}.$

The competitive equilibrium defined in Definition 2 is then completely characterized by these portfolios on the one hand, and equations (A), (C), (3a), (3b), (3c) on the other. Note that equation (3a) obtains from aggregating all consumers' budget constraint (at equality), using market clearing for the real asset (MC_A) and all borrowing contracts (MC_B):

$$\int_i n_A^i p di + \int_i \int_{\phi} n_B^i(\phi) q(\phi) d\phi di + \int_i n_C^i di = \int_i w di \quad \Rightarrow \quad pS + \int_i n_C^i di = w.$$

Noting that only agents with $p_{t+1}^i \in [m, \pi'']$ buy cash in quantity $n_C^i = w$ allows to show equation (3a) from:

$$\int_i n_C^i di = wF(\pi'').$$

Finally, note that the market clearing equation for financial contracts (C) obtains with these notations from (MC_B):

$$\int_i n_B^i(\phi) di = 0 \quad \Rightarrow \quad -\frac{w}{p - q(x)} f(\Gamma(x)) d\Gamma(x) + \frac{w}{q(x)} f(x) dx = 0.$$

Computational Remarks. As I will show in Section 1.6, closed form expressions for cutoffs, portfolios, as well as all prices in the model can be obtained in the case where the density of beliefs is uniform over the segment $[m, M]$. In general, if the density of beliefs $f(\cdot)$ takes any shape, closed form expressions can of course not be obtained. The equilibrium however displays some common properties over all these density functions, which are the subject of the following sections: allocative interest rates (Section 1.3), Pareto distributions for leverage of shape two for bounded distributions $f(\cdot)$ (Section 1.4), and finally a generally non-monotonic relationship between leverage and returns (Section 1.5).

In all cases, the equilibrium can be very easily obtained on the computer as follows. First, it is possible to transform the two first-order differential equations derived above (A) and (C) into one second-order differential equation (see Appendix B.1.1):

$$\left(\frac{f(\Gamma(x))}{f(x)} \Gamma'(x) \right)' (\Gamma(x) - x) + \frac{f(\Gamma(x))}{f(x)} \Gamma'(x) + \left(\frac{f(\Gamma(x))}{f(x)} \right)^2 \Gamma'(x)^2 = 0.$$

Initial conditions for equations (A) and (C) can be expressed as two conditions on $\Gamma(\cdot)$:

$$\Gamma(\pi'') = \pi \quad \text{and} \quad \Gamma'(\pi'') = \frac{p - \pi''}{\pi''} \frac{f(\pi'')}{f(\pi)}.$$

Given (π'', p, π) the full schedule for the assignment function $\Gamma(\cdot)$ is now determined, as well as the full schedule for bond prices, through $r(\cdot)$ given by:

$$r(x) = \frac{x}{p} \left[1 + \frac{f(\Gamma(x))}{f(x)} \Gamma'(x) \right].$$

Finally, (π'', p, π) are given numerically by equations (3a), (3b), (3c) when closed form solutions are not available. Computationally, the full solution to the differential equation need not be calculated to get at the cutoffs: only $\Gamma(\pi)$ and $\frac{f(\Gamma(\pi))}{f(\pi)} \Gamma'(\pi)$ are calculated, since the cutoffs obtain from those two values only:

$$\frac{1 - F(\pi'')}{p} = \frac{S}{w}, \quad \frac{\pi}{p} \left[1 + \frac{f(\Gamma(\pi))}{f(\pi)} \Gamma'(\pi) \right] = \frac{\pi - \pi''}{p - \pi''}, \quad M = \Gamma(\pi).$$

But before giving an example where all these quantities can be calculated in closed form (Section 1.6) let us turn first to some general properties of the competitive equilibrium defined above.

1.3 Allocative Interest Rates

Allocative Interest Rates. Interest rates turn out to be strictly greater than the rate of return to the storage technology in this model, even though agents are risk-neutral and lenders expect to make no losses from their investments. There are excess returns on loan contracts. The intuition for why interest rates are higher than $R = 1$, or the returns to Cash, in this model, is that interest rates play an allocative role in excluding pessimistic borrowers from contracts allowing a high leverage ratio. Interest rates here therefore do not reflect either time preference (the model contains none), or risk, as they usually do, but they reflect the fact that borrowers must part with some of their expected surplus, and give it to lenders, in order to prevent more pessimistic borrowers to borrow with those contracts they want most.

The Credit Spread Puzzle. The fact that risk premia are too high to be explained by default probabilities and losses upon default is an important and well-established anomaly from the point of view of finance theory (Cochrane (2011)). For example, Collin-Dufresne et al. (2001) show that variables that should in principle determine credit spread changes have little if any explanatory power. Another example among others is the puzzle in Bartolini et al. (2010), that

repurchase agreements' rates on Mortgage Backed Securities secured by government-sponsored agencies, and private-label MBS were not meaningfully different from rates on unsecured inter-bank loans. The question is then why banks use secure funding if that does not allow them to get a pricing advantage. Many papers have tried to address this problem, yet is able to fully resolve the puzzle quantitatively (Huang and Huang (2012)), at least without assuming unrealistically high parameters for risk aversion for example. Another example is Chen et al. (2008), who use habit formation in preferences to explain the Baa-Aaa credit spread. More closely related to this paper, Albagli et al. (2013) use a noisy Rational Expectations Model to show that it can account for at least part of the puzzle, specifically for high investment grade bonds. As in this paper, this is derived in a framework with risk-neutral agents.

Discussion. Note that because loans with higher leverage (or lower margins) are those which default for a higher range of realization of the collateral, "allocative premia" can very much look like "risk premia" to an econometrician observing the link between loan "riskiness", as measured by their implicit leverage ratios, and interest rates. However, it is important to note that even though lenders do not expect even high leverage ratio loan contracts to default, it is possible that they will according to the objective probability measure, that is given the realization of p_{t+1} . How much allocative interest rates can explain of the credit spread puzzle depends then on the objective probability measure for p_{t+1} relative to endogenous threshold π , under which some lenders will face default. Note however that this does not take anything away from the fact that disagreement will be the parameter towards which credit spreads should be judged, rather than risk aversion. Note also that for some probability measures on p_{t+1} at least, high-quality bonds (with high margins) will never default. This result is interesting because the credit spread puzzle is more salient for high quality bonds. In the extension with Short-Sales Contracts in Section 2.1, it will be shown that even the highest quality bonds must earn some excess return in order to compensate lenders from not shorting.

1.4 Pareto Distributions for Leverage Ratios (Low Disagreement)

It is useful to define first a transformation of the density of beliefs to be able to investigate the case of disagreement h going to zero for a given belief structure. Consider a transform $f_h(\cdot)$ of the density function of beliefs $f(\cdot)$ given by:

$$\forall x \in [M - (M - m)h, M], \quad f_h(x) = \frac{1}{h} f\left(\frac{x - M}{h} + M\right). \quad (f_h)$$

That is, the original distribution is scaled so that the shape of the distribution is preserved, so that effectively disagreement goes to zero (using a change of variable $X = M + (x - M)h$). Therefore $\{f_h(\cdot)\}_{h \in [0,1]}$ are distributions of beliefs closer to the benchmark of common prior when disagreement h goes to zero. Note that if $f(\cdot)$ is a density on $[m, M]$, then $f^h(\cdot)$ also is a density on $[M - (M - m)h, M]$ for $h \in [0,1]$. From now on, all quantities denoted by a

lowerscript h will refer to Economy \mathcal{E}^B where the density of beliefs is given by $f_h(\cdot)$ defined in (f_{*h*}). Note also that the cumulative of density $f_h(\cdot)$ is then given as $F_h(x) = F\left(\frac{x-M}{h} + M\right)$ for $x \in [M - (M - m)h, M]$.

Function l_h represents the leverage ratio of borrowers y as a function of the lender x with whom they were matched, in Borrowing Economy with a density of beliefs $f_h(\cdot)$. But in order to determine the distribution of leverage ratios across borrowers, it is useful to express leverage as a function of borrowers' beliefs y using $x = \Gamma_h^{-1}(y)$. The resulting function is denoted by $L_h(\cdot)$, with $l_h(x) = (L_h \circ \Gamma_h)(x) = L_h(y)$ and therefore given by:

$$L_h(y) = \frac{\Gamma_h^{-1}(y)}{p_h r_h(\Gamma_h^{-1}(y)) - \Gamma_h^{-1}(y)}.$$

The study of the cumulative distribution function for leverage ratios of borrowers is the subject of the following developments.

Taylor Expansions. The most important result to get at is why the leverage ratio of borrowers who are able to borrow from lenders with beliefs close to $x = \pi_h$ goes to infinity when disagreement h tends to zero. Note that the highest leverage factor in Borrowing Economy \mathcal{E}^B with structure of beliefs $f_h(\cdot)$ is the leverage factor $L_h(M)$ because of assortative matching. From equation (3b), the denominator of this leverage ratio is given as a function of cutoffs and the asset price:

$$p_h r_h(\pi_h) - \pi_h = \pi_h'' \frac{\pi_h - p_h}{p_h - \pi_h''} \quad \Rightarrow \quad L_h(M) = \frac{\pi_h p_h - \pi_h''}{\pi_h'' \pi_h - p_h}.$$

Note first that the cutoffs π_h, π_h'' and the asset price p_h are comprised between $M - (M - m)h$ and M and therefore $p_h, \pi_h, \pi_h'' \rightarrow M$ when $h \rightarrow 0$. Because both the numerator and the denominator go to zero, one needs to go further into the Taylor expansions to show that maximum leverage goes to infinity. The following lemma shows that maximum leverage actually behaves like an inverse squared term of disagreement.

Lemma 1. *Consider Borrowing Economies with structure of beliefs $f_h(\cdot)$ defined in (f_{*h*}). When disagreement h goes to zero, the cutoffs π_h, π_h'' , the asset price p_h , and $\pi_h - p_h$ are such that:*

$$\pi_h'' = M - (M - m)h - o(h), \quad \pi_h = M - O(h^2), \quad p_h = M - O(h^2), \quad \text{and} \quad \pi_h - p_h = O(h^3)$$

Therefore, when disagreement h goes to zero, the maximum and minimum leverage ratios of borrowers $L_h(M)$ and $L_h(\pi_h)$ are such that:

$$L_h(M) = \frac{\pi_h p_h - \pi_h''}{\pi_h'' \pi_h - p_h} = O\left(\frac{1}{h^2}\right) \quad L_h(\pi_h) = \frac{\pi_h''}{p_h - \pi_h''} = O\left(\frac{1}{h}\right).$$

Proof. See Appendix B.1.3. □

The full proof of this lemma is in Appendix B.1.3, however it is useful to give an intuition for all of these approximations. For concreteness, one can see an illustration of these Taylor expansions for equilibrium cutoffs on the special case of flat priors on Figure 2 (that is, $f(x) = 1/M$ for

$x \in [0, M]$). On this figure, one can see that cutoff p_t'' is tangent to the $y = M(1 - h) = m$ line. The economic interpretation is that as disagreement goes to zero, all agents' funds are invested in the Real Asset. The reason is that borrowers are then able to leverage themselves with very low margins from market clearing equation for financial contracts (C), and so the price of the asset is getting closer to that of the maximum beliefs M . Only the beliefs of the most optimistic agents are then effectively expressed. This also explains why π_h and p_h are much closer to M than π_h'' , as the aggregate measure of lenders is much greater than that of borrowers. This can very clearly be seen graphically on Figure 3: when disagreement goes to zero, the assignment function $\Gamma(\cdot)$ becomes more and more flat on a greater portion of the x -axis. The reason why the difference $\pi_h - p_h$ between cutoff π_h and the asset price p_h is an order of magnitude lower than π_h and p_h , is that the Taylor expansions of both the asset price and cutoff π_h have the same second-order term in the Taylor expansion. This result itself follows from the fact that as disagreement goes to zero, interest rates on loan contracts tend to one according to a squared term of disagreement, as can clearly be seen in the case of flat priors on Figure 4. From equation (3b), one concludes immediately that they have the same second-order Taylor expansions. Finally, the maximum leverage ratio of borrowers follows from the Taylor expansions of its numerator, a linear term of disagreement, and its denominator, a cubic term; the minimum leverage ratio of borrowers follows from the fact that the numerator tends to M when disagreement goes to zero, while the denominator goes to zero as a linear term of disagreement.

Pareto Distributions. With these Taylor expansions in mind, we are now equipped to state the two main formal results concerning Pareto distributions, that concern the behavior of the distribution of leverage ratios when disagreement goes to zero. Proposition 2 concerns a bounded density of beliefs $f(\cdot)$ on $[m, M]$, and Proposition 3 considers the case of a singularity at the most optimistic beliefs M . In a dynamic, 3-period model, this singularity can come from the outcome of the static model leading to a Pareto distribution for leverage ratios and of end-of-period wealth for the most optimistic agents (see Section 3.2 for a dynamic extension of the model). Denote $\|f(l)\|_\infty^{[a,b]}$ as the supremum of $|f(l)|$ on $[a, b]$.

Proposition 2 (Limiting Pareto Distribution of Shape Two for Leverage Ratios of Borrowers). *Let the density of beliefs $f(\cdot)$ be differentiable, continuous and bounded on $[m, M]$, with $f(M) > 0$.*

The distribution of leverage ratios for borrowers tends to a Pareto distribution with shape two in the upper tail, when disagreement goes to zero. Formally, denoting by $G^h(\cdot)$ the distribution function for the leverage ratio of borrowers with density of beliefs $f^h(\cdot)$:

$$\exists A_h, \quad \|l^2(1 - G_h(l)) - A_h\|_\infty^{[L_h(M)/2, L_h(M)]} \xrightarrow{h \rightarrow 0} 0,$$

Proof. See Appendix B.1.4. □

Proposition 3 (Limiting Pareto Distribution of Other Shapes and Attracting Zipf's Law). *Let the density of beliefs $f(\cdot)$ be differentiable, continuous and such that $f(y) \sim_{y \rightarrow M} (M - y)^{-1/\alpha}$, that is wealth is a Pareto distribution of coefficient α on the support of beliefs around M .*

The distribution of leverage ratios for borrowers tends to a Pareto distribution with shape β given as the harmonic mean of 1 and $-\alpha$, when disagreement goes to zero. Formally, if β is given as:

$$\frac{1}{\beta} = \frac{1}{2} \left[1 - \frac{1}{\alpha} \right].$$

Then denoting by $G_h(\cdot)$ the distribution function of the leverage ratio of borrowers with density of beliefs $f_h(\cdot)$:

$$\exists B_h, \quad \left\| l^\beta (1 - G_h(l)) - B_h \right\|_{\infty}^{[L_h(M)/2, L_h(M)]} \xrightarrow{h \rightarrow 0} 0,$$

As a result, speculation with heterogenous beliefs and collateral constraints in economy \mathcal{E}^B increases skewness when wealth is less skewed than Zipf ($\alpha > 1$), and decreases skewness when wealth is initially more skewed than Zipf ($\alpha < 1$).

Proposition 2 says that when the distribution of wealth on beliefs' support is initially bounded, as leverage ratios go to infinity, the distribution of these leverage ratios tends to behave like one over a square of the leverage ratio $1 - G_h(l) \sim A_h/l^2$. The reader might worry that the qualification needed for the result, that disagreement should go to zero, limit its applicability. However proceeding by means of an example as is done in Section 1.6, and looking in particular at Figure 5 shall convince the reader of the importance of this result. In particular, the prediction for the Pareto coefficient is obtained irrespective of the precise shape of the function of beliefs, as is stated in this proposition and illustrated using two different forms of beliefs' functions in the Online Appendix. In practice, it is in fact well known and intuitive that economic Pareto distributions are always truncated, as Figure 6 for the distribution of leverage ratios of hedge funds shows (see Gabaix (2009)). This also happens in this model, as long as disagreement is strictly positive (but does not in random growth models, for example). Moreover, it should be noted that as disagreement goes to zero, one approaches the benchmark of common prior. Taking disagreement as being low amounts to studying the economy's behavior not too far from the benchmark.

Intuition for Pareto Distribution of Shape Two. Before going into the details of the mathematical proof of the fact that the distribution of leverage converges to that of an asymptotic Power Law of shape two¹¹, it is worth giving first an intuition for it. The leverage ratio, in the space of pessimists' beliefs, is proportional to a ratio of one over the difference between the collateral price and the price of the bond, which goes to zero when more optimistic agents are

¹¹A graphical intuition for the Pareto distribution of tail coefficient two is also given in an Online Appendix.

lending. The reason for why margins go to zero is that the most optimistic lenders would almost be willing to buy the asset at the going price. When the distribution of beliefs is sufficiently regular, this difference can be approximated by a uniform distribution in the limit, because it is close to a difference between lenders' beliefs and a constant (up to an interest rate term though, which is negligible in the limit). The reason why the exponent of this Pareto is two, and not one, is that what is measured is the leverage ratio in borrowers' space $L_h(y)$, and not $l_h(x)$ in lenders' space. Because the assignment function $\Gamma_h(\cdot)$ has a derivative close to zero for the most optimistic lenders ($\Gamma'_h(\pi_h) \approx 0$), a lot of lenders are matched with very few borrowers asking for close to zero margins, and therefore the measure of the corresponding lenders is higher than the measure of borrowers. Going from lenders' space to borrowers' space then involves a square transformation. Therefore, the distribution has a shape exponent of exactly two.

Intuition for Pareto Distribution of other Shapes and the role of Zipf's law. The last part of the previously given intuition, going from the lenders' space to the borrowers' space, heavily relies on the fact that the distribution of wealth is bounded for the most optimistic beliefs. If initially, borrowers are distributed on the support of beliefs according to Zipf's Law, then the assignment function becomes linear, which follows from equation (C). In other words, there is a "congestion effect", as borrowers ask loans with low margins but they are so numerous that the number of lenders needed to supply their demand for capital is higher; and the two effects actually exactly offset each other. With such a linear assignment function, the leverage ratio now has a degenerate distribution: every borrower is able to achieve the same leverage ratio, so that $\beta = +\infty$ if β denotes the Pareto tail coefficient for leverage ratios. Of course, the case of Zipf's Law is only knife-edge. If the distribution of wealth on beliefs' support is initially more skewed than Zipf, one can then understand why the congestion effect becomes stronger than the skewness coming from the shape of the leverage factor. The leverage function then follows a Pareto with a negative coefficient. More precisely, in the following mathematical intuition I explain why if the initial distribution of wealth on the support of beliefs is such that $f(y) \sim (M - y)^{-1/\alpha}$, then the distribution of leverage is a Pareto with shape β given as the harmonic mean of 1 and $-\alpha$:

$$\frac{1}{\beta} = \frac{1}{2} \left[1 - \frac{1}{\alpha} \right].$$

Conversely, if the distribution of wealth on beliefs' support is initially less skewed than Zipf, then

Mathematical Intuition for Pareto Distribution of Shape Two. Proposition 2 states that for low enough disagreement, the implied distribution of leverage of borrowers at the competitive equilibrium outlined in proposition 1 can be made arbitrarily close to a distribution whose asymptotic behavior is a Pareto distribution of shape two. The full proof of this result is rather technical, but given its importance, it is worth trying to get at a mathematical intuition for it. To show this intuition simply, let me define an equivalence relation around x_0 between two real-valued variable f and g denoted as $f(x) \sim_{x_0} g(x)$, such that $f(x) \sim_{x_0} g(x)$ if and only if $\frac{f(x)}{g(x)} \rightarrow m \neq 0$. If the distribution of wealth on beliefs' support is bounded around M , from

equation (C):

$$f_h(\Gamma_h(x))\Gamma'_h(x) = \frac{p_h r_h(x) - x}{x} f_h(x).$$

Note that for the behavior of the upper tail of the distribution of leverage ratios, what matters is the behavior of the assignment function around the most optimistic beliefs for lenders of borrowers ($x \rightarrow \pi_h$ and $y \rightarrow M$). Because margins $\frac{p_h r_h(x) - x}{x}$ go to zero when one goes to these most optimistic beliefs, for x in the neighborhood of π_h what matters is how fast the numerator goes to zero:

$$\Gamma'_h(x) \sim_{\pi_h} p_h r_h(x) - x.$$

When disagreement tends to zero, integrating between x and π_h gives:

$$\Gamma_h(\pi) - \Gamma_h(x) \sim_{\pi_h} (p_h r_h(x) - x)^2.$$

More precisely, the fact that disagreement tends to zero here means that the term $(p_h r_h(\pi_h) - \pi_h)^2$ is neglected. Using the assignment function $\Gamma_h(\cdot)$ to write the same thing in borrowers' space, and that $M = \Gamma_h(\pi_h)$:

$$M - y \sim_M (p_h r_h(\Gamma_h^{-1}(y)) - \Gamma_h^{-1}(y))^2.$$

Finally:

$$L_h(y) = \frac{\Gamma_h^{-1}(y)}{p_h r_h(\Gamma_h^{-1}(y)) - \Gamma_h^{-1}(y)} \sim_M \frac{1}{\sqrt{M - y}}.$$

Therefore, leverage follows a Pareto distribution of shape two in the upper tail. The reader can also refer to the [Online Appendix](#) for a graphical intuition of this Pareto distribution with a tail coefficient equal to two.

The full Power Law distribution with infinite leverage for some borrowers obtains only when disagreement goes to zero, or approaching the benchmark of common priors. In contrast, as we will see in section 1.6 and in particular on Figure 5, when disagreement is low but finite, the distribution will be close to a Pareto distribution, up to a term depending on the magnitude of disagreement.

Note that as disagreement h goes to zero, the economy moves closer to the benchmark of common prior. This result means that Pareto distributions for the upper tail of leverage appear as long as there is even an infinitesimal amount of disagreement between agents. This finding also suggests a fragility of the benchmark of common priors, where all agents invest in the asset in proportion of their wealth. This benchmark is not the limit when disagreement goes to zero of an heterogeneous priors model.

At the qualitative level, this model teaches us that when disagreement is low, very highly leveraged entities, whose empirical counterparts may be thought as hedge funds or shadow banks, are not at all pathological but naturally emerge as long as there is some disagreement in the economy. There is a strong force in the economy which pushes financial intermediaries, but also

entrepreneurs, to leverage themselves as much as possible. Of course, an open question is why these institutions did not appear earlier, if high leverage is always the outcome of any amount of belief disagreement. The institutional and regulatory environments certainly have played an important role.

Mathematical Intuition for Pareto Distribution of other Shapes and the role of Zipf's law. The result about leverage can even be generalized when there is some singularity at the most optimistic beliefs. I now look at the case where the distribution of wealth on beliefs support at the most optimistic belief is such that $f(y) \sim (M - y)^{-1/\alpha}$, that is the distribution of wealth is a Pareto distribution of shape α in the upper tail at the most optimistic beliefs. This proposition will prove useful for thinking about what happens to the distribution of leverage at the second round of a speculation game, when wealth is already a Power Law around the most optimistic beliefs (see section 3.2).

In the previous example, the case where $\alpha = +\infty$ was examined (boundedness of the wealth distribution function). Here, I allow for any parameter of the shape distribution. Interestingly, the leverage distribution still is right skewed when the initial distribution of wealth on beliefs' support is not "too right skewed" (speaking loosely, it is going to be true *up to the Zipf law*). In contrast, when the initial distribution is more right skewed than the Zipf law, speculation will lead to a decrease in right skewness. Thus Zipf Law will provide a lower bound to the Pareto-Lorenz coefficient.

The proof for this result goes through in the same way as with unbounded beliefs. As before the intuition for the result can be given to try and understand the stability and "attracting" property of Zipf's law. If α is the Pareto coefficient on wealth at time t , denote by β is the Pareto coefficient on return at time t which is also the Pareto coefficient for leverage. Therefore:

$$L_h(y) = \frac{\Gamma_h^{-1}(y)}{p_h r_h(\Gamma_h^{-1}(y)) - \Gamma_h^{-1}(y)} \sim_M (M - y)^{-1/\beta} \Rightarrow M - \Gamma_h(x) \sim_{\pi_h} (p_h r_h(x) - x)^\beta$$

Differentiating gives the distribution for $\Gamma_h'(\cdot)$ around π_h :

$$\Gamma_h'(x) \sim_{\pi_h} (p_h r_h(x) - x)^{\beta-1}$$

Using $y = \Gamma_h(x)$ in the density function for wealth, with Pareto exponent α , that is $f(y) \sim (M - y)^{-1/\alpha}$, one gets:

$$f_h(\Gamma_h(x)) \sim_{\pi_h} (M - \Gamma_h(x))^{-1/\alpha} \sim_{\pi_h} (p_h r_h(x) - x)^{-\beta/\alpha}.$$

Finally, β is then determined by the accounting equation (C):

$$\begin{aligned} \Gamma_h'(x) &= \frac{f_h(x)}{f_h(\Gamma_h(x))} \frac{p_h r_h(x) - x}{x} \Rightarrow (p_h r_h(x) - x)^{\beta-1} \sim (p_h r_h(x) - x)^{\beta/\alpha} (p_h r_h(x) - x) \\ &\Rightarrow \frac{1}{\beta} = \frac{1}{2} \left[1 - \frac{1}{\alpha} \right]. \end{aligned}$$

This congestion effect is actually rather intuitive. If hedge funds became too wealthy relative to their brokers, they cannot be leveraging themselves as much. A stationary distribution

therefore emerges endogenously from self-equilibrating forces. This is interesting because firm sizes (and even city sizes) empirically follow a Zipf law, at least in the upper tail. In section 3.2, I look more closely at the dynamics of the wealth distribution as time evolves, under different demographic structures.

1.5 Non Monotonic Relationship Between Leverage and Realized Returns

Borrowers' Realized Returns. Conditional on a price realization in $t + 1$ for the risky asset, denoted by p_{t+1} , the realized return on wealth $ROE^B(p_{t+1}, \Gamma^{-1}(y))$ of borrowers with beliefs $p_{t+1}^i = y$ is given by a weighted average of the return on the risky asset and the return on borrowed funds (see Appendix B.1.1), from equation (R_b). Note in addition that borrowers are protected by limited liability, as they can default on their promises, and therefore their return on wealth can never be lower than zero. In the case of no-default (that is, if $p_{t+1} \geq \Gamma^{-1}(y)$), then this return on equity is given by:

$$ROE^B(p_{t+1}, \Gamma^{-1}(y)) = \frac{p_{t+1}}{p} (1 + l(\Gamma^{-1}(y))) - r(\Gamma^{-1}(y))l(\Gamma^{-1}(y)) = \frac{p_{t+1} - \Gamma^{-1}(y)}{p - q(\Gamma^{-1}(y))}$$

In contrast, default arises when the borrowers' equity is completely wiped out, which happens for price realizations $p_{t+1} < \Gamma^{-1}(y)$ and in that case $ROE^B(p_{t+1}, \Gamma^{-1}(y)) = 0$.

The derivative of this function can be calculated, and shows that realized returns have a maximum for $y = p_{t+1}$, or the optimist who was right about the price of the asset (which is how prices of contracts were constructed). More precisely, I show in the Appendix B.1.2 that the derivative of $ROE^B(p_{t+1}, \Gamma^{-1}(y))$, the ex-post cross-sectional distribution of returns conditional on realized price p_{t+1} , has the sign of $p_{t+1} - y$:

$$\left(\frac{x}{r(x)} \right)' = \frac{r(x) - r'(x)x}{r(x)^2} > 0 \quad \Rightarrow \quad \text{sign} \left[\frac{dROE^B(p_{t+1}, \Gamma^{-1}(y))}{dy} \right] = \text{sign} [p_{t+1} - y].$$

This property is natural in the context of this model: the optimist making the highest return conditional on a price realization in $t + 1$ p_{t+1} has to be the one who was entertaining that belief at time t . Therefore, when ex-post realized returns are in the support of beliefs, there is a non-monotone relationship between leverage and ex-post realized returns. This is a result that a simpler model would not have delivered, in which ex-post returns would have been proportional to leverage, or to the quantity of assets held.

Lenders' Realized Returns. The average return on wealth for a lender with beliefs $p_{t+1}^i = x$, conditional on a price realization p_{t+1} for the risky asset, denoted by $ROE^L(p_{t+1}, x)$, is given by equation (R_l):

$$ROE^L(p_{t+1}, x) = \frac{1}{q(x)} \min\{p_{t+1}, x\}$$

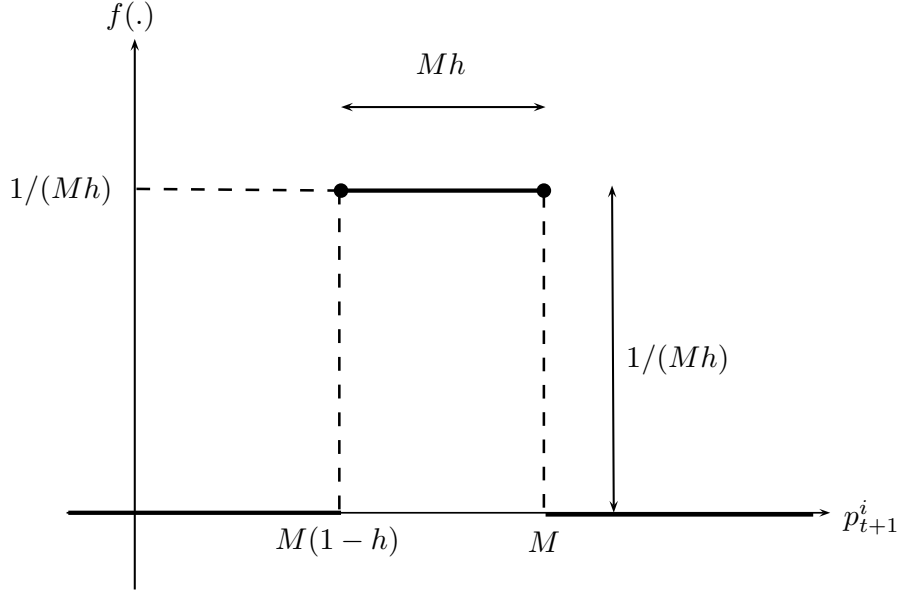
Cash Investors' Realized Returns. The return of cash investors does not depend on their beliefs nor on return on the risky asset, and is always equal to the expected return on cash

$R = 1$.

$$ROE^C(p_{t+1}) = 1.$$

1.6 Numerical Example with Flat Priors

Figure 1: PARAMETRIZATION OF THE DENSITY OF BELIEFS WITH FLAT PRIORS



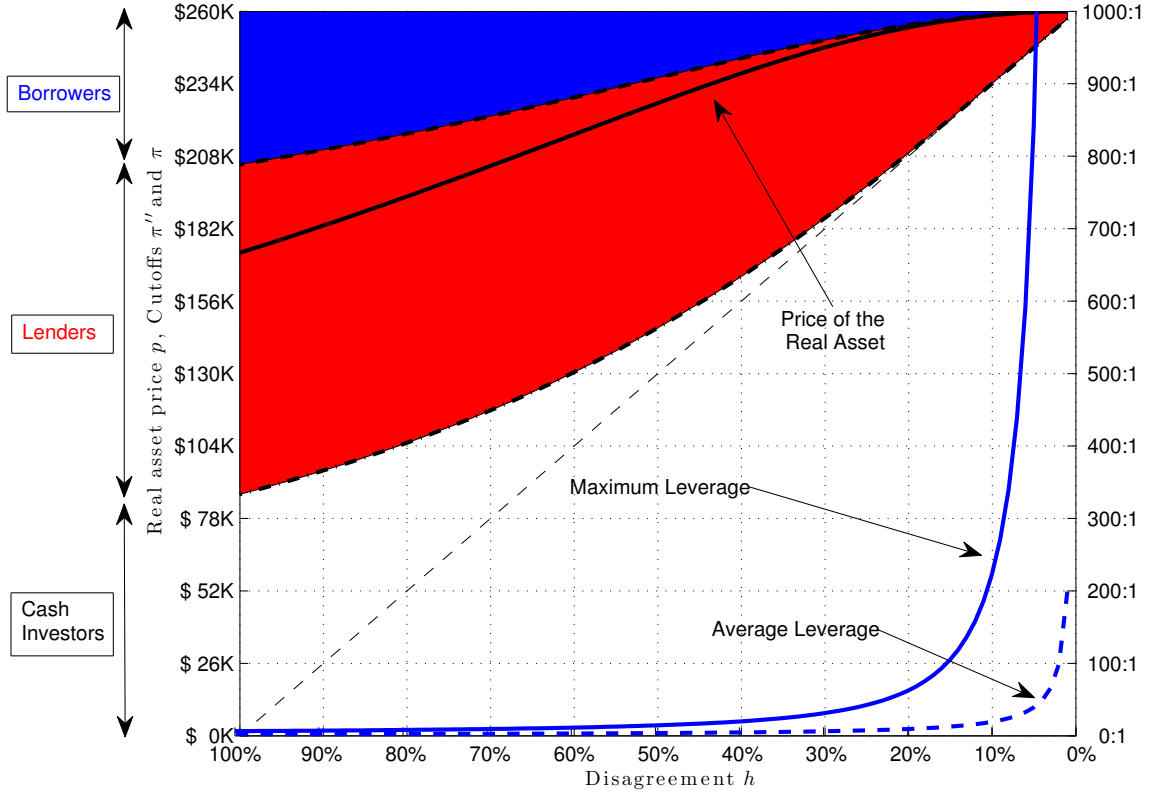
Note: This figure is a plot of the density of beliefs given by $f(x) = \frac{1}{Mh} \mathbb{1} [M(1-h), M] (x)$.

In this section, I solve for the equilibrium given in proposition 1 in the case of flat priors (Figure 1). I then derive and illustrate the value of cutoffs as a function of disagreement through Figure 2), the shape of the assignment function on Figure 3, the function giving allocative premia $r(.) - 1$ on Figure 4, the leverage countercumulative distribution function showing the Pareto distribution property on a log-log scale on Figure 5, the empirical counterpart on Figure 6, and the values for leverage on Figure 7.

The flat prior distribution for beliefs is a natural density of beliefs to look at because the uniform distribution on $[a, b]$ is the maximum entropy distribution among all continuous distributions contained in the interval $[a, b]$.¹² Corollary 1 of Proposition 1 gives the assignment function, assignment interest rates function and prices when agents have beliefs distributed over $[M(1-h), M]$ according to density function $f(x) = 1/(Mh)$ for all $x \in [M(1-h), M]$. Denoting by $\mathbb{1}(\cdot)$ the indicator function, we then have: $f(x) = \frac{1}{Mh} \mathbb{1} [M(1-h), M] (x)$, and $F(x) = \frac{x-M(1-h)}{Mh} \mathbb{1} [M(1-h), M] (x) + \mathbb{1} [M, +\infty] (x)$ (see Figure 1).

¹²Maximizing entropy therefore minimizes the amount of prior information contained in the distribution.

Figure 2: COMPARATIVE STATICS WITH RESPECT TO DISAGREEMENT h IN ECONOMY \mathcal{E}^B



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$ (see Figure 1). The price of the asset p and cutoffs π and π'' are given by corollary 1, as a function of h . Note the graphical visualisation of previously derived Taylor expansions in Lemma 1: $\pi_h'' = M - O(h)$, $\pi_h = M - O(h^2)$, $p_h = M - O(h^2)$, $\pi_h - p_h = O(h^3)$, $L_h(M) = O(1/h^2)$. The minimum belief is $m = M(1-h)$. For low disagreement, speculation allows all agents' endowments to be invested in the asset, through borrowers' increasing leverage.

Corollary 1. Assume a flat priors belief structure, that is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, and $M = w/S$. Then at the equilibrium of economy \mathcal{E}^B , the assignment function and assignment interest rates are given as a function of the asset price p , and cutoffs π'' and π by:

$$\Gamma(x) = \frac{1}{2} \left(-2x - \hat{\pi} - \tilde{\pi} + \sqrt{2\hat{\pi}} \sqrt{(4x + \tilde{\pi} + \hat{\pi})} \right) \quad r(x) = \frac{x}{p} \sqrt{\frac{2\hat{\pi}}{4x + \hat{\pi} + \tilde{\pi}}},$$

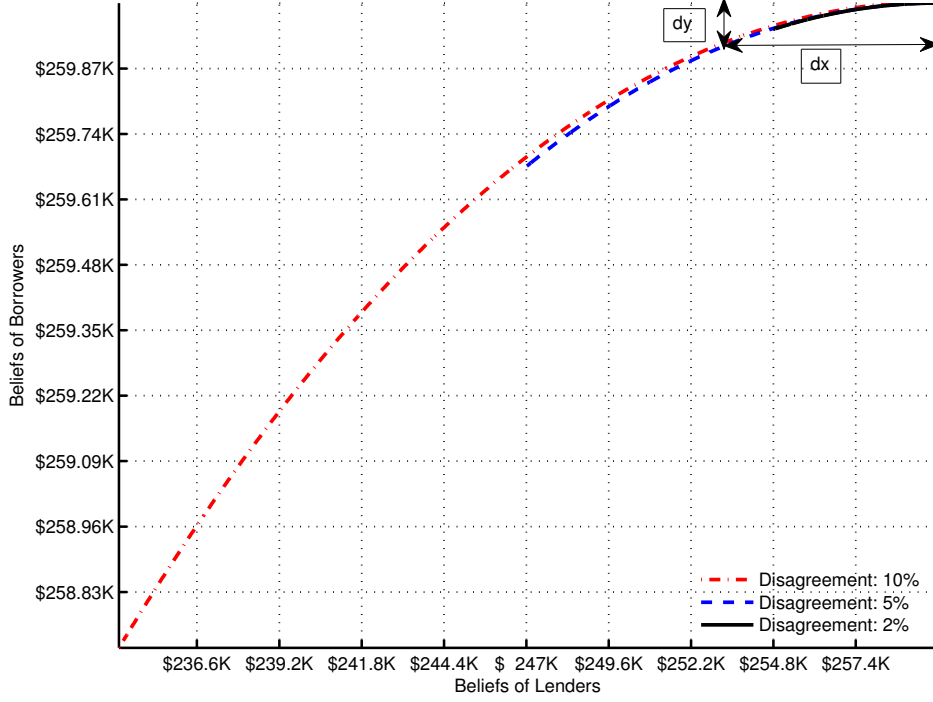
with $\hat{\pi} = \frac{p^2}{\pi''} \frac{\pi - \pi''}{p - \pi''}$, and $\tilde{\pi} = -2\pi'' - 2\pi + \sqrt{\hat{\pi}^2 + 4\hat{\pi}\pi'' - 4\hat{\pi}\pi}$.

The asset price p , the cutoffs π'' and π are a function of h and M with:

$$p = \frac{1 + h + 2h^2 + 2h^3 - \sqrt{(-1+h)^2(1+2h^2)}}{2h + h^2 + 4h^3 + 2h^4} M \quad \pi'' = \frac{1 + 2h^2 + \sqrt{(-1+h)^2(1+2h^2)}}{2 + h + 4h^2 + 2h^3} M$$

$$\pi = \frac{1}{2 + 4h^2} \left[\sqrt{(-1+h)^2(1+2h^2)} + (1 + 2h^2) \left(1 + \sqrt{\frac{h^2}{1+2h^2}} \right) \right] M.$$

Figure 3: ASSIGNMENT OF MODERATES TO OPTIMISTS THROUGH LOAN CONTRACTS



Note: The density of beliefs is again $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 10\%, 5\%, 2\%$ successively (see Figure 1). Note that for very optimistic borrowers and lenders, that is beliefs for lenders $x \rightarrow \pi^-$, or beliefs for borrowers $y \rightarrow M^-$, a very small interval of borrowers dy trade with a large number of lenders dx . Mathematically, this is because $\frac{dy}{dx} = \Gamma'(\pi) \approx 0$. Hence, a small measure of very optimistic borrowers effectively trades with a corresponding very large number of lenders, which is why their leverage is, to the limit where disagreement goes to zero, infinite.

Proof. See Appendix B.1.5. □

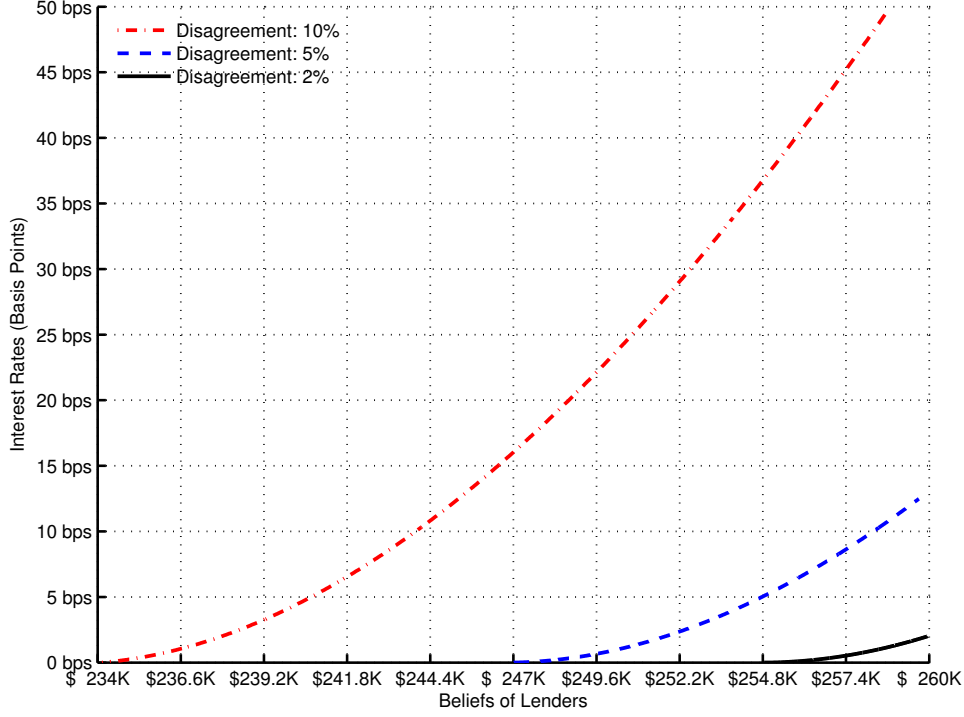
Corollary 1 therefore states that with a flat prior distribution of beliefs $f(\cdot)$, everything in the model can be expressed in closed form. From equations (A) and (C) in proposition 1, and using the shape of the density function, one gets the following non linear second order differential equation for the assignment function:

$$\forall x \in [\pi'', \pi], \quad (\Gamma(x) - x)\Gamma''(x) + \Gamma'(x) + \Gamma'(x)^2 = 0,$$

together with two initial conditions $\Gamma(\pi'') = \pi$ and $\Gamma'(\pi'') = \frac{p - \pi''}{\pi''}$. That this non-linear second-order differential equation has a closed form solution given by corollary 1 is a priori unexpected, as solutions of second-order differential equations cannot usually be expressed in closed form albeit for very special cases. See Appendix B.1.5 for a derivation of the solution of this differential equation.

The price of the real asset p and cutoffs π and π'' can then be calculated as a function of h , and are proportional to M : they are given explicitly by Corollary 1. Once again, closed form

Figure 4: ALLOCATIVE PREMIA - SPREADS AS A FUNCTION OF LENDERS' BELIEFS



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1} [M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 10\%, 5\%, 2\%$ successively (see Figure 1). Return is plotted as a function of the beliefs of lenders. Note that this return is not a reward for losses upon default, nor risk, but plays an allocative role in allowing matching between lenders and borrowers.

solutions obtain from using closed form solutions for $\Gamma(\cdot)$ as well as for $r(\cdot)$ in the set of three equations (3a), (3b) and (3c) - again, this is shown in Appendix B.1.5.

Expected Return on Equity. Facing these interest rates, borrowers expect to have a return on equity as a function of their types equal to:

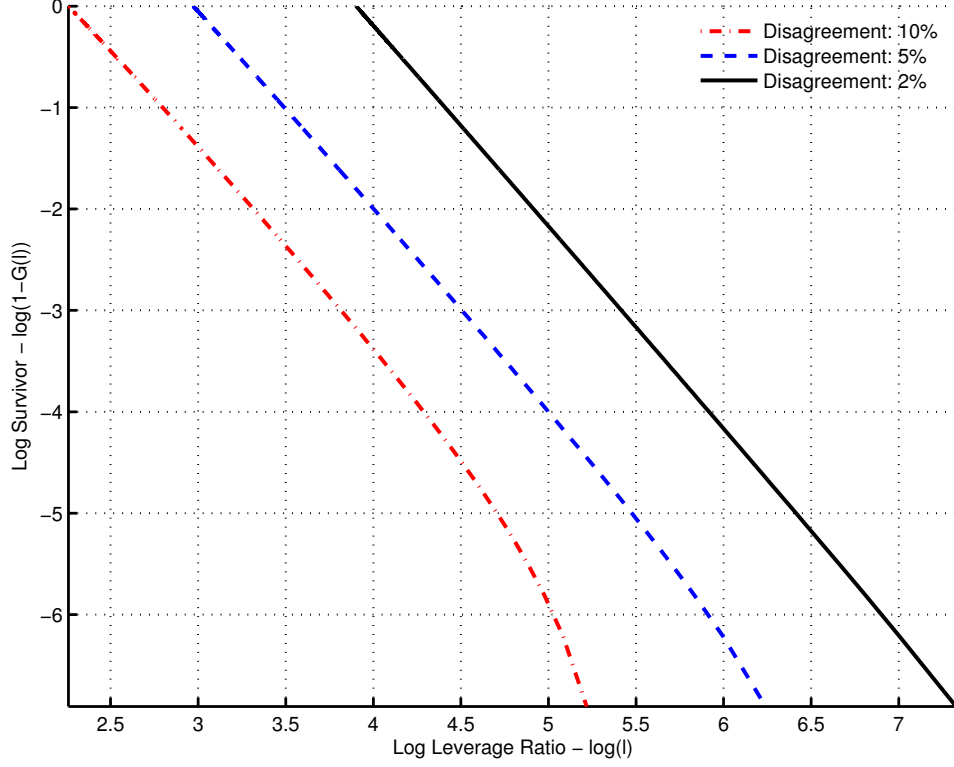
$$ROE^e(y) = \frac{\hat{\pi} \left(\sqrt{2} \sqrt{\hat{\pi} (-\tilde{\pi} + \hat{\pi} - 4y)} - \tilde{\pi} + \hat{\pi} - 4y \right)}{p \left(\sqrt{2} \sqrt{\hat{\pi} \left(2\sqrt{2} \sqrt{\hat{\pi} (-\tilde{\pi} + \hat{\pi} - 4y)} - \tilde{\pi} + 3\hat{\pi} - 4y \right)} - 2\hat{\pi} \right)}.$$

Proof. See Appendix B.1.6. □

Note that even for very highly levered borrowers, with $y = M$, the expected return on equity remains finite because of allocative interest rates. On the one hand, more optimistic borrowers secure loans with lower margins; but on the other hand, they must give a greater share of these expected returns to lenders.

Distribution of Leverage Ratios. The distribution of leverage is very highly skewed, as can be seen on a regular scale (Figure 7). On a log-log scale it is possible to note the asymptotic

Figure 5: THEORETICAL CROSS-SECTIONAL DISTRIBUTION OF LEVERAGE RATIOS



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1} [M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 10\%, 5\%, 2\%$ successively (see Figure 1). This plot represents on a log-log scale the countercumulative distribution function of borrowers' leverage, when the number of them is 10^3 . The survivor function converges to that of a Pareto with $\alpha = 2$. One can compare this to the empirical counterpart on Figure 6.

Pareto property (Figure 5). The distribution of leverage can be expressed in closed form:

$$G(l) = \left(1 - \frac{M}{M - \pi} + \frac{\hat{\pi} - \tilde{\pi}}{4(M - \pi)} \frac{1}{M - \pi}\right) - \frac{\hat{\pi}}{2(M - \pi)} \frac{1}{(l + 1)^2}.$$

Proof. See Appendix B.1.6. □

Interest Rates. Finally, interest rates implicitly paid by borrowers on their loan contracts range from $r(\pi'') = 1$ to $r(\pi)$ given by:

$$r(\pi) = \frac{\pi}{p} \sqrt{\frac{2\hat{\pi}}{4\pi + \hat{\pi} + \tilde{\pi}}}.$$

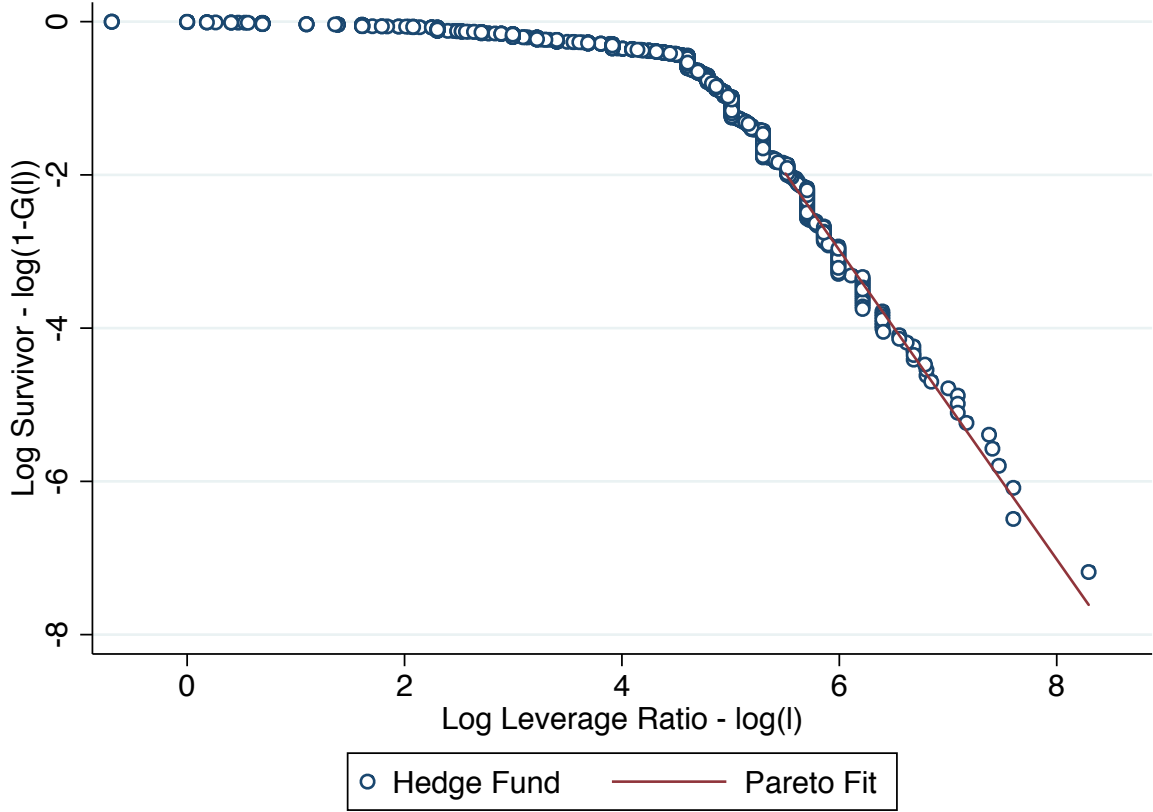
Ex-post Return on Equity. As in the general case, the ex-post cross-sectional distribution of returns for borrowers is such that:

$$\text{sign} \left[\frac{dROE^B(p_{t+1}, \Gamma^{-1}(y))}{dy} \right] = \text{sign} [p_{t+1} - y].$$

Therefore, the return on equity peaks at $y = p_{t+1}$, for the lender who was right about the asset's value, and there is a non-monotone relationship between leverage and ex-post realized returns on equity.

Figure 6: EMPIRICAL CROSS-SECTIONAL DISTRIBUTION OF LEVERAGE RATIOS.
HEDGE FUNDS IN THE TASS HEDGE FUND DATABASE, AUGUST 2006.

Red line is OLS Regression of leverage log survivor function $\log(1 - G(l))$ on log leverage $\log(l)$ for $\log(l) > 5$ or $l > 148.4$. **Point estimate:** -2.018 (std: 0.2, $R^2 = 98\%$).

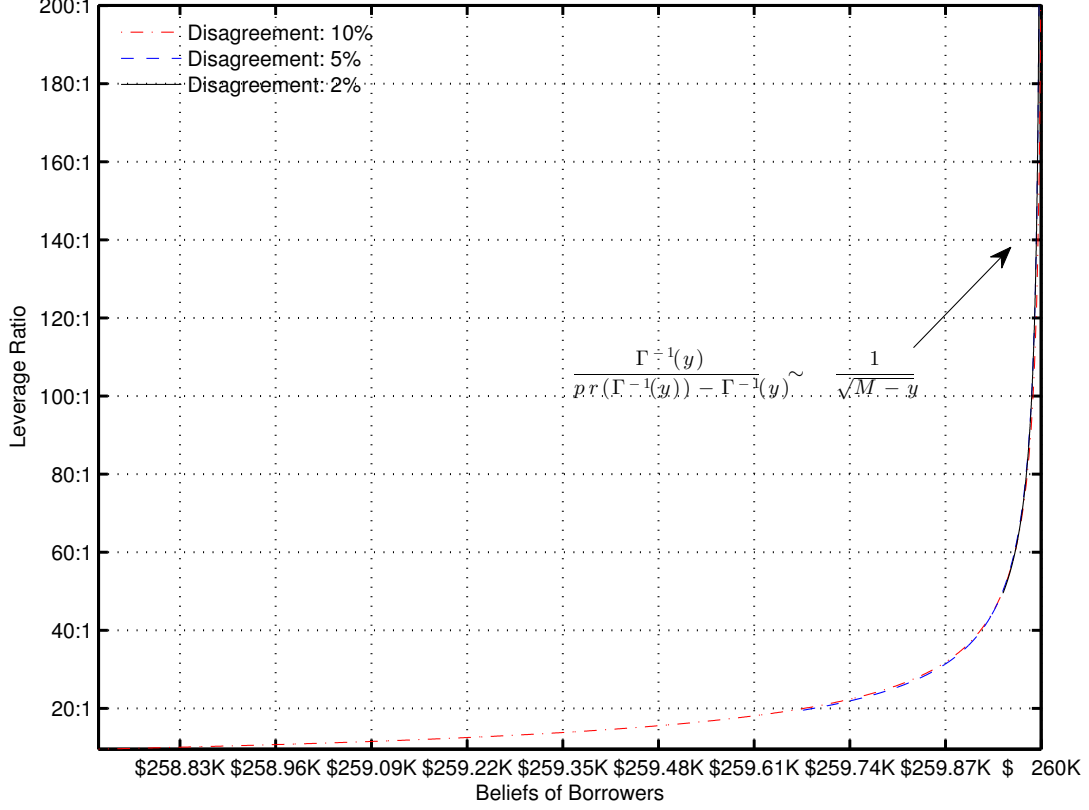


Note: Source: TASS Lipper Hedge Fund Database. The pattern of this leverage distribution is not specific to August 2006, but true in all months all over 1993 – 2010, and across each category of hedge fund. The model counterpart of this empirical countercumulative distribution plotted on a log-log scale is Figure 5. The standard error is note the "naive" OLS standard error but is computed according to Gabaix and Ibragimov (2011)'s method.

Depending on realized base excess returns $\frac{p_{t+1}}{p}$ on the Real Asset, one can graph the end of period distribution of returns. There are three cases: negative excess returns, low positive excess returns, and high positive excess returns. In the case of extreme positive return realizations, the cross-section of returns inherits the Pareto property of leverage ratios. When the distribution is initially egalitarian, the cross section will be a limiting Pareto distribution of shape parameter two.

Comparative statics. The comparative statics with respect to the disagreement parameter $h \equiv \frac{S}{w}H$ are summarized in Figure 2. The maximum leverage ratio is given by the expression derived earlier of $L(M)$, and the average leverage ratio is given by the amount of borrowers'

Figure 7: LEVERAGE AS A FUNCTION OF BORROWERS' BELIEFS



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 10\%, 5\%, 2\%$ successively (see Figure 1). This plot is the counterpart on a regular scale of borrowers' leverage ratios, which are very large for the most optimistic borrowers.

wealth divided by lenders' wealth:

$$L^{max} = L(M) = \frac{\pi}{\pi''} \frac{p - \pi''}{\pi - p} \quad L^{avg} = \frac{F(\pi) - F(\pi'')}{1 - F(\pi)} = \frac{\pi - \pi''}{M - \pi}.$$

Numerical Example. Take the fundamental price of one house in the United States in $t+1$ to be equal to $p_{t+1} = \$252K$. Assume the real sage interest rate is $R = 0$, therefore agents with "fundamentalists" beliefs will buy only if the price if below $M(1-h) = \$252K$. Assume there is a continuum of agents overvaluing housing, with the most optimistic agent overvaluing housing by a bit more than 3%, with $M = \$260K$, and $H = \$8K$, the stock of housing being $100M$ units. Assume also that each household has $\$260K$ to transfer over the course of a lifetime, which he can do either by investing in the storage technology or in the asset. Assume there is a flat prior distribution for the expectation of housing prices. Agents have beliefs distributed over $[\$252K, \$260K]$ according to density function $f(x) = 1/H$. In the model, $w = \$26T$ ($\approx 180\%$ GDP) and $S = 100M$.¹³ In the case, the scaled measure of disagreement is $h = \frac{S}{w}H \approx 3.07\%$.

¹³Note that the model has the unrealistic feature that it contains only one asset; this is why for concreteness

The assignment returns function is plotted on Figure 4, where the type of the moderate is expressed by the haircut this lender is asking for. Note again, and very importantly, that here excess returns over the risk free rate have nothing to do with risk aversion, even though they are negatively correlated to haircuts in equilibrium (see Figure 4, remember that higher x corresponds to a higher expected price for the asset, and hence a lower haircut). On Figure 4, it so appears as if more risky loans had lower prices. However, there is no notion of risk here, as lenders are de facto risk-neutral. Optimists nevertheless expect to pay an excess return over the risk-free rate, because that allows them to leverage into an asset which has an even higher expected excess return to them.

Then, denoting by $\{p\}^{\mathcal{E}^B}$ the price of housing in economy \mathcal{E}^B , I get that:

$$\{p\}^{\mathcal{E}^B} \approx \$259.88K \quad \{\pi''\}^{\mathcal{E}^B} \approx \$252.2K.$$

That is, all agents with beliefs for prices of housing lower than $\$252.2K$ sit out from both financial and Real Asset markets, and invest in Cash, those with beliefs between $\$252.2K$ and $\$259.88K$ are lenders who lend to borrowers, and finally borrowers with beliefs between $\$259.88K$ and $\$260K$ use their own funds as well as borrowed funds to invest in housing. Note that the price of housing is higher than it would be without leverage. Denote by \mathcal{E}^\emptyset such an economy where no financial contract can be written. Optimistic agents would only be able to invest their own funds which in the case of flat priors would give a price of the asset equal to: $\{p\}^{\mathcal{E}^\emptyset} = \frac{1}{H+\frac{w}{S}} M \frac{w}{S} \approx \$252.25K$ in this numerical example (this calculation comes easily from equating asset demand and asset supply: $w(1-F(p)) = Sp$). This is the leverage effect described by Geanakoplos (2003), enabling optimists to bet more. Again, the new result from this paper is that the matching of optimists and pessimists occurs through an assignment function whose expression is given in 1, and drawn on Figure 3 for this example.

Calibration. Compare then Figure 5 representing leverage on a log-log scale in this very simple example of flat priors, with Figure 6 plotting the empirical distribution of leverage of Hedge Funds at a certain point in time (August 2006, but this is true for all months present in the TASS Hedge Fund database, that is April 1993 - September 2010). The model fits the upper part of the leverage distribution extremely well, and the point estimate for the tail coefficient is 2.03, with a standard error of 0.2, computed according to Gabaix and Ibragimov (2011)'s method, using $std = b\sqrt{\frac{2}{N}}$, with N the number of observations.¹⁴ Disagreement h can then be calibrated to the data, for example through the maximum or the minimum leverage in the upper tail and in August 2006, is estimated to be around $h \approx 1.9\%$ through both methods. Note however that the exact value for disagreement depends on the structure of beliefs. The calibration above is therefore conditional on the structure of beliefs being uniform. See the [Online Appendix](#) for an illustration of the distributions of leverage ratio when the distribution of beliefs is not uniform.

the stock of wealth is calibrated to the stock of housing wealth. Wealth/income ratios are more in the range of 3 – 4 in the United States, but the stock of housing is approximately 1.8 times GDP.

¹⁴Moreover, note that leverage ratios are self-reported here, which may explain accumulation points towards round numbers, like 200 or 300 for example.

2 Extensions

This section extends the results of section 1 to the case where short-sales are possible, as well as securitization. They can be skipped for faster reading and the reader can go directly to some applications in section 3.

2.1 Short-Sales Economy \mathcal{E}^S

In the simple economy considered in section 1, short-selling the Real Asset was not possible, following a long tradition in the disagreement models literature since Miller (1977) and Harrison and Kreps (1978). The main reason, was one of simplicity of exposition. Moreover, that there are short-sales constraints is certainly true for specific houses, and some argue for newly created firms with IPO lockups (see Ofek and Richardson (2003) for evidence on the dotcom bubble). However, the results of the first section are valid, and even strengthened, when short-sales are allowed in their different forms, as I show in this section: the model does not only lead to Pareto distributions for wealth conditional on positive extreme return realizations, but also conditional on negative extreme return realizations, because short-sellers are also able to take on a very high leverage: one immediately thinks of John Paulson in the 2006 – 2009 housing bust episode (Lewis (2010)).

From this Short-Sales section also follow three additional important results. First, lending and short-selling can coexist in equilibrium, while one shortcoming in models with beliefs disagreement and endogenous margins is that pessimists always *a priori* want to short rather than lend, from Geanakoplos (1997) to Simsek (2013). Second, credit spreads are quantitatively higher with short-sales than without because interest rates are not only allocative but must also compensate lenders for not shorting. The safest loans therefore also command a credit spread relative to the risk free rate, in line with empirical evidence and in contrast with the model without short-sales in Section 1 (see Figure 4 for the case of flat priors). And third, some agents invest, unlevered, in securities, at the competitive equilibrium. That most agents do buy securities out of their own funds is consistent with empirical evidence once again, especially on equity markets where shorting is possible; on housing markets on the contrary where shorting is not possible, lending is more prevalent.¹⁵

2.1.1 Setup

The setting is the same as in Section 1: two periods, a continuum of agents with same initial wealth, a need to transfer wealth from t to $t + 1$. In addition to Cash, the asset in finite supply and the collateralized loan contracts, I now allow agents to agree to collateralized Short-Sales contracts, using Cash as collateral. In practice, such borrowing can potentially occur in two ways: either through the outright lending of securities (see D’Avolio (2002) for a description), or through selling synthetic contracts promising the replicate the payoff of an asset. Both of them

¹⁵Of course, a lot of borrowing on housing markets also comes from indivisibilities together with the preference for owner-occupied housing (intrinsic or tax-induced).

are equivalent in terms of payoffs, even though creating synthetic securities may prove more costly in real world financial markets with transaction costs. I abstract here from transaction costs considerations, and so I will alternatively talk about "buying synthetic contracts" as well as "lending securities", as those two are equivalent. Note however that I maintain the assumption that securitization, or lending against loan contracts, is not possible or sufficiently costly that agents would not do it in equilibrium. Anticipating a bit on Section 2.2, this is true when disagreement is relatively low compared to these costs.

Short-Sales Contract. Formally, the short-seller sells a Short-Sales Contract which in practice can be achieved by borrowing a Real Asset, and selling this asset, or selling an asset one does not own. On the other side of the trade, the agent buying the Short-Sales Contract is called the *synthetic buyer*, or the *asset lender*: when lending a Real Asset in a short-sales arrangement, one is effectively buying a promise from a borrower to return this asset in the future. This promise, aside from the risk of default of the counterparty, replicates the payoffs the corresponding lent contract: for example, the short-seller must return the dividends to the lender when he collects them. Formally, I define a Short-Sales Contract in the Short-Sales Economy \mathcal{E}^S as:

Definition 3 (Short-Sales Contract, Economy \mathcal{E}^S). *A Short-Sales Contract $\beta \equiv (1, \gamma)_s$ in economy \mathcal{E}^S is a promise of a transfer of Cash corresponding to the payoff of 1 unit of asset in period $t + 1$, collateralized by γ units of Cash. γ is the Cash-collateral of the Short-Sales Contract. The Short-Sales Contract is denoted by $(\gamma)_s$.*

A Short-Sales Contract therefore replicates the payoff of the Real Asset, aside from default risk. So perhaps counterintuitively, a short-seller *sells* Short-Sales Contracts. This convention is symmetric to that of the Borrowing Contract, where a borrower sells Borrowing Contracts. Note that for Short-Sales Contracts, the normalization is taken with respect to the amount promised, instead of being taken with respect to the collateral used to secure the transaction. The reason is that beliefs are expressed as a function of the expected price of the asset p_{t+1}^i , so it is simpler to choose this convention. However, note once again that the normalization is without loss of generality. From now on, the Short-Sales Contract $(1, \gamma)_s$ will be denoted only by the value for the Cash collateral $(\gamma)_s$. As for Borrowing Contracts, Short-Sales Contracts with Cash collateral γ are traded in an anonymous market at a competitive price $q_s(\gamma)$, and payment is only enforced by collateral which was put in the form of Cash: agents default whenever the value of the Cash collateral, on which agents have homogenous beliefs (equal to γ for everyone) is greater than the value of the asset which was promised as delivery in $t + 1$. Thus the payoff from buying contract $(\gamma)_s$ is $\min\{p_{t+1}, \gamma\}$. In period t , the Short-Sales Contract is sold by the short-seller, who gets $q_s(\gamma)$ units of Cash in exchange for the contract. Note that this time, lenders are therefore going to require margins from short-sellers in excess of the current price of the asset, when they buy the Short-Sales Contract - not only the proceeds of the sale of the asset has to stay as collateral, but some extra Cash must be given by the lender. Note also that the payoff of the Short-

Sales Contracts seems to take the same form as that of the Borrowing Contract of Definition 1, however an important difference is that in equilibrium $\gamma \geq p$, while for the borrowing contract $\phi \leq p$. This is because the Borrowing Contract is bought by an agent who is relatively pessimistic about the asset, while the Short-Sales Contract is bought by an agent who is relatively optimistic.

Equilibrium. In Economy \mathcal{E}^S , agents choose their positions in Assets n_A^i , Borrowing and Short-Sales Contracts $n_B^i(\cdot), n_S^i(\cdot)$, and Cash n_C^i , so as to maximize their expected wealth in period $t+1$ according to their subjective beliefs p_{t+1}^i about the Real Asset (W), subject to their budget constraint (BC), their collateral constraint (CC), and their cash-collateral constraint for Short-Sales Contracts (CCC):

$$\max_{(n_A^i, n_B^i(\cdot), n_C^i, n_S^i(\cdot))} n_A^i p_{t+1}^i + \int_{\phi} n_B^i(\phi) \min\{\phi, p_{t+1}^i\} d\phi + n_C^i + \int_{\gamma} n_S^i(\gamma) \min\{\gamma, p_{t+1}^i\} d\gamma \quad (\text{W})$$

$$\text{s.t.} \quad n_A^i p + \int_{\phi} n_B^i(\phi) q(\phi) d\phi + n_C^i + \int_{\gamma} n_S^i(\gamma) q_s(\gamma) d\gamma \leq w \quad (\text{BC})$$

$$\text{s.t.} \quad \int_{\phi} \max\{-n_B^i(\phi), 0\} d\phi \leq n_A^i \quad (\text{CC})$$

$$\text{s.t.} \quad \int_{\gamma} \max\{-\gamma n_S^i(\gamma), 0\} d\gamma \leq n_C^i \quad (\text{CCC})$$

$$\text{s.t.} \quad n_A^i \geq 0, \quad n_C^i \geq 0$$

Note again that when $n_S^i(\gamma) > 0$, agent i buys Short-Sales Contract $(\gamma)_s$, and is therefore an asset lender (or a synthetic buyer). When $n_S^i(\gamma) < 0$, agent i sells Short-Sales Contract $(\gamma)_s$, and is therefore a short-seller. Each time a short-seller sells one unit of Short-Sales Contract, he needs to own γ units of cash collateral. This gives equation (CCC).

As in Economy \mathcal{E}^B , a *Competitive Equilibrium* for Economy \mathcal{E}^S is a price p for the Real Asset and a distribution of prices $q(\cdot)$ for all traded Borrowing Contracts (ϕ) and a distribution of prices $q_s(\cdot)$ for all traded Short-Sales Contracts $(\gamma)_s$, and portfolios $(n_A^i, n_B^i(\phi), n_C^i, n_S^i(\gamma))$ for all agents i in the Real Asset, Borrowing Contracts, Cash, and Short-Sales Contracts such that all agents i maximize expected $t+1$ period wealth according to their subjective beliefs (W), subject to budget constraint (BC), collateral constraint (CC), cash-collateral constraint (CCC) and markets for the Real Asset, Borrowing Contracts, and Short-Sales Contracts clear:

$$\int_i n_A^i di = S, \quad (\text{MC}_A)$$

$$\forall \phi, \quad \int_i n_B^i(\phi) di = 0. \quad (\text{MC}_B)$$

$$\text{and } \forall \gamma, \quad \int_i n_S^i(\gamma) di = 0. \quad (\text{MC}_S)$$

2.1.2 Equilibrium

Proposition 4 is the counterpart for the Short-Sales Economy of Proposition 1 for the Borrowing Economy. It allows to investigate the robustness of the findings in Proposition 1, and in particular

the Power Law distribution for leverage and the excess returns on Borrowing Contracts, but also to study short-selling and lending together in a disagreement model, something which was not possible before.

Proposition 4 (Equilibrium of Economy \mathcal{E}^S). *At a competitive equilibrium of Economy \mathcal{E}^S ,*

1. *The space of agents' beliefs $[m, M]$ is partitioned through π'' , π , π' and $\hat{\pi}$ into five intervals:*
 - *Agents i with beliefs $p_{t+1}^i \in [m, \pi'']$ (short-sellers) invest in Cash and sell Short-Sales Contracts.*
 - *Agents i with $p_{t+1}^i \in [\pi'', \pi]$ (Cash lenders) buy Borrowing Contracts.*
 - *Agents i with $p_{t+1}^i \in [\pi, \pi']$ (synthetic buyers, asset lenders) buy Short-Sales Contracts.*
 - *Agents i with $p_{t+1}^i \in [\pi', \hat{\pi}]$ (unlevered buyers) buy the asset.*
 - *Agents i with $p_{t+1}^i \in [\hat{\pi}, M]$ (borrowers) buy the asset and sell Borrowing Contracts.*
2. *Synthetic buyers with beliefs y buy contracts $(y)_s$ with Cash collateral y , sold by short-sellers of type x with $x = \Gamma_s(y)$. $\Gamma_s(\cdot)$ is a strictly increasing function from beliefs of synthetic buyers in $[\pi, \pi']$ to short-sellers in $[m, \pi'']$ (positive assortative matching).*
3. *The implicit interest rate of a Short-Sales Contract (y) , $r_s(\cdot)$, defined as the excess return relative to the Real Asset and the assignment function $\Gamma_s(\cdot)$ are solutions of a system of two first-order Ordinary Differential Equations:*

$$p(y - \Gamma_s(y))r'_s(y) + pr_s(y) - \Gamma_s(y)r_s(y)^2 = 0 \quad \text{with} \quad r_s(\pi') = 1 \quad (A'_s)$$

$$f(\Gamma_s(y))\Gamma'_s(y) = \frac{yr_s(y) - p}{p}f(y) \quad \text{with} \quad \Gamma_s(\pi') = \pi''. \quad (C'_s)$$

4. *Lenders with beliefs x buy Borrowing Contracts (x) , sold by borrowers of type $y = \Gamma(x)$. $\Gamma(\cdot)$ is a strictly increasing function from beliefs of lenders in $[\pi'', \pi]$ to beliefs of borrowers in $[\hat{\pi}, M]$ (positive assortative matching). The implicit interest rate of traded contract with face value x , $r(\cdot)$, and the assignment function $\Gamma(\cdot)$ are solutions of a system of two first-order Ordinary Differential Equations:*

$$x(\Gamma(x) - x)r'(x) - \Gamma(x)r(x) + pr(x)^2 = 0 \quad \text{with} \quad r(\pi'') = \frac{\pi' - \pi''}{\pi' - p} \quad (A')$$

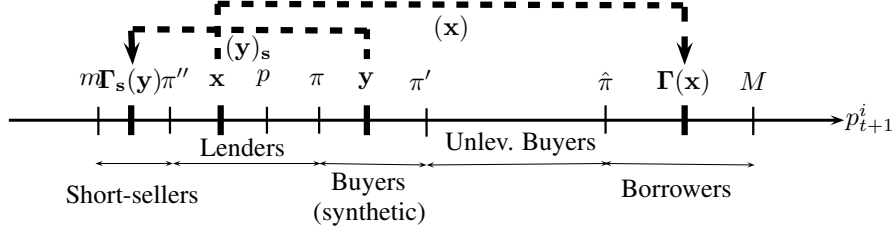
$$f(\Gamma(x))\Gamma'(x) = \frac{pr(x) - x}{x}f(x) \quad \text{with} \quad \Gamma(\pi'') = \hat{\pi}. \quad (C')$$

5. π'' , π , π' , $\hat{\pi}$ and p (the asset price) are solution to:

$$(a) \quad \frac{1 - F(\pi')}{p} + \frac{F(\pi) - F(\pi'')}{p} = \frac{S}{w}, \quad (b) \quad \frac{\hat{\pi}}{p} = \frac{\hat{\pi} - \pi''}{p - \pi''} \frac{\pi' - \pi''}{\pi'}, \quad (c) \quad r(\pi) = \frac{\pi}{p} r_s(\pi),$$

$$(d) \quad M = \Gamma(\pi) \quad \text{and} \quad (e) \quad m = \Gamma_s(\pi). \quad (2)$$

ECONOMY \mathcal{E}^S : BORROWING, SHORT-SALES, NO PYRAMIDING



Note: Lenders of type x and borrowers of type $\Gamma(x)$ are effectively matched through Borrowing Contracts (x) , as in Economy \mathcal{E}^B . Synthetic buyers or asset lenders with beliefs $p_{t+1}^i = y$ buy Short-Sales contracts $(y)_s$ with Cash collateral y and promising to return one unit of asset at time $t + 1$, sold by short-sellers with beliefs $p_{t+1}^i = \Gamma_s(y)$. Synthetic buyers of type y and short-sellers of type $\Gamma_s(y)$ are thus *effectively matched* through Short-Sales Contracts $(y)_s$, even though Short-Sales Contracts are also traded anonymously.

Discussion. In contrast to what happens in the Borrowing Economy \mathcal{E}^B , the Short-Sales Economy \mathcal{E}^S differs from economies considered by Geanakoplos (1997) and Simsek (2013) even for the first part of the proposition, concerning agents' types. In Geanakoplos (1997) as well as in his subsequent work, short-selling is ruled out altogether. Simsek (2013) considers Short-Sales Contracts, but only in the context where at least some fraction of investors are exogenously forbidden to buy Borrowing Contracts. In contrast, I do not restrict any agent from buying or selling debt and short contracts in economy \mathcal{E}^S . Three results stand out. First, in equilibrium, pessimists do not necessarily want to short, but some of them do lend. As in economy \mathcal{E}^B , the channel by which they are incentivized to do so is the interest rate channel: borrowers give a part of their returns to lenders, not only to exclude other borrowers, but also to make lenders indifferent between lending and shorting. This is captured by the initial condition of the assignment differential equation (A') $r(\pi'') = \frac{\pi' - \pi''}{\pi' - p}$, which replaces the initial condition found in economy \mathcal{E}^B , and given by $r(\pi'') = 1$. The outside option of lenders is to sell the asset short and not to invest in Cash, which allows them to capture a higher surplus from lenders at the competitive equilibrium. A direct consequence of this is that in contrast to Economy \mathcal{E}^B , the lowest value for allocative premia is given by $r(\pi'') - 1$ which is strictly greater than zero. Therefore, even the safest loans whose empirical counterparts can be thought of as AAA tranches command an excess return over the safe interest rate, in line with empirical evidence. Moreover, as we shall see later through the numerical example, short-sales are shown to be the exception rather than the norm, especially when disagreement is low, which can potentially explain why short-sales are quite rarely observed in financial markets. Low short-selling volume (short interest) is not necessarily evidence of short-sales constraints. Second, not all borrowers are levered in equilibrium, which is again a feature of real world financial markets, as not all stock investors buy on margin for example, and something that previous disagreement model did not

predict. The intuition, expressed by equation (2b), is that in this model it is costly to borrow, because lenders have a better outside option, which is to short. All borrowers with beliefs in $[\pi', \hat{p}]$ do not find it profitable to leverage themselves into the asset because the costs are too high. Third, interest rates on Short-Sales Contracts, materialized by different rebate rates on real world financial markets, also play an allocative role of matching short-sellers to synthetic buyers, just as allocative interest rates. High rebate rates are thus not necessarily an indication of short-sales constraints, just as low short-selling volumes. The mathematical details of the proof of Proposition 4 are in Appendix B.2, but it is useful to try to get to some intuition for the results.

Agents' Behavior. The arrow above represents the types of investment agents make as a function of beliefs, at the equilibrium of economy \mathcal{E}^S , which is stated in the first part of proposition 4. Since the types of agents is different from Geanakoplos (1997) and Simsek (2013), it is useful to give an intuition for it. The crucial thing to note is that most pessimistic traders are actually bullish about Cash relatively to the Real Asset. Hence, just as borrowers want to be levered into the Real Asset using the safe asset in Economy \mathcal{E}^B (and, as we shall see later, in this Economy \mathcal{E}^S) because they are bullish about the Real Asset, pessimists do not just invest their equity in Cash here as they do in Economy \mathcal{E}^B , they also borrow selling Short-Sales contracts. They lever them up into Cash using the Real Asset: short-sellers expect that the implicit interest rate they will pay on such borrowing, which depends on the price of the asset in $t + 1$, is lower than the return they will get in Cash, which is one. Corresponding to these borrowers, there are lenders of securities, whose assets are a promise to be paid next period an amount equal to the price of the asset then. In terms of payoffs, it is therefore akin to a Real Asset if one abstracts from default risk. In equilibrium, such synthetic buyers will only buy Short-Sales contracts that are fully secured relative to their beliefs, and therefore will expect to get the same payoff as the Real Asset (up to a rebate rate, which I will talk about later). This is why they are called synthetic buyers. The intuition for why synthetic buyers have beliefs below that of unlevered buyers in equilibrium is that they are relatively less pessimistic about the prospects of short-sellers, and therefore ask for less Cash-collateral. In equilibrium, competitive forces will be such that they will be buying Short-Sales contracts sold by short-sellers. I will now turn in more detail to points two to five of Proposition 4, focusing on the main differences with the Borrowing Economy \mathcal{E}^B : the determination of traded Short-Sales Contracts between short-sellers and synthetic buyers (point two), implicit interest rates on Short-Sales Contracts (rebate rates) and corresponding assignment function (point three), interest rates and assignment function for Borrowing Contracts (point four, very briefly, as it is very much like in Economy \mathcal{E}^B), and finally to cutoffs (point five).

Traded Short-Sales Contracts. Denote lenders of securities by their beliefs about the future price of the Real Asset $y = p_{t+1}^i$ and short-sellers, or borrowers of securities by $x = p_{t+1}^i$. The set of contracts that short-sellers and lenders can potentially agree to is indexed by all possible amount of Cash of collateral $y \in \mathbb{R}_+$ for one unit of asset promised by short-sellers. By a symmetric line of reasoning as for lenders/borrowers in Economy \mathcal{E}^B , it is possible to

reduce this potential triple dimensionality to one, and to show that a lender of securities will necessarily buy a Short-Sales contract with the amount of Cash collateral y equal to beliefs about the future price of the Real Asset (see Appendix B.2 for a proof). Using the notation defined in Definition 3, a lender with beliefs y buys Short-Sales Contract $(y)_s$. Denote by $q_s(y)$ the price of this contract, and by $r_s(y)$ the implicit excess return relative to a "Real Asset" given by $r_s(y) = p/q_s(y)$. Contract $(y)_s$ promises an asset in $t + 1$ to its owner, but is sometimes cheaper than a Real Asset in equilibrium, and therefore commands an excess return relative to the Real Asset. These excess returns play an allocative role in matching short-sellers to the lenders; in real world financial markets, the counterpart to these excess returns is low rebate rates on Cash collateral to short-sellers: Cash collateral stays with the lender, and gives the safe return (it is usually invested in highly liquid, low risk securities, such as money market funds or Treasuries) but only part of that return is transferred to the short-seller. This is how the lender earns return $r_s(y)$.

| A | L | |
|-----|--------------|---|
| y | $y - q_s(y)$ | E |
| | $q_s(y)$ | D |

The implicit leverage ratio for this contract with face value y denoted by $l_s(y)$ is then given by the amount of Cash that a Short-Sales Contract allows to invest in with one unit of personal funds:

$$l_s(y) = \frac{q_s(y)}{y - q_s(y)} = \frac{p}{yr_s(y) - p}.$$

For a short-seller, selling Short-Sales Contract $(y)_s$ allows to raise $q_s(y)$ in Cash, and requires y in collateral. Therefore, $y - q_s(y)$ has to be put in equity by a short-seller to invest y in Cash, a trade shown on the Figure on the left. One unit of equity allows to sell $1/(y - q_s(y))$ Short-Sales Contracts $(y)_s$. The corresponding amount of Cash raised by this sale is therefore $\frac{q_s(y)}{y - q_s(y)}$. Short-seller's expected return with belief x is then given by the mean of expected return on Cash with weight $1 + l_s(y)$, which is equal to one for all lenders, and expected return given to asset lenders $x/q_s(y) = x/pr_s(y)$ with weight the size of the borrowed funds $-l_s(y)$. Note that the expected return paid to asset lenders is both composed of the return on the asset, which short-sellers expect to be lower than the returns to Cash, and the excess return $r_s(y)$. Denoting by $W_s^x(x, y)/w$ the return for a Short-Seller with beliefs x conditional on selling a Short-Sales Contract $(y)_s$ one can therefore write:

$$\frac{W_s^y(x, y)}{w} = 1 + l_s(y) - \frac{x}{p}r_s(y)l_s(y).$$

Once again, one can show that this function is strictly supermodular in x and y , which gives an assortative matching between short-sellers and synthetic buyers, represented by means of a function $\Gamma_s(\cdot)$ taking an asset lender as argument, represented by his beliefs, and mapping him to a short-seller.

Interest Rates and Assignment Function for Short-Sales Contracts. Two first-order differential equations and two initial conditions determine functions $\Gamma_s(\cdot)$ and $r_s(\cdot)$. The first initial condition is given by the fact that the most optimistic synthetic lender with beliefs π' must be matched with the most optimistic short-seller with beliefs π'' , so that $\Gamma_s(\pi') = \pi''$. The second is a pasting equation, which equates the returns from investing, unlevered, in one unit of Real Asset and buying a Short-Sales Contract, the two of which must make agent with beliefs π' indifferent. Therefore the excess return earned by this lender must be zero, which gives $r_s(\pi') = 0$. Equation (A'_s) expresses the fact that excess returns $r_s(\cdot)$ are such as short-sellers are selling the contracts with a collateral such that they assortatively matched to asset lenders. Equation (C'_s) is a market clearing equation for Short-Sales Contracts. The derivations being similar to that of the matching between borrowers and lenders in Borrowing Economy \mathcal{E}^B , I refer to Appendix B.2 for more detail.

Borrowing Contracts. As in Borrowing Economy \mathcal{E}^B , there are both leveraged investors and lenders in this economy, and assortative matching between them in the competitive equilibrium ensured by equations (A') and (C') . The only difference is the initial condition for the excess premium function $r(\cdot)$, given by an indifference equation for the most pessimistic lenders. Instead of having to be made indifferent with the return of Cash as in Economy \mathcal{E}^B , here their outside option of to short-sell the asset and they must hence given a higher return equal to $r(\pi') = \frac{\pi' - \pi''}{\pi' - p}$, the return they would get from shorting. This is important, as it suggests that excess premia on bonds do not only come from the necessary exclusion of too pessimistic borrowers, but that it also incentivizes a pessimist to lend instead of shorting. Therefore, even the "safest" loans commands an excess return on the risk free rate, in line with empirical evidence.

Cutoffs. Finally, the equations linking the cutoffs between them in Economy \mathcal{E}^S can be derived from the following observations. Equation (2a) is a market clearing equation, expressing that lenders, unlevered buyers and borrowers' own funds are all used to purchase the Real Asset. Sp is asset supply, while $w(F(\pi) - F(\pi'')) + w(F(\hat{\pi}) - F(\pi')) + w(1 - F(\hat{\pi}))$ is asset demand and therefore:

$$\frac{1 - F(\pi')}{p} + \frac{F(\pi) - F(\pi'')}{p} = \frac{S}{w}.$$

Equation (2b) says that agents with beliefs about the price of the asset equal to $\hat{\pi}$ shall be indifferent between investing levered and unlevered into the asset. What makes such a thing possible without $\hat{\pi} = p$, is that the interest rate paid by the most pessimistic borrower is not one as in Borrowing Economy \mathcal{E}^B , but is such that $r(\pi'') > 1$. The indifference equation writes as:

$$\frac{\hat{\pi}}{p} = \frac{\hat{\pi} - \pi''}{p - \frac{\pi''}{r(\pi'')}} \Rightarrow \frac{\hat{\pi}}{p} = \frac{\hat{\pi} - \pi''}{p - \pi''} \frac{\pi' - \pi''}{\pi'}.$$

The second equation results from the initial condition for the assignment equation (C') : $r(\pi'') = \frac{\pi' - \pi''}{\pi' - p}$. Similarly, equation (2c) is also a "pasting" equation, expressing that returns

from lending and from buying a synthetic asset, which are composed of both the return to the asset and the excess returns coming from lower than one rebate rates, is:

$$r(\pi) = \frac{\pi}{p} r_s(\pi).$$

Finally, equation (2d) expresses the fact that the most optimistic borrower is matched with the most optimistic lender, so that $M = \Gamma(\pi)$. Equation (2e) says that the most optimistic asset lender is matched with the most optimistic short-seller, so $m = \Gamma_s(\pi)$.

2.1.3 Pareto Distributions for Leverage Ratios

Finally, as in economy \mathcal{E}^B , there is a limiting Pareto Distribution for leverage ratios of short-sellers, in addition to the limiting Pareto of borrowers, which still obtains. For low disagreement, leverage is very high both for optimists who are able to borrow from lenders with beliefs lower than but close to $x = p$, and for short-sellers who are able to borrow from asset lenders (or synthetic buyers) with beliefs higher than and close to $y = p$. More precisely, for a sufficiently regular density function of beliefs - one sufficient condition is that it be differentiable on the neighborhood of p , but it can in fact be even less regular than that - leverage is shown once again to have a limiting Pareto distribution of shape parameter two. The generalization of result 2 is given below.

Proposition 5 (Limiting Pareto Distribution for Leverage Ratios of Borrowers and Short-Sellers for low Disagreement). *Let the density of beliefs $f(\cdot)$ be differentiable, continuous and bounded. The distribution of leverage ratios for borrowers tends to a Pareto distribution with shape two, when disagreement goes to zero. Formally, denoting by $I^h(\cdot)$ the distribution function for the leverage ratio of borrowers with density of beliefs $f^h(\cdot)$:*

$$\exists C_h, \quad \|l^2(1 - I_h(l)) - C_h\|_{\infty}^{[L_h(M)/2, L_h(M)]} \xrightarrow{h \rightarrow 0} 0,$$

where $\|f(l)\|_{\infty}$ denotes the supremum of $|f(l)|$ where f is defined, when it exists.

Similarly, the distribution of leverage ratios for short-sellers tends to a Pareto distribution with shape two, when disagreement goes to zero. Formally, denoting by $J^h(\cdot)$ the distribution function for the leverage ratio of short-sellers with density of beliefs $f^h(\cdot)$:

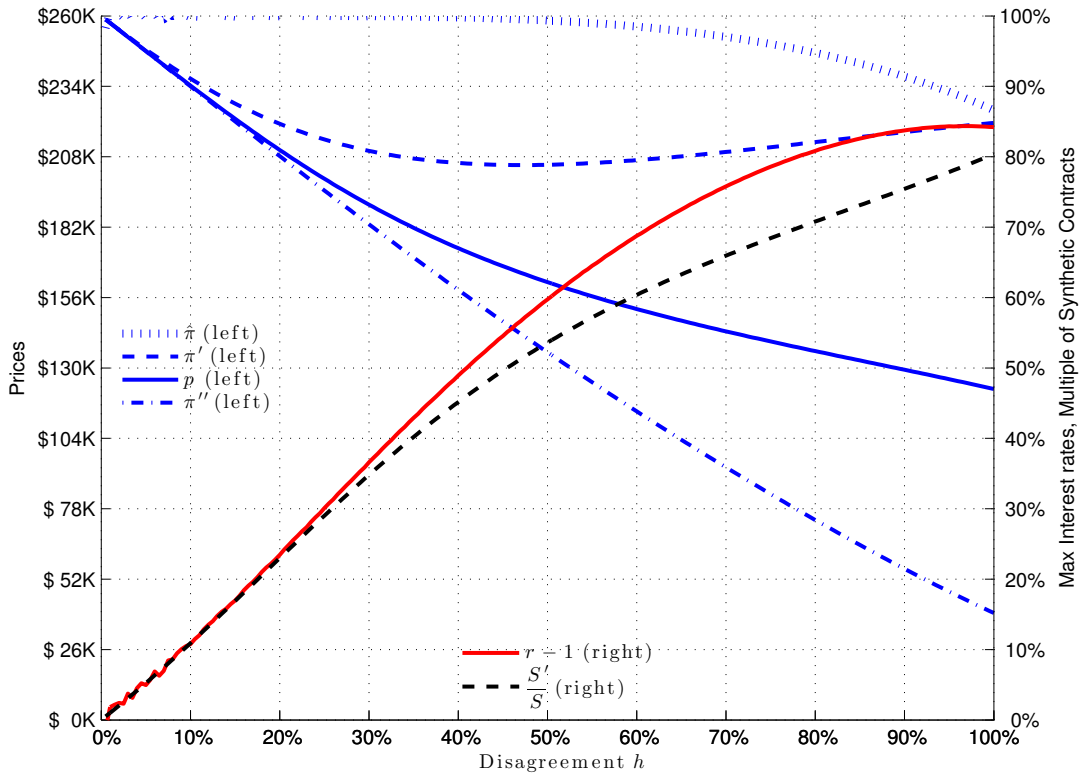
$$\exists D_h, \quad \|l^2(1 - J_h(l)) - D_h\|_{\infty}^{[L_{sh}(M)/2, L_{sh}(M)]} \xrightarrow{h \rightarrow 0} 0,$$

Note that the Pareto distribution for borrowers generalizes to short-sales economies, even though it is relatively harder for them to borrow because they must give part of their returns to potential short-sellers, so that they lend instead of sell short. The intuition behind this result is that close to p , the return that pessimists get out of shorting is low, so that this does not prevent borrowers to leverage a lot into the asset. The proof for this is exactly symmetric to that in the Borrowing Economy \mathcal{E}^B .

2.1.4 Numerical Example with Flat Priors

Let us come back to the numerical example developed in Section 1.6 for the case of the Borrowing Economy \mathcal{E}^B . The assumed belief structure is one of flat priors with $f(x) = \frac{1}{h} \mathbb{1}[M(1-h), M](x)$, and $F(x) = \frac{x - M(1-h)}{H} \mathbb{1}[M(1-h), M](x)$. A corollary of lemma 4, using equations (2a), (2b), (2c), (2d), and (2e), is that cutoffs can be calculated as a function of the parameters of this belief distribution M and H , and in particular comparative statics with respect to disagreement $h = \frac{S}{w}H$ are shown on Figure 8. In the [Online Appendix](#), the derivation and calculation of these cutoffs is explained more in detail.

Figure 8: COMPARATIVE STATICS WITH RESPECT TO DISAGREEMENT h IN ECONOMY \mathcal{E}^S



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$ (see Figure 1). The price of the asset p and cutoffs π'' , π , π' , \hat{p} are given by Proposition 4, as a function of h . The minimum belief is $m = M(1-h)$. Note that with short-sales, short-sellers take extreme positions and the price of the asset goes to the minimum belief m .

Of course, the price with short-sales is lower than without short-sales. It is straightforward to show, using equations (2a), (2b) and (2c), (2d) and (2e), that this is general.

2.2 Securitization Economies \mathcal{E}_n^B

In Borrowing Economy \mathcal{E}^B , the contracting set was exogenously restricted to a set where agents can only collateralize units of the Real Asset. I therefore also explicitly assuming that the securitization of loans, and the lending against these loans (which also are collateral) was not

possible. This considerably simplified the exposition of the model, and allowed to go quicker to the main results of the paper. It was also interesting to look at this case first because in practice, many asset markets seem to be well described by this assumption. For example, in the United States, Regulation T and SEC's Rule 15c3 limit Prime Brokers' use of rehypothecated collateral from a client. Similarly, securitization in the housing market is a fairly recent phenomenon, the extent of which was broadened in the nineties in the United States but that is still not possible in most countries around the world for regulatory reasons. Apart from institutional and regulatory obstacles, securitization entails some cost, which as we will see later, have to be weighted against its expected benefits: for example Greenwood and Scharfstein (2013) estimate the output from securitization to have neared one percent of GDP in most years between 2001 and 2007.¹⁶

However, in economy \mathcal{E}^B , lenders earn a higher return than cash because they earn an allocative premium, and would therefore ideally leverage themselves using cash into Borrowing Contracts. In this section, I allow agents to do such a thing. A real-world example of these contracts is Mortgage-Backed Securities, which were a prominent actor in the 2008 – 2009 financial crisis, and are loans which are themselves made against mortgages. I show both that the main results of Section 1 go through with pyramiding lending arrangements, but also that collateral chains, or "pyramiding lending arrangements" to quote Geanakoplos (1997) lead to an increase in asset prices relative to the Borrowing Economy \mathcal{E}^B .

2.2.1 Securitization Economy \mathcal{E}^B_2

In this section, I work in the same setting as in Economy \mathcal{E}^B (see Section 1.1), except I allow agents to agree to collateralized Borrowing Contracts not only using the Real Asset as collateral, but also using Borrowing Contracts collateralized by the Real Asset as collateral, which I refer to as *Type-2 Borrowing Contracts*. Formally,

Definition 4 (Type-2 Borrowing Contract, Economy \mathcal{E}^B_2). *A Type-2 Borrowing Contract $(\phi')_2$ in economy \mathcal{E}^B_2 is a promise of $\phi' \geq 0$ units of Cash in period $t+1$, collateralized by one unit of Borrowing Contract (ϕ) , such that $\phi' \leq \phi$. ϕ' is referred to as the face value of the Borrowing Contract.*

Note that restricting ourselves to $\phi' \leq \phi$ is without loss of generality, as Type-2 Borrowing Contracts with $\phi < \phi'$ have the same payoffs as Borrowing Contracts (ϕ') , no matter what p_{t+1} is. The reason is that the holder of the Borrowing Contract always wants to default for $\phi < p_{t+1} < \phi'$. Note also that the face value ϕ of the Borrowing Contract backing the Type-2 Borrowing Contract $(\phi')_2$ is irrelevant, as long as $\phi' \leq \phi$. The reason is that the collateral matters only for price realization such that $p_{t+1} \leq \phi'$ and that for those price realizations, the corresponding Borrowing Contract also is in default, such that only the collateral matters. This is why the *Type-2 Borrowing Contract $(\phi')_2$* can be referred to by its face value only. The price of this Type-2 Borrowing Contract will be denoted by $q_2(\phi')$.

¹⁶They also calculate that fixed income securities grew from totaling 57% of GDP in 1980 to 182% of GDP in 2007, and 58% of the growth of fixed income securities came from securitization.

As in economy \mathcal{E}^B , agent i chooses his position in the Real Asset n_A^i , a menu of financial Borrowing Contracts (ϕ) denoted by distribution $n_B^i(\cdot)$, a menu of Type-2 Borrowing Contracts denoted by $n_{B2}^i(\cdot)$, and Cash n_C^i , in order to maximize his expected wealth in period $t + 1$ according to his subjective beliefs p_{t+1}^i about the Real Asset (W), subject to his budget constraint (BC), collateral constraint (CC) and collateral constraint for Type-2 Borrowing Contracts (CC2):

$$\begin{aligned}
& \max_{(n_A^i, n_B^i(\cdot), n_{B2}^i(\cdot), n_C^i)} n_A^i p_{t+1}^i + \int_{\phi} n_B^i(\phi) \min\{\phi, p_{t+1}^i\} d\phi + n_C^i + \int_{\phi'} n_{B2}^i(\phi') \min\{\phi', p_{t+1}^i\} d\phi' \quad (\text{W}) \\
& \text{s.t.} \quad n_A^i p + \int_{\phi} n_B^i(\phi) q(\phi) d\phi + n_C^i + \int_{\phi'} n_{B2}^i(\phi') q(\phi') d\phi' \leq w \quad (\text{BC}) \\
& \text{s.t.} \quad \int_{\phi} \max\{-n_B^i(\phi), 0\} d\phi \leq n_A^i \quad (\text{CC}) \\
& \text{s.t.} \quad \int_{\phi'} \max\{-n_{B2}^i(\phi'), 0\} d\phi' \leq \int_{\phi} n_B^i(\phi) d\phi \quad (\text{CC2}) \\
& \text{s.t.} \quad n_A^i \geq 0, \quad n_C^i \geq 0
\end{aligned}$$

Note that each time a borrower/lender sells of unit of Type-2 Borrowing Contract, he needs to own unit of Borrowing Contract hence equation (CC2). A *Competitive Equilibrium* for Economy \mathcal{E}^B_2 is then a price p for the Real Asset, a distribution of prices $q(\cdot)$ for all traded Borrowing Contracts (ϕ), of prices $q_2(\cdot)$ for all traded Type-2 Borrowing Contracts (ϕ') and portfolios $(n_A^i, n_B^i(\phi), n_{B2}^i(\phi'), n_C^i)$ for all agents i in the Real Asset, Borrowing Contracts, Type-2 Borrowing Contracts and Cash, such that all agents i maximize expected $t+1$ period wealth according to their subjective beliefs (W), subject to budget constraint (BC), collateral constraints (CC) and (CC2), and markets for the Real Asset Borrowing Contracts and Type-2 Borrowing Contracts clear:

$$\begin{aligned}
& \int_i n_A^i di = S, \quad (\text{MC}_A) \\
& \forall \phi, \quad \int_i n_B^i(\phi) di = 0. \quad (\text{MC}_B) \\
& \text{and } \forall \phi', \quad \int_i n_{B2}^i(\phi') di = 0. \quad (\text{MC}_B)
\end{aligned}$$

Proposition 6 is the counterpart for the Securitization Economy \mathcal{E}^B_2 of Proposition 1 for the Borrowing Economy.

Proposition 6 (Equilibrium of Economy \mathcal{E}^B_2). *At a competitive equilibrium of Economy \mathcal{E}^B_2 ,*

1. *The space of agents' beliefs $[m, M]$ is partitioned through π_1'', π_2'' and π into four intervals such that:*

- *Agents i with beliefs $p_{t+1}^i \in [m, \pi_2'']$ (cash investors) invest in Cash.*
- *Agents i with beliefs $p_{t+1}^i \in [\pi_2'', \pi_1'']$ (lenders of Type-2) buy Type-2 Borrowing Contracts.*

- Agents i with $p_{t+1}^i \in [\pi_1'', \pi]$ (lenders) buy Borrowing Contracts and sell Type-2 Borrowing Contracts.
 - Agents i with $p_{t+1}^i \in [\pi, M]$ (borrowers) buy the Asset and sell Borrowing Contracts.
2. Lenders with beliefs x buy Borrowing Contracts with face value x , sold by borrowers with beliefs $y = \Gamma_1(x)$. $\Gamma_1(\cdot)$ is a strictly increasing function from beliefs of lenders in $[\pi_1'', \pi]$ to beliefs of borrowers in $[\pi, M]$ (positive assortative matching). Lenders of type-2 with beliefs z buy Borrowing Contracts of type-2 with face value z , sold by lenders with beliefs $x = \Gamma_2(z)$. $\Gamma_2(\cdot)$ is a strictly increasing function from beliefs of type-2 lenders in $[\pi_2'', \pi_1'']$ to beliefs of lenders in $[\pi_1'', \pi'']$ (positive assortative matching).
 3. The implicit interest rate of Borrowing Contract with face value x , $r_1(x)$, the implicit interest rate of Type-2 Borrowing Contract with face value z , $r_2(z)$, and the assignment functions $\Gamma_1(x)$ and $\Gamma_2(z)$ are solutions of a system of four first-order Ordinary Differential Equations:

$$x(\Gamma_1(x) - x)r_1'(x) - \Gamma_1(x)r_1(x) + pr_1(x)^2 = 0 \quad \text{with} \quad r_1(\pi_1'') = r_2(\pi_1'') \quad (A_1)$$

$$f(\Gamma_1(x))\Gamma_1'(x) = \frac{pr_1(x) - x}{x} \frac{xr_2(\Gamma_2^{-1}(x))}{xr_2(\Gamma_2^{-1}(x)) - \Gamma_2^{-1}(x)} f(x) \quad \text{with} \quad \Gamma_1(\pi_1'') = \pi. \quad (C_1)$$

$$z(\Gamma_2(z)r_1(\Gamma_2(z)) - z)r_2'(z) - \Gamma_2(z)r_1(\Gamma_2(z))r_2(z) + \Gamma_2(z)r_2(z)^2 = 0$$

$$\text{with} \quad r_2(\pi_2'') = 1 \quad (A_2)$$

$$f(\Gamma_2(z))\Gamma_2'(z) = \frac{\Gamma_2(z)r_2(z) - z}{z} f(z) \quad \text{with} \quad \Gamma_2(\pi_2'') = \pi_1''. \quad (C_2)$$

4. π_1'', π_2'', π and p (the asset price) are solution to:

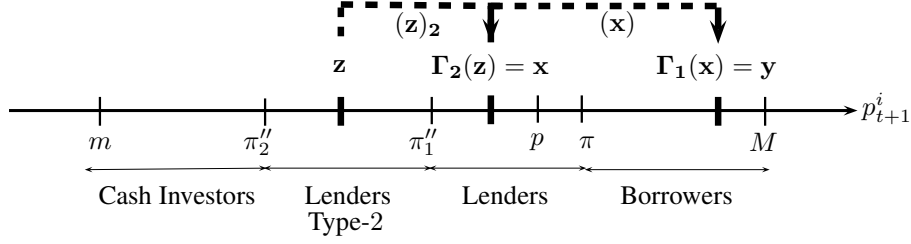
$$(a) \quad \frac{1 - F(\pi_2'')}{p} = \frac{S}{w}, \quad (b) \quad r_2(\pi) = \frac{\pi - \pi_1''}{p - \pi_1''}, \quad (c) \quad M = \Gamma_1(\pi),$$

$$\text{and} \quad (d) \quad \pi = \Gamma_2(\pi_1''). \quad (3)$$

This proposition in many way resembles proposition 1, albeit for some (mostly minor) differences detailed in 6. The main difference is that Borrowing Contracts can now be used by Lenders to finance themselves from even less optimistic agents. The intuition for why they want to do that in equilibrium is that borrowers give them part of their returns because of the allocative argument, and that lenders therefore want to purchase as many of these contracts as possible, possibly using Borrowing Contracts themselves as collateral.¹⁷

¹⁷This can clearly be seen on Balance Sheets available on the [Online Appendix](#).

ECONOMY $\mathcal{E}^{\mathcal{B}}_2$: BORROWING, AND SECURITIZATION



Note: Lenders with beliefs $p^i_{t+1} = x$ and borrowers with beliefs $\Gamma_1(x)$ are effectively matched through Borrowing Contracts (x) , as in Economy $\mathcal{E}^{\mathcal{B}}$. Because agents with beliefs $p^i_{t+1} = x$ earn part of Borrowers' excess return, they also want to leverage themselves from pessimists with lower beliefs than them. When agents can agree to collateralized Borrowing Contracts of Type-2 (that is, Borrowing Contracts collateralized by other Borrowing Contracts), Type-2 Lenders with beliefs z and lenders with beliefs $\Gamma_2(z) = x$ are effectively matched through Type-2 Borrowing Contracts $(z)_2$. Agents with beliefs x thus borrow from agents z using a collateralized Borrowing Contract as collateral, and a loan with face value z .

Comparative Statics with Flat priors. On Figure 9, the price p is given in the case of flat priors, or $f(x) = \frac{1}{h} \mathbb{1}[M(1-h), M](x)$ (see Figure 1), with $M = \$260K$ and for different $H = Mh$ on the x axis. From this numerical example, one can see very clearly and intuitively that for all values of h , further borrowing against Borrowing Contracts raises Asset Prices:

$$\{p\}^{\mathcal{E}^{\mathcal{B}}_3} > \{p\}^{\mathcal{E}^{\mathcal{B}}_2} > \{p\}^{\mathcal{E}^{\mathcal{B}}}.$$

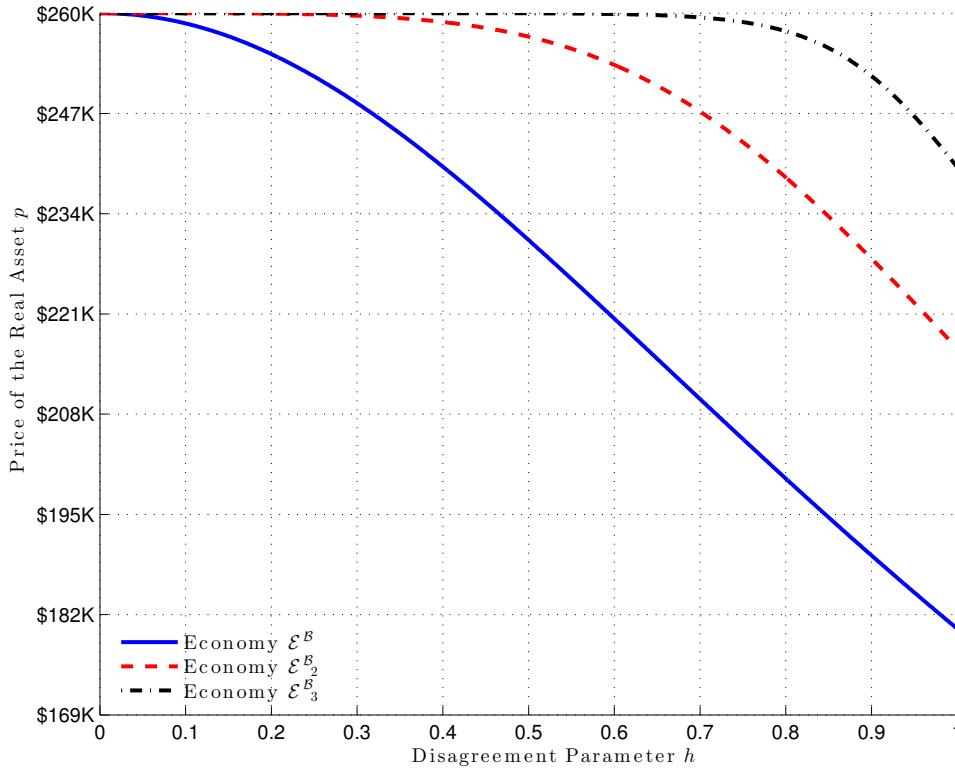
Interestingly, the results on collateralization chains obtained in this section are therefore quite similar from the ones obtained in Holmstrom and Tirole (1998) type models, in which liquidity is scarce because future capital income is not entirely pledgeable. To the best of my knowledge, it is the first time that what Geanakoplos (1997) calls *pyramiding lending arrangements* are modeled in the context of a disagreement model.

The intuitive reason for this effect, as in Fostel and Geanakoplos (2012), is that allowing for more layers of pyramiding is akin to making markets more complete, as agents de facto buy "slices" of contracts, which are more and more tailored to their beliefs. More precisely, a lender in economy $\mathcal{E}^{\mathcal{B}}_2$, buying Borrowing Contract (x) and selling Type-2 Borrowing Contract $(z)_2$ effectively buys the Real Asset conditional on $p^i_{t+1} \in [z, x]$.

2.2.2 Securitization Economy $\mathcal{E}^{\mathcal{B}}_\infty$

As was shown previously, successive rounds of rehypothecation or recollateralization lead to increasing asset prices when the number of "tranches" increases, or the collateral can be promised several times. De facto, these arrangements look very much like the tranching of collateral. The

Figure 9: COMPARATIVE STATICS WITH RESPECT TO DISAGREEMENT h IN ECONOMIES \mathcal{E}^B , \mathcal{E}^B_2 AND \mathcal{E}^B_3



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$. The price of the asset p is increasing when more layers of lending are possible. In Economy \mathcal{E}^B , Borrowing Contracts can only be collateralized by Real Assets. In Economy \mathcal{E}^B_2 , Borrowing Contracts can also be used as collateral, and corresponding Borrowing Contracts are referred to as Type-2 Borrowing Contracts (see Definition 4). Finally, in Economy \mathcal{E}^B_3 , Type-2 Borrowing Contracts can also be used as collateral.

reason is the following: when the number of layers increases, each agent de facto buys the "tranche" of the collateral that he thinks more likely, by choosing the face value of the loan corresponding to his beliefs and selling the other "states" to more pessimistic agents.

This can be generalized when the number of elements in this pyramid goes to infinity. In that case, agents with beliefs $p_{t+1}^i = x$ buy a Type- n Borrowing Contract $(x)_n$ with face value x , and sell a Type- $n + 1$ Borrowing Contract $(x - dx)_{n+1}$ with face value equal to $x - dx$ using the previous contract as collateral. In the limit where n goes to ∞ , the price of the Real Asset goes to M for all values of the disagreement parameter.

An open question however is to extend the models below to both short-selling and pyramiding arrangements. These are important questions, given that Credit Default Swaps on bonds have been used to bet against the housing market in 2007 (see Lewis (2010)). I leave this to future research.

3 Two Applications

3.1 Margin Requirements

It is sometimes argued that imposing margin requirements on asset-based borrowing would throw "sand in the wheels" in the run-up to a bubble, and prevent optimists from bidding prices up. For example, regulators can impose minimum margins (or equivalently maximum Loan-To-Value ratios) on the housing market. The effect of such regulations is studied in economy \mathcal{E}_m^B with Borrowing Contracts only in section 3.1.1.

However, outside of the real estate market, very few market participants nowadays borrow the Real Asset using collateralized loan as in Section 1. To be leveraged in a long position, they more often buy a futures option or a forward contract, which combine a long position and leverage. And of course, financial institutions buy many more complex financial instruments, so that regulators do not regulate margin directly. Rather, they use Value-at-Risk, a proxy for margin. As a consequence, they necessarily also limit margins on Short-Sales Contracts. I study in Section 3.1.2 the effect of margins in such an economy where margins also apply to Short-Sales Contracts (Economy \mathcal{E}_m^S).

3.1.1 Borrowing Economy with required margins \underline{m} \mathcal{E}_m^B

With margin requirements or leverage caps, lenders and borrowers are constrained in the type of loans they can buy or sell. Thus the only difference with the Borrowing Economy in Section 1 is that the margin they take on their loan must at least be equal to a value m , so that for all Borrowing Contracts (ϕ) , it must be that:

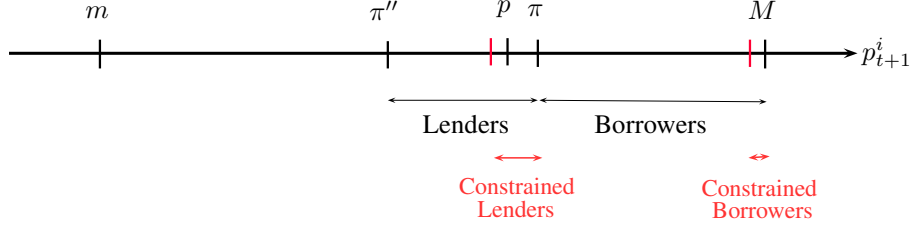
$$\frac{p - \frac{\phi}{r(\phi)}}{\frac{\phi}{r(\phi)}} \geq \underline{m}$$

In other words, a margin requirement regulation is a limitation on the set of Borrowing Contracts that is traded. A natural generalization of proposition 1 with margin requirements is then stated in a similar result, which would have equation (C) as the only changing equation. It should now be equation (C^m), which takes into account that minimum margin requirements may be binding and restrict the set of contracts available to some agents:

$$f(\Gamma(x))\Gamma'(x) = \max \left\{ \frac{pr(x) - x}{x}, \underline{m} \right\} f(x) \quad \text{with} \quad \Gamma(\pi'') = \pi. \quad (C^m)$$

In particular, in the example studied earlier (see, for example, section 1.6), that is, $f(x) = \frac{1}{h} \mathbb{1}[M(1-h), M](x)$, one gets without difficulty the prices as in Section 1. As can be seen on Figure 10, imposing margins is effective in reducing asset prices in economy \mathcal{E}_m^B . Interestingly, this intuition can however be reversed when short-sales are allowed. Because imposing margins prevent short-sellers from taking risks to sell the asset short, prices can be higher in equilibrium with the imposition of a margin requirement. In the next section, I show that such can be the case, taking the example of flat priors once again as an illustration.

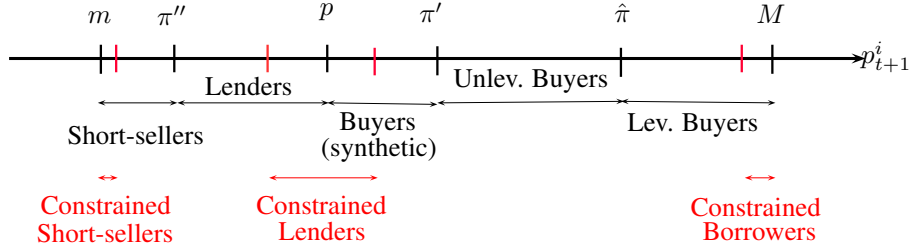
ECONOMY $\mathcal{E}_{\underline{m}}^{\mathcal{B}}$: BORROWING, NO SHORT-SALES AND MINIMUM MARGINS \underline{m}



3.1.2 Short-Sales Economy with required margins \underline{m} $\mathcal{E}_{\underline{m}}^{\mathcal{S}}$

Short-sellers also are borrowers of the asset, and margin requirements will also apply to their borrowing (below).

ECONOMY $\mathcal{E}_{\underline{m}}^{\mathcal{S}}$: BORROWING, SHORT-SALES AND MINIMUM MARGINS \underline{m}



In practice, this may be because caps on margins (or on leverage ratios, equivalently) may be implemented through Value-at-Risk methods, which are as sensitive to long as to short positions. Therefore both Borrowing Contracts traded (ϕ) and Short-Sales Contracts (γ) must be such that:

$$\frac{p - \frac{\phi}{r(\phi)}}{\frac{\phi}{r(\phi)}} \geq \underline{m} \quad \text{and} \quad \frac{\gamma - \frac{p}{r_s(\gamma)}}{\frac{p}{r_s(\gamma)}} \geq \underline{m}.$$

The equilibrium of the Short-Sales Economy with minimum margins \underline{m} , denoted as $\mathcal{E}_{\underline{m}}^{\mathcal{S}}$, is therefore determined by Proposition 4, with the only equations changing being the clearing equation (C'_s) which is now ($C^{m'}_s$), and equation (C') which is now ($C^{m'}$). They now write:

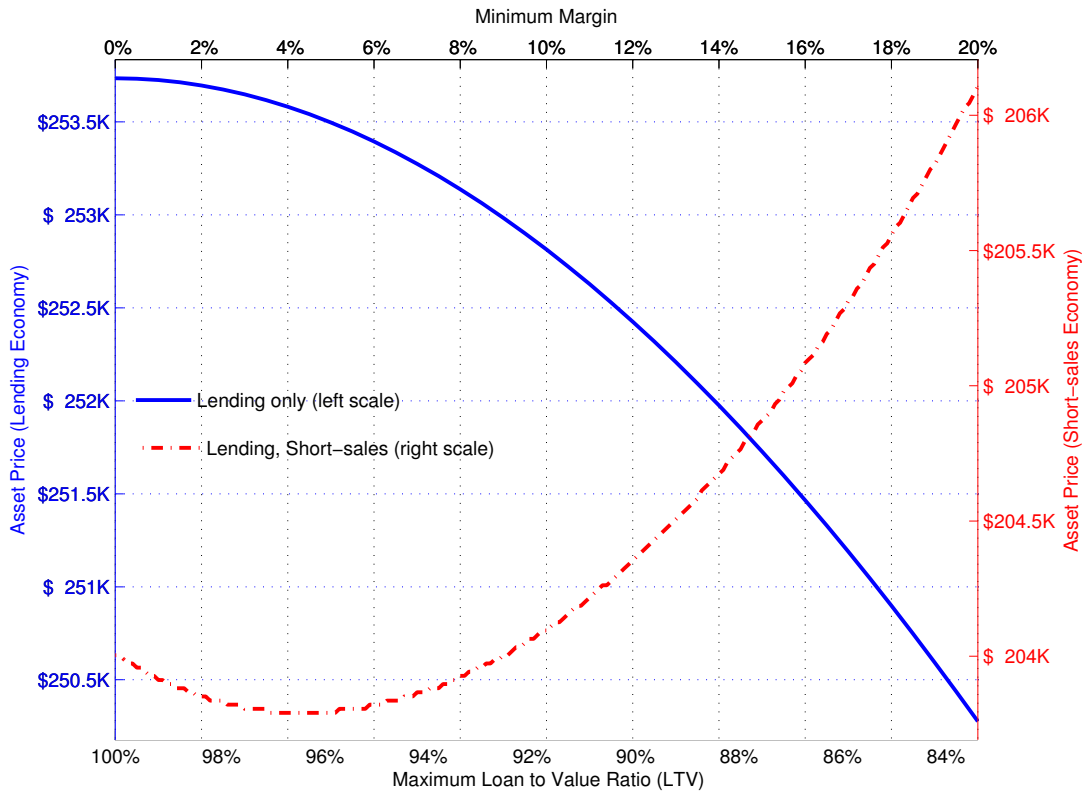
$$f(\Gamma_s(y))\Gamma'_s(y) = \max \left\{ \frac{y r_s(y) - p}{p}, \underline{m} \right\} f(y) \quad \text{with} \quad \Gamma_s(\pi') = \pi''. \quad (C^{m'}_s)$$

$$f(\Gamma(x))\Gamma'(x) = \max \left\{ \frac{p r(x) - x}{x}, \underline{m} \right\} f(x) \quad \text{with} \quad \Gamma(\pi'') = \hat{\pi}. \quad (C^{m'})$$

These modifications to Proposition 4 in the case of margin requirements take into account that minimum margin requirements may be binding, for Borrowing Contracts and for Short-Sales Contracts, and restrict the set of contracts available to some agents.

In particular, for the case of flat priors, prices can be calculated and they are plotted on Figure 10. One can see in particular that more stringent margin requirements do not necessarily lead to lowering asset prices. As can be seen on Figure 14 in Appendix A, the mechanism goes through short-sellers' limited ability to sell short when margin requirements are imposed. This result is potentially interesting for policymakers who are sometimes tempted to use margin requirements as a means to dampen asset price rises in periods that they identify as "irrationally exuberant."

Figure 10: EFFECT OF MARGIN REQUIREMENTS IN ECONOMIES \mathcal{E}_m^B AND \mathcal{E}_m^S



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 23\%$. Without short-sales, margins have the intuitive effect of lowering asset prices. However, with short-sales, margins here can contribute to raising asset prices. Intuitively, this is because margin also throw "sand in the wheels" of short-sellers also.

3.2 Some Dynamics of the Wealth Distribution

The aim of this section is to show that the static model developed above can be fruitfully extended to study the skewness of the wealth distribution. In particular, I will show that the model developed in section 1 for the Borrowing Economy \mathcal{E}^B can be used to generate long run distributions of wealth with Pareto-Lorenz coefficients between one and three. The intuition

for this result runs as follows. In section 1, it was shown that an heterogeneous beliefs model with collateral constraints generates Power Law distributions for leverage of shape two when the distribution of wealth on the support of beliefs is initially bounded, and that it generates Power Law distributions for leverage with a different shape when wealth is initially more or less skewed than a uniform distribution, which can happen precisely when the process is repeated many times. In particular, when the distribution of wealth is initially skewed towards borrowers in such a way that the Pareto-Lorenz coefficient is two on the support of beliefs, then the distribution of leverage has a Pareto shape of 4, because of the congestion effect described in section 1. When the process is repeated many times, and agents have infinite lives, the distribution of wealth tends therefore to be skewed according to Zipf's law. When agents are finitely lived, then there are two opposing forces at play which explain intermediary values for Pareto-Lorenz coefficients. The first is that wealth tends to be characterized by a Zipf Law in the long run, but the second is that agents from new generations coming in either all have the same wealth, or have no correlation between their wealth and their beliefs. This force tends to lead to a wealth distribution less skewed than Zipf. I show that in the long run, the Pareto Lorenz coefficient nevertheless reaches a steady-state.

To focus on the main message, I look at Borrowing Economy \mathcal{E}^B of section 1, and on a special case for the structure of dividends and beliefs. Time runs from $t = 0$ to $t = T$. Interest rates are normalized to $R = 1$, like before. The dividend about which agents disagree in all periods from 0 to $T - 1$ is dividend d_T to be served in period T , with a large number of periods $T \gg 1$. Agent i has expectation about dividend in period T denoted by d_T^i . There are intermediary dividends in the interim period d_1, d_2, \dots, d_{T-1} and all agents agree on the values for these dividends. However, I assume that at each interim point in time $t \in \{1, \dots, T - 1\}$, a large shock to dividends served $d_t \geq \max_i p_i - p_t$ in period t occurs, so that a Pareto distribution for returns is generated in each period (see section 1.5). Of course, less extreme shocks can occur, but they do not change the Pareto tail exponent of wealth, because they lead to a bounded distribution for return; therefore a period here must be interpreted as the typical time between two events outside of the support of beliefs. To avoid the additional motive for overvaluation in Harrison and Kreps (1978), agents' beliefs about future agents' beliefs are that they will conform to their expectations of the dividends d_T . Thus, expectations of an agent with beliefs i in period t about $t + 1$'s period price is:

$$p_{t+1}^i = \sum_{j=t+1}^{T-1} d_j + p_T^i.$$

I examine different cases for the demographics of this economy, which can be interpreted as different cases for the frequency of large shocks (the "turbulence" of the economy).

3.2.1 Unique Generation $N = T$

In the first case, a unique cohort of agents arrives at $t = 0$ with a bounded distribution of wealth on the support of beliefs, so that denoting by $(\alpha_t)_{t=0,1,\dots,T}$ the Pareto coefficient of the wealth

distribution on the support of beliefs with $f_t(y) \sim (M_t - y)^{-1/\alpha_t}$, the initial Pareto coefficient for wealth is $\alpha_0 = +\infty$. Denote by $(\beta_t)_{t=0,1,\dots,T-1}$ the Pareto coefficient on the leverage distribution in period t , and hence on realized returns in period $t+1$. Then the distribution of leverage in the first period has tail exponent $\beta_0 = 2$ as shown in section 1.5. The resulting wealth distribution in period one is given by $f_1(y) = RoE_0(y)f_0(y) \sim RoE_0(y) \sim (M_t - y)^{-1/2}$. More generally, the density of beliefs in period t is given by:

$$f_{t+1}(y) = RoE_t(y)f_t(y)$$

Therefore, the Pareto coefficient of the wealth distribution in period $t+1$ is given by:

$$(M_t - y)^{-1/\alpha_{t+1}} = (M_t - y)^{-1/\alpha_t} (M_t - y)^{-1/\beta_t} \Rightarrow \frac{1}{\alpha_{t+1}} = \frac{1}{\alpha_t} + \frac{1}{\beta_t}$$

From result 3, the coefficient β_t on the leverage distribution is given by $\frac{1}{\beta_t} = \frac{1}{2} \left[1 - \frac{1}{\alpha_t} \right]$, and therefore:

$$\frac{1}{\alpha_{t+1}} = \frac{1}{\alpha_t} + \frac{1}{2} \left[1 - \frac{1}{\alpha_t} \right] = \frac{1}{2} \left[1 + \frac{1}{\alpha_t} \right].$$

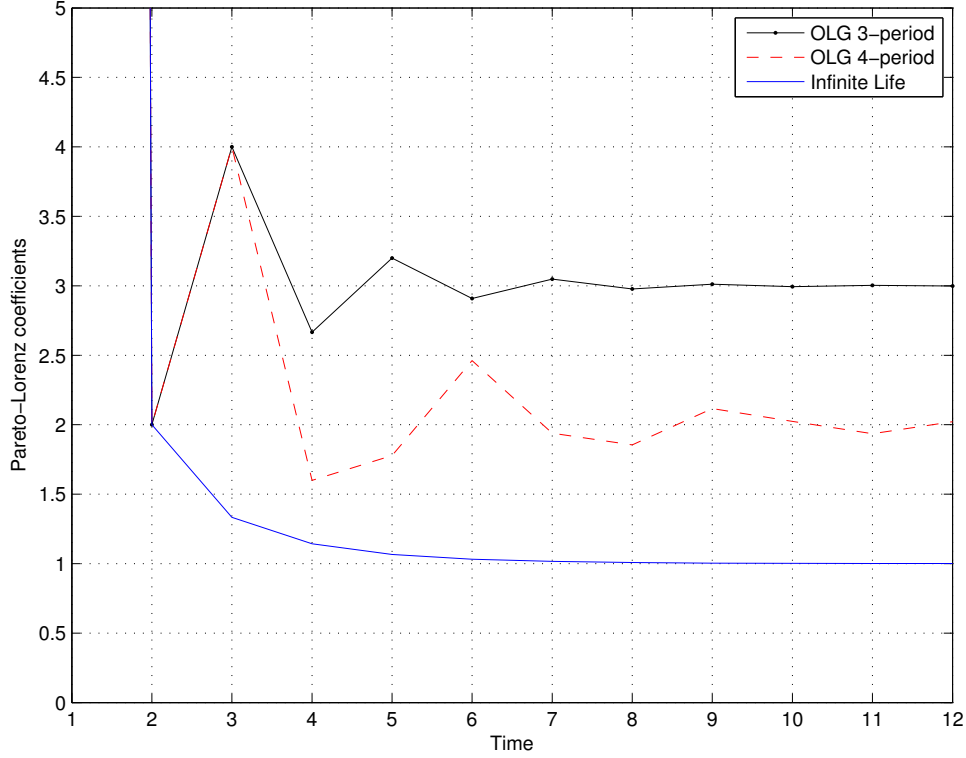
The Pareto coefficient in period $t+1$ is therefore the harmonic mean of the coefficient in period t and one. When the number of periods is large, it therefore converges to one (see Figure 11), while the Pareto coefficient for leverage on the support of beliefs diverges to $+\infty$ (see Figure 15). This result is actually intuitive from result 3: because speculation with leverage increases skewness when the distribution is initially "less skewed than Zipf", and increases skewness when it is "more skewed than Zipf", Zipf's law arises as the long term stationary distribution when speculation is repeated several times. This is interesting as Zipf's law is known to describe well the upper tail of the size of relatively long-lived entities, such as firm and city sizes. Unlike in random growth theory, these fat-tailed distributions here do not require Gibrat's law, reflecting barriers or other types of statistical frictions, which are absolutely necessary in random growth models so that the distribution does not become degenerate, or that its variance does not grow without bound (see Gabaix (1999) for example). The mapping from the model to these questions is however not entirely straightforward, and I leave it to future research.

3.2.2 OLG Economies

At the opposite extreme of the single generation economy, the case of the OLG economy with two period lives is one when a new generation arrives each period, invests in line with its beliefs, and then consumes. For each new cohort $t = 0, 1, \dots, T$, the initial Pareto coefficient for the distribution of wealth on the support of beliefs is given by $\alpha_t = +\infty$. In that case, it is straightforward to see that the distribution of leverage will always be a distribution with tail exponent two, and that this shall only translate into consumption inequality.

Consider now the case of an OLG economy with three period lives, which can be interpreted as a higher frequency of extreme dividend realizations. Denote a generation by its time of birth s , and α_t^s the Pareto exponent on the wealth distribution of generation s at time t . Since agents all

Figure 11: SIMULATED SEQUENCE OF PARETO-LORENZ COEFFICIENTS α_t FOR WEALTH AND TOP (CAPITAL) INCOMES



Note: Simulated Pareto-Lorenz coefficients for wealth and top (capital) incomes. Inequality is greater when the Pareto-Lorenz coefficient α_t is lower (and minimized for $\alpha_t = +\infty$). Note that top (capital) incomes can be interpreted as returns to entrepreneurship in general, or as proprietary traders' salaries, when they are in part paid through bonuses. An increase in the duration of agents' lives in the model can be interpreted as a higher preference of leaving bequests, as well as for greater "financial turbulence", in the sense of more frequent return realizations outside of the support of beliefs.

start in life with the same wealth, for all t , $\alpha_t^t = +\infty$. When middle-aged, from the compounding equation given above, the wealth distribution on the support of beliefs is given by:

$$\frac{1}{\alpha_{t+1}^t} = \frac{1}{\beta_t} + \frac{1}{\alpha_t^t}.$$

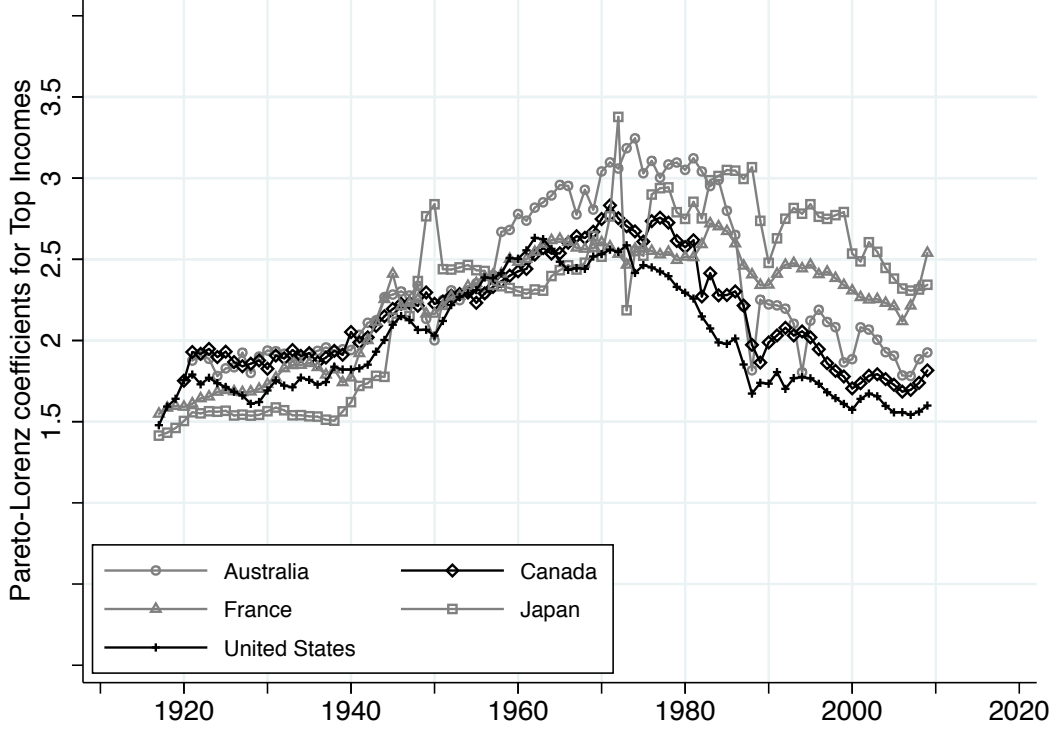
As a consequence, the sequence of Pareto coefficients for middle aged wealth at time t $\{(\alpha_t^{t-1})_{t \in \mathbb{N}^*}\}$ is solution to:

$$\frac{1}{\alpha_{t+1}^t} = \frac{1}{2} \left[1 - \frac{1}{\alpha_t^{t-1}} \right].$$

It is straightforward to show that this sequence of Pareto-Lorenz coefficients converges to three (see Figure 11). Before generalizing to the case of G period lives, it is useful to review a last special case, that of 4 period lives. At time $t+2$, two rounds of shocks matter for the distribution:

$$\frac{1}{\alpha_{t+2}^t} = \frac{1}{\beta_{t+1}} + \frac{1}{\alpha_{t+1}^t} = \frac{1}{\beta_{t+1}} + \frac{1}{\beta_t} + \frac{1}{\alpha_t^t}.$$

Figure 12: EMPIRICAL EVOLUTION OF PARETO-LORENZ COEFFICIENTS FOR TOP INCOMES IN A SAMPLE OF ADVANCED ECONOMIES (1920-2010). SOURCE: WORLD TOP INCOME DATABASE



Note: This plot shows the evolution of Pareto-Lorenz coefficients α , as taken from the World Top Income Database (Atkinson et al. (2009), see also Alvaredo et al. (2013)). The sample is restricted to the countries having the longest series on record.

At time $t+1$, the distribution was such that β_{t+1} was given by Pareto coefficients of generation $t+1$. The young had $\alpha_{t+1}^{t+1} = +\infty$, the second generation $\alpha_{t+1}^t = \beta_t$ since it experienced only the returns at time t , and the third generation:

$$\frac{1}{\alpha_{t+1}^{t-1}} = \frac{1}{\beta_t} + \frac{1}{\beta_{t-1}}.$$

A curcial point is that the old generation in that case has the more skewed wealth distribution on the support of beliefs, which determines the upper-tail of the distribution of returns in period $t+1$:

$$\frac{1}{\beta_{t+1}} = \frac{1}{2} \left[1 - \frac{1}{\alpha_{t+1}^{t-1}} \right].$$

Similarly the upper tail of the distribution of returns in period t is given by:

$$\frac{1}{\beta_t} = \frac{1}{2} \left[1 - \frac{1}{\alpha_t^{t-2}} \right].$$

The iteration equation for $\{(\alpha_{t+1}^{t-1})_{t \in \mathbb{N}^*}\}$ is therefore:

$$\frac{1}{\alpha_{t+2}^t} = \frac{1}{2} \left[1 - \frac{1}{\alpha_{t+1}^{t-1}} \right] + \frac{1}{2} \left[1 - \frac{1}{\alpha_t^{t-2}} \right].$$

The Pareto coefficient for wealth converges to two. More generally, the result can be generalized to the case with G period lives, in which case:

$$\frac{1}{\alpha_{t+G-2}^t} = \frac{1}{2} \sum_{i=1}^{G-2} \left[1 - \frac{1}{\alpha_{t+G-2-i}^{t-i}} \right].$$

In this case of G period lives, when the number of interim periods becomes large $T \rightarrow +\infty$, the coefficient on Pareto distributions for wealth becomes more skewed in the long run, and converges to $\frac{2}{G-2} + 1$. Empirically, Klass et al. (2006) provide evidence for Pareto-Lorenz coefficients of wealth distributions between 1.7 and 1.1 when estimated for the United States on the Forbes 400, which corresponds to G between 5 and 22. The model can therefore rationalize this increase in wealth inequality at the top from a longer horizon of agents (e.g. higher taste for bequests), or through a more "turbulent" economy, as a period was defined as the average duration between two extreme events. However, there are obviously other competing explanations for the recent rise in wealth inequality (for example, see Piketty (2011)). How much comes from this financial mechanism of entrepreneurial returns to investment is a very important question left for future research.

4 Conclusion

In this paper, I have developed a new model of investor disagreement with a continuum of types, which has helped me solve a number of theoretical questions that are relevant empirically. The cross-sectional distribution of leverage is a Pareto distribution of shape two when the distribution of wealth on beliefs' support is initially bounded, a prediction which has strong empirical support in the monthly TASS hedge fund database, across 1991 – 2010. Excess returns on bonds emerge, which do not only compensate lenders for risk, but which are necessary in equilibrium to allocate the most optimistic moderates to more optimistic borrowers, and allow them to lever more than more pessimistic ones.

A very simple dynamic model was developed in Section 3.2 to try and investigate the dynamics of the wealth distribution when capital incomes are given by power laws as in the static model. It was shown that speculation can increase skewness of the wealth distribution, but also that the mechanism developed in this paper puts some bound on inequality, and that the wealth distribution can never become more skewed than Zipf's law. The reason is that as optimistic/pessimistic hedge funds, or optimistic entrepreneurs become more wealthy, they also become too numerous to borrow from pessimists. In the limit when the wealth distribution follows Zipf's law, leverage becomes the same for all of them. Whether that is the reason why firm and city sizes, which are relatively long lived entities, follow Zipf law in the upper tail is an interesting avenue for future research. One limitation of the model is that the period $t+1$ returns were largely assumed to be

exogenous in the model, in the static as well as in the dynamic version where shocks on dividends were determining the returns. The dynamic implications need to be further investigated, and in particular the endogenous Power distribution of wealth on beliefs' support would generate an interesting amplification mechanism of learning dynamics.

Given the relative simplicity of the model, and the minimal assumptions required to derive the Pareto distribution for the leverage of hedge funds, this paper suggests that models with a continuum of beliefs together with collateral constraints could be useful to study the returns to entrepreneurship, and the top income distribution more generally. This is interesting because since Pareto (1897), economists have sought to explain why earnings distributions were always skewed to the right and had a fat tail: that is, the top percentile of earners always account for a disproportionate share of total earnings. Selection, sorting, or human capital effects all without doubt contribute to the skewness of the income distribution, but the only theories which can explain the Paretian nature of the upper tail of the earnings distribution are stochastic models following Gibrat (1931)'s law. The intuition for why such models generate skewed distributions is rather simple: wealthy agents, or large firms, are those who were repeatedly lucky, and had a successive number of very good draws of return on capital for wealth, or growth for firms. This paper gives a new intuition for why top income distributions are skewed to the right in a Paretian manner: some agents are particularly optimistic or pessimistic, and they are able to take large bets because their optimistic beliefs allow them to leverage with agents who ask for very low margins. The reason for why those borrowers, be they entrepreneurs or traders, are able to take large bets is that the most optimistic of the lenders would almost be willing to buy the assets themselves at the going price. When they turn out to be right about investment payoffs, they can make very large positive excess returns ex-post.

The model could therefore perhaps help understand better the recent increase in income and wealth inequality across advanced economies, coming from entrepreneurs as well as investment bankers' unequal access to credit. For Piketty and Saez (2003) the explanation is regulation like top income taxes as well as the change in attitudes towards inequality. Skill-biased technical change or globalization are other candidate explanations, but it is still hard to understand why inequality is coming from the very high end of the distribution (top 1%, or top 0.1% even, see Atkinson et al. (2009) for example), and why the pattern is much more pronounced in the Anglo-Saxon countries. In contrast, the model developed in this paper can potentially rationalize an increase in inequality following a decrease in margin requirements for both entrepreneurs and traders, which might proxy more generally for banking deregulation. In the case of entrepreneurs, deregulation then lets them borrow and invest more according to their optimistic beliefs. In the case of traders, deregulation can for example allow proprietary trading desks at investment banks to use the banks' regulatory capital for leveraging into speculative bets. Whether either, or both, of these entrepreneurial and investment-banking interpretations of my model can in fact explain the observed recent increase in top income and wealth inequality is a promising question for future work. But one can read Kaplan and Rauh (2009) and Bell and Van Reenen (2013) as suggestive evidence of the latter.

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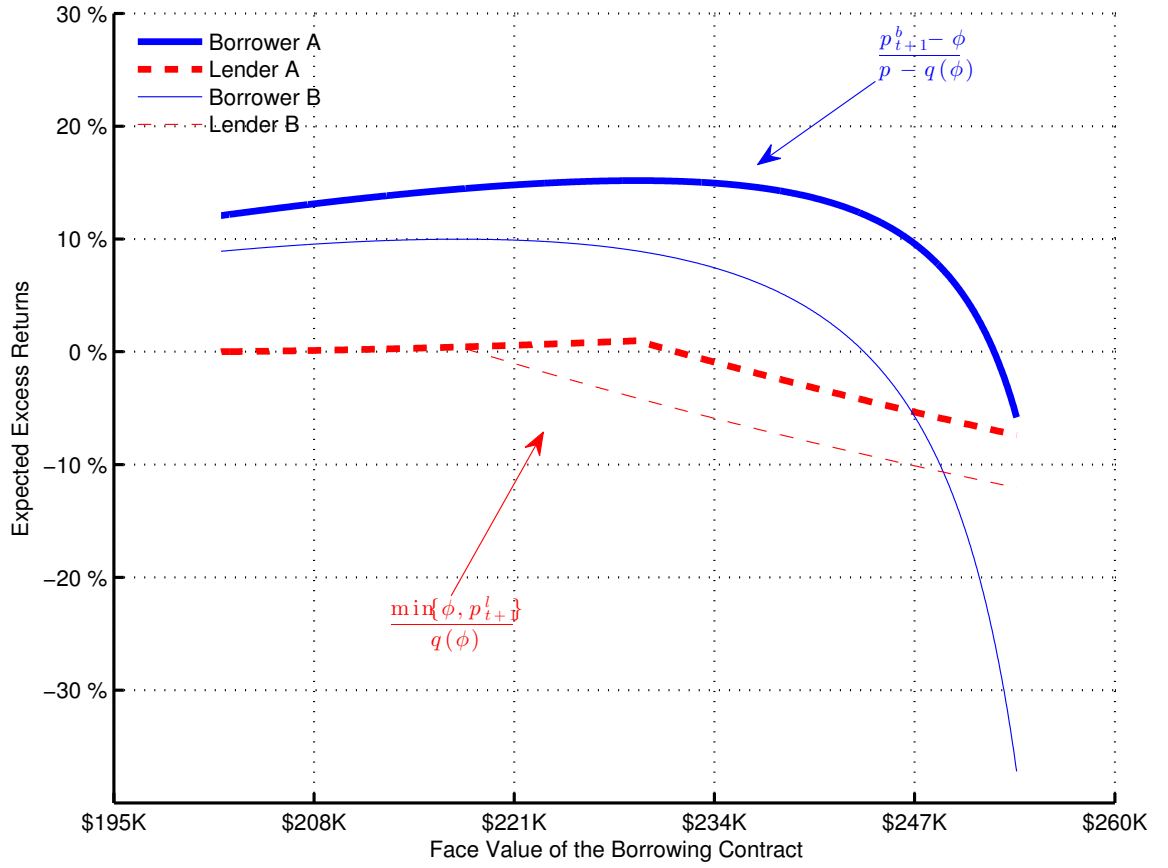
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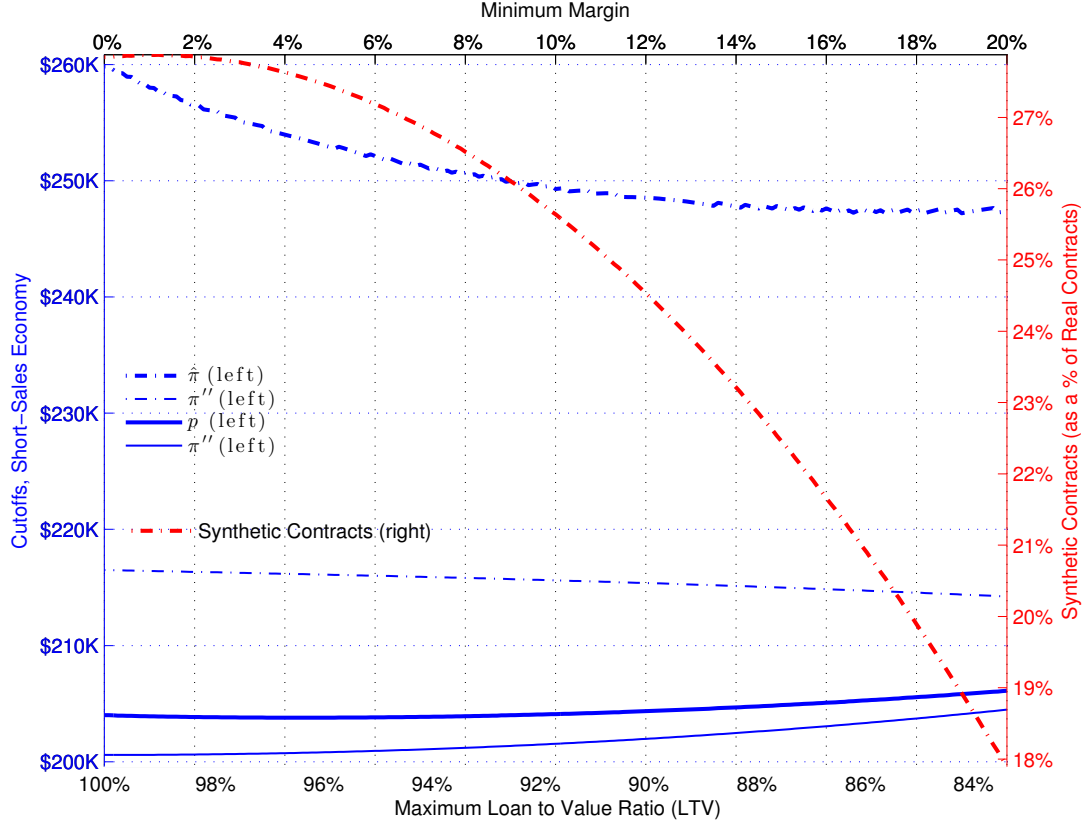
A Figures

Figure 13: BORROWERS' AND CORRESPONDING LENDERS' PROBLEMS IN BORROWING ECONOMY \mathcal{E}^B



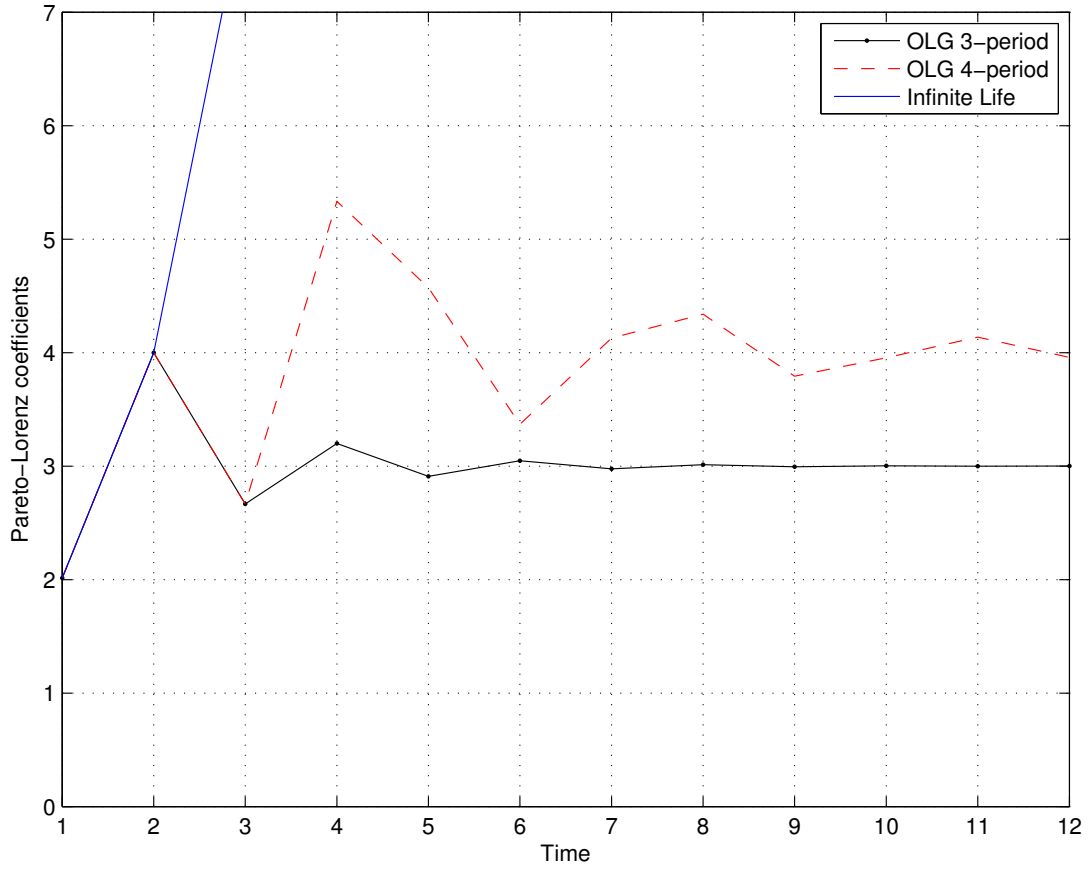
Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 23\%$, so $m = \$200K$. Borrower A and Lender A have beliefs such that $p_{t+1}^b = \Gamma(p_{t+1}^l)$ so that they choose the same contracts and effectively Lender A lends to Borrower A.

Figure 14: EFFECT OF MARGIN REQUIREMENTS IN ECONOMY \mathcal{E}_m^S : MECHANISM



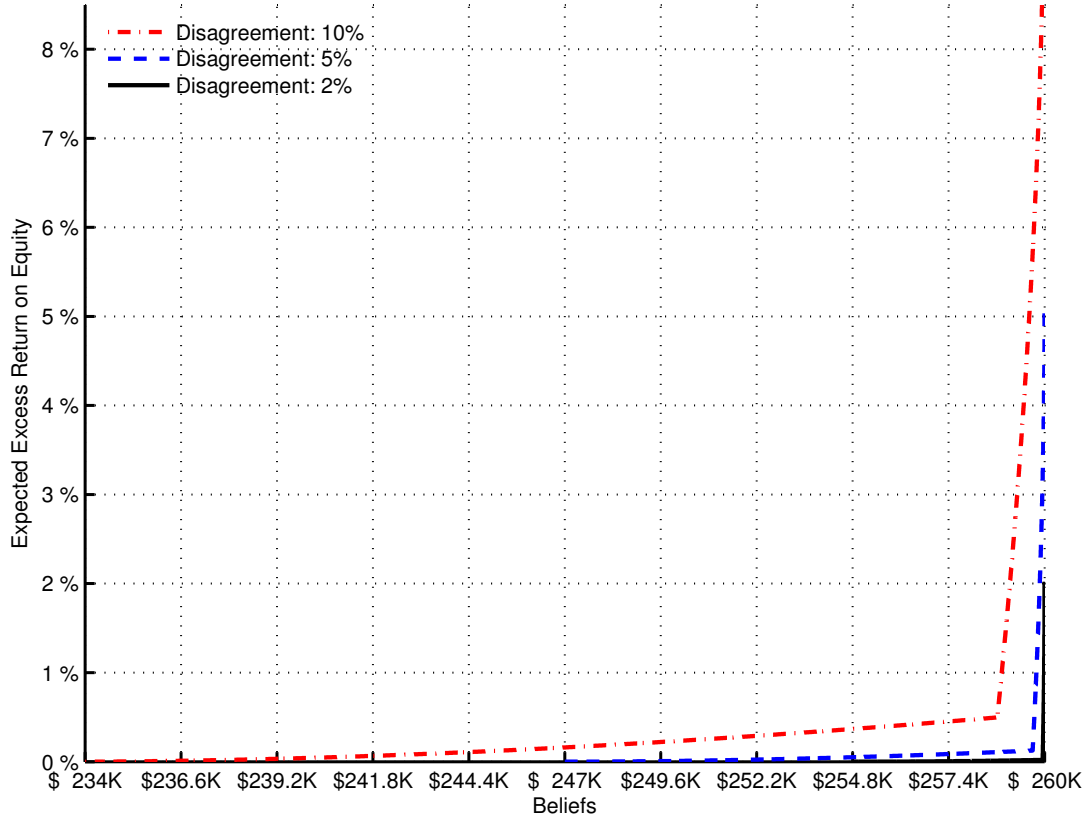
Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 23\%$. This graph gives an intuition for why margin requirements first decrease asset prices and then increase them when short-sales are allowed: for low values of \underline{m} , the elasticity of the number of synthetic contracts to margin requirements is very low (right scale). Speaking loosely, one can see on the graph that the cutoffs associated to short-selling, π' , p and π'' only just move, so that most of the effect of margin requirements occurs through restricting lending (curve associated to $\hat{\pi}$). However, when minimum margins reach $\underline{m} \approx 4\%$, the decrease of implicit synthetic contracts due to margin requirements preventing short-sellers and synthetic buyers from exploiting all the gains from trade overtakes this first effect which, as can be seen on this graph through $\hat{\pi}$, still is present but only fades away.

Figure 15: TAIL COEFFICIENTS β FOR LEVERAGE AND RETURNS



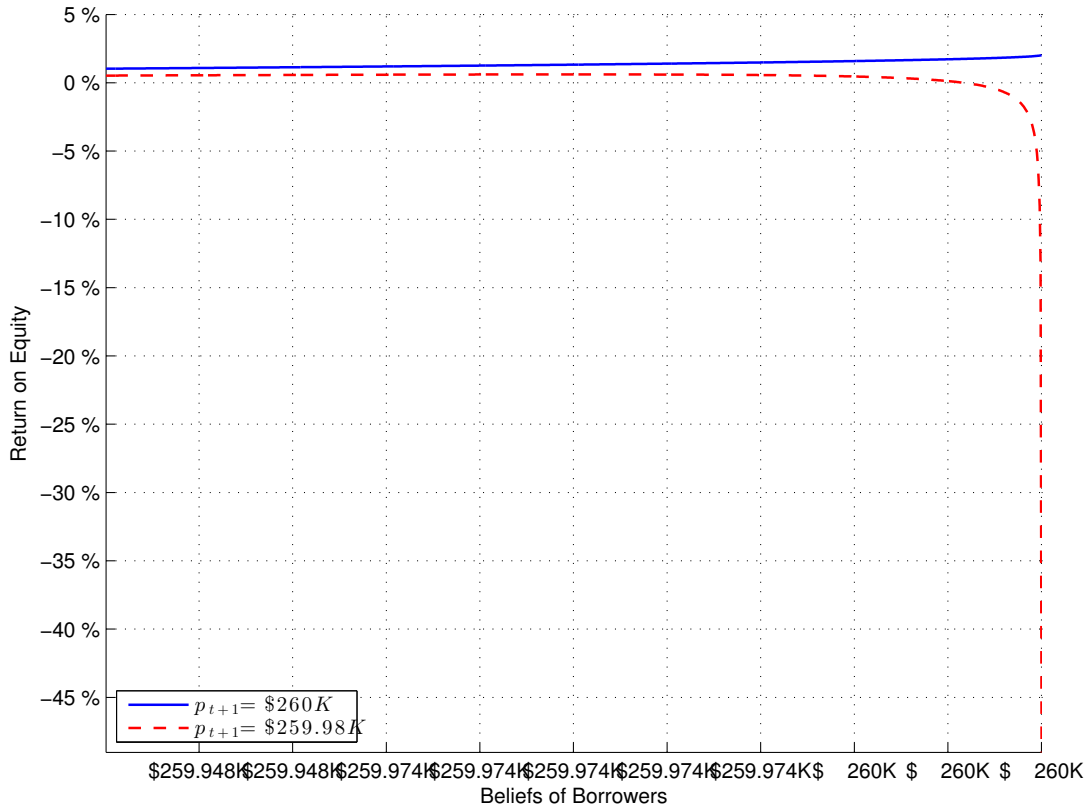
Note: Refer to Figure 11 for the Pareto-Lorenz coefficients on Wealth and Top Incomes. Note that both leverage and ex-post returns on wealth have lower skewness when the distribution of wealth on the support of beliefs is initially skewed. As explained in Section 1.4, this is because of the attracting property of Zipf Law. To the limit, when the distribution of wealth on the support of beliefs is Zipf, leverage is the same for all borrowers, and $\beta_t = +\infty$.

Figure 16: EXPECTED EXCESS RETURN ON EQUITY IN BORROWING ECONOMY \mathcal{E}^B



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1}[M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 10\%, 5\%, 2\%$ successively. This Figure represents the expected Return on Equity for cash investors, borrowers and lenders at the competitive equilibrium of Economy \mathcal{E}^B . Using the notations of Appendix B.1.1, for lenders $ROE^L(p_{t+1}^i, p_{t+1}^i) = r(p_{t+1}^i)$ since they choose $(\phi) = (p_{t+1}^i)$ at a competitive equilibrium. For borrowers $ROE^B(p_{t+1}^i, p_{t+1}^i) = \frac{p_{t+1}^i - \Gamma^{-1}(p_{t+1}^i)}{pr(p_{t+1}^i) - p_{t+1}^i} p_{t+1}^i$. Note that an agent with beliefs $p_{t+1}^i = \pi$ is indifferent between lending and borrowing from the pasting equation (3b): the curve is continuous at this point.

Figure 17: END OF PERIOD RETURNS FOR BORROWERS - POSITIVE EXCESS RETURNS IN THE RANGE OF OPTIMISTS' BELIEFS



Note: The density of beliefs is $f(x) = \frac{1}{Mh} \mathbb{1} [M(1-h), M](x)$, with $M = w/S = \$260K$, and $h = 2\%$. This Figure represents the shape of end of period returns for borrowers, when the price realization p_{t+1} is in the range of optimists' beliefs, that is $p_{t+1} \in [\pi, M]$. The example once again is The continuous line represents end of period returns when $p_{t+1} = M$, that is the most optimistic borrower was right about the future price of the asset. He then makes the highest return. The dotted line represents end of period returns when $p_{t+1} = \frac{\pi+M}{2}$. The returns are highest for the borrower who was right about the future price of the asset. Very levered investors with $p_{t+1}^i = M^-$ make high negative excess returns.

B Proofs

B.1 Proofs for section 1 - Borrowing Economy \mathcal{E}^B

B.1.1 Proofs for Proposition 1 - Section 1.2

As in the main text, $ROE^B(p_{t+1}^i, \phi)$ denotes the return on equity for a borrower with beliefs p_{t+1}^i selling a Borrowing Contract of type (ϕ) , which is a contract with face value ϕ and one unit of collateral. A first way of getting this return on equity is to use agents' problem. If only contract ϕ is used, the collateral constraint (CC) can be used to replace $n_B(\cdot)$ in the budget constraint (BC), and expressed the fact that each unit of real asset that the borrower owns is going to be used in equilibrium to back a Borrowing Contract:

$$-\int_{\phi} n_B^i(\phi) d\phi = n_A^i \quad \text{and} \quad n_C^i = 0 \quad \Rightarrow \quad n_A^i(p - q(\phi)) = w.$$

Replacing the equilibrium quantity of real asset, borrowing contracts, and cash used in equilibrium in the agent's maximization of wealth (W) gives:

$$\phi = \arg \max_{\phi} \frac{p_{t+1}^i - \phi}{p - q(\phi)} w.$$

A more intuitive reason for this result is provided in the main text, and involves the leverage ratio:

$$l(\phi) = \frac{q(\phi)}{p - q(\phi)}.$$

Then $ROE^B(p_{t+1}^i, \phi)$ is given as the mean of the return on the asset, with weight the size of the purchase for each unit of equity $1 + l(\phi)$, and the return given to lenders $r(\phi) = \phi/q(\phi)$, with weight the size of the borrowed funds $-l(\phi)$:

$$ROE^B(p_{t+1}^i, \phi) = \frac{p_{t+1}^i}{p} (1 + l(\phi)) - r(\phi)l(\phi) = \frac{p_{t+1}^i - \phi}{p - q(\phi)}. \quad (R_b)$$

Denote by $ROE^L(p_{t+1}^i, \phi)$ the return on equity for a lender with beliefs p_{t+1}^i . Then, with one unit of wealth he can purchase $1/q(\phi)$ units of contracts of type ϕ , each of which provide an expected payoff to him equal to $\min\{p_{t+1}^i, \phi\}$. Therefore:

$$ROE^L(p_{t+1}^i, \phi) = \frac{1}{q(\phi)} \min\{p_{t+1}^i, \phi\}. \quad (R_l)$$

Finally, denote by $ROE^C(p_{t+1}^i)$ the return on equity for a Cash investor. By assumption the returns to Cash are normalized to one so its expected return does not depend on the expected return on the Real Asset:

$$ROE^C(p_{t+1}^i) = 1.$$

Given his beliefs p_{t+1}^i , an agent i maximizes his end of period wealth or equivalently, his return to equity. He chooses to be a lender, a borrower or a Cash investor and conditional on being a

lender/borrower, the type ϕ of contract he buys/sells.

Partition of Agents. Note that the sets of beliefs corresponding to borrowers, lenders and cash investors are necessarily convex because of the form of each agents' optimization program. One comes easily to a contradiction by assuming initially that there exist (x, y) and $z \in]x, y[$ with (x, y) being invested in the Real Asset and z buying financial contract, for example, or through a revealed preference argument. Moreover, note that because of the assumed bound on agents' beliefs $M \leq w/S$ (itself coming from market clearing in $t + 1$) the three sets of agents are necessarily present in equilibrium, from the market clearing equation (3a) which is proved in the main text $w(1 - F(\pi'')) = Sp$. Note first that some agents need to be buying the Real Asset in equilibrium, for the market for the Real Asset to clear.

Assume that there are no lenders in this economy. Then we would have $w(1 - F(p)) = Sp$ from market clearing. If there was no cash investors, we would have $p = m$ so $p = w/S$ and hence $M = w/S = m$, which contradicts the heterogenous beliefs assumption. This configuration with cash investors and investors in the Real Asset would not be an equilibrium, because asset investors with $p_{t+1}^i > p$ would have an incentive to sell borrowing contracts to cash investors, giving them the return of cash, to make a higher return. Hence, there are lenders in this economy.

Assume now that there are only lenders and borrowers in this economy. Then from market clearing and $\pi'' = m$ agents' funds must all be invested in the Real Asset, and so $w(1 - F(m)) = w = Sp$. Therefore $p = w/S$. Once again, this leads to $M = w/S$, and there would be no Real Asset investors in this economy, who are in the segment $[p, M]$ reduced to a set of measure zero, a contradiction as lenders would have no one to buy their Borrowing Contracts from.

Finally, at the end of this proof, we shall verify that given prices, every agent is indeed choosing his type optimally. (see subsection "Final check" below)

Solution to Lender's problem. A first remark is that the price of the loan $q(\phi)$ of a contract of type ϕ is necessarily increasing in the face value of the loan ϕ , if such contract is traded. The reason is that no optimist would ever sell a loan with a higher face value and a lower price, which would give him both less money and a higher repayment - this can be seen on the formula above for the return of a borrower (R_b). A second remark is that the implied interest rate on the loan $r(\phi) = \phi/q(\phi)$ is also increasing in the face value of the loan if the contract is traded, which is coming once again from the borrower's problem:

$$ROE^B(p_{t+1}^i, \phi) = \frac{p_{t+1}^i - \phi}{p - \frac{\phi}{r(\phi)}} = \frac{p_{t+1}^i - p}{p - \frac{\phi}{r(\phi)}} + 1 + \frac{\frac{1}{r(\phi)} - 1}{p - \frac{\phi}{r(\phi)}} \phi.$$

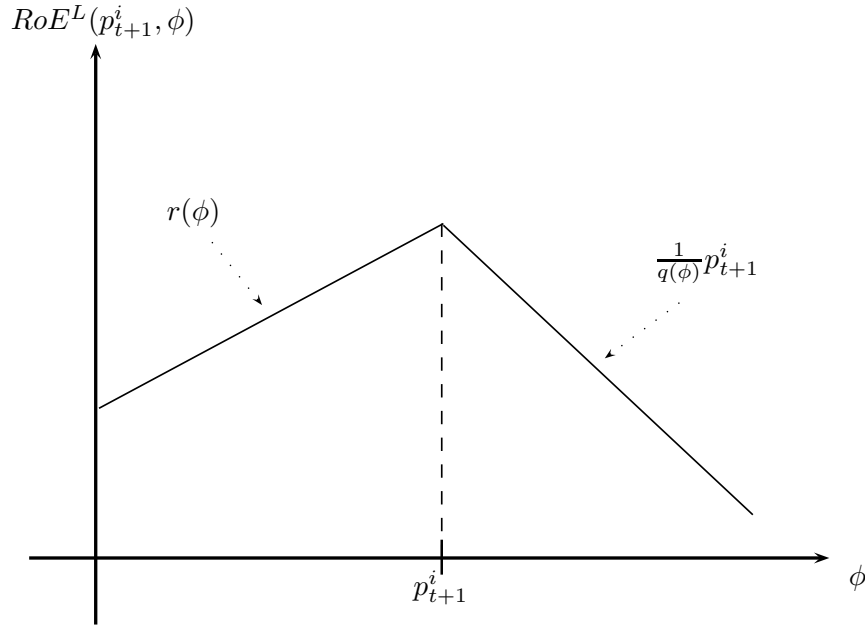
A borrower would never sell a contract with a lower face value ϕ and a higher implied return $r(\phi)$, since his return on equity would then unambiguously be lower, as from the above expression $ROE^B(p_{t+1}^i, \cdot)$ is decreasing in $r(\phi)$ and increasing in ϕ . This is actually intuitive, as a lower face value ϕ means a lower leverage, and if $r(\phi)$ was decreasing also a higher implicit cost of borrowing. Hence $r'(\phi) > 0$. Since this is the case, from the expression for the return of a lender

we have:

$$\begin{aligned} \frac{\partial ROE^L(p_{t+1}^i, \phi)}{\partial \phi} &= r'(\phi) > 0 \quad \text{for } \phi \leq p_{t+1}^i \\ \frac{\partial ROE^L(p_{t+1}^i, \phi)}{\partial \phi} &= p_{t+1}^i \frac{-q'(\phi)}{q(\phi)^2} < 0 \quad \text{for } \phi \geq p_{t+1}^i \end{aligned}$$

Therefore, a lender with beliefs p_{t+1}^i necessarily buys a contract with a face value corresponding to his expectation of future period's price $\phi = p_{t+1}^i$. This is in fact very intuitive (see Figure 18). If the face value of the loan is lower than his beliefs then the lender always chooses the contract with the highest face value, because loans with lower margins come with higher interest rates in equilibrium ($r(\phi)$ is increasing). If the face value of the loan is higher than his beliefs about the future price of the asset, then the lender expects default for sure and so any contract with $\phi \geq p_{t+1}^i$ gives one unit of collateral. In that case there is no point in choosing a contract with a higher face value, which is more expensive ($q(\phi)$ is increasing), but gives the same wealth in period $t + 1$.

Figure 18: CHOICE OF A BORROWING CONTRACT (ϕ) FOR LENDERS WITH BELIEFS p_{t+1}^i



Note that a lender will not invest in the asset nor in the storage technology so that $n_A^i = 0$ and $n_C^i = 0$. From the Budget Constraint (BC) in agents' problem, and the fact that the agent will choose a contract with face value $\phi = p_{t+1}^i$, this results in:

$$\int_{\phi} n_B^i(\phi) q(\phi) d\phi = w \quad \Rightarrow \quad \forall \phi, \quad n_B^i(\phi) = \frac{w}{q(p_{t+1}^i)} \delta_{p_{t+1}^i}(\phi).$$

A lender with beliefs $p_{t+1}^i \in [\pi'', \pi]$ therefore chooses portfolios given by:

$$n_A^i = 0, \quad n_B^i(\cdot) = \frac{w}{q(p_{t+1}^i)} \delta_{p_{t+1}^i}(\cdot), \quad n_C^i = 0.$$

Solution to Borrower's problem. Conditional on being a borrower, an agent with beliefs p_{t+1}^i chooses a Borrowing Contract (ϕ) to maximize:

$$ROE^B(p_{t+1}^i, \phi) = \frac{p_{t+1}^i - \phi}{p - q(\phi)} = \frac{p_{t+1}^i - \phi}{pr(\phi) - \phi} r(\phi).$$

The first order condition for this maximization program is that:

$$\frac{\partial ROE^B(p_{t+1}^i, \phi)}{\partial \phi} = \frac{(-r(\phi) + (p_{t+1}^i - \phi)r'(\phi))(pr(\phi) - \phi) - (pr'(\phi) - 1)(p_{t+1}^i - \phi)r(\phi)}{(pr(\phi) - \phi)^2}.$$

The optimality condition for a borrower with beliefs p_{t+1}^i simplifies into:

$$\phi(p_{t+1}^i - \phi)r'(\phi) - p_{t+1}^i r(\phi) + pr(\phi)^2 = 0.$$

In equilibrium, an optimist with beliefs $p_{t+1}^i = y$ must be incentivized to choose a contract ϕ such that he is matched to moderate x with $y = \Gamma(x)$, therefore a borrower $\Gamma(x)$ must be incentivized to choose contract $(\phi) = (x)$ that lender x buys. This gives equation (A):

$$\forall x \in [\pi'', \pi], \quad x(\Gamma(x) - x)r'(x) - \Gamma(x)r(x) + pr(x)^2 = 0$$

From the previous derivations, the portfolio of a borrower with beliefs $p_{t+1}^i \in [\pi, M]$ is then given by:

$$n_A^i = \frac{w}{p - q(\phi')}, \quad n_B^i(\cdot) = -\frac{w}{p - q(\phi')} \delta_{\phi'}(\cdot), \quad n_C^i = 0,$$

with ϕ' given by: $\phi' = \arg \max_{\phi} \frac{p_{t+1}^i - \phi}{p - q(\phi)}.$

Final check. Finally, it is important to check whether given the above defined prices, borrowers would not rather be lenders or cash-investors given their beliefs, and the same for lenders and cash-investors.

Cash investors with beliefs $p_{t+1}^i < \pi'' < p$ do not want to buy the asset which gives them a lower expected return than cash, let alone do leveraged investing into this asset. They do not want to lend either since if they did they would choose the lowest leverage ratio loans from the above reasoning (all of them give them a unit of collateral in $t + 1$ in expectation, and the lowest leverage loans are the cheapest), and such lowest leverage ratio loans have face value π'' with $q(\pi'') = \pi''$ so they give $ROE^L(p_{t+1}^i, \pi'') = p_{t+1}^i / \pi'' < 1$ to cash investors.

Lenders with beliefs $p_{t+1}^i \in [\pi'', \pi]$ do not want to invest in cash because they expect to make $r(p_{t+1}^i) > 1$ from lending. The fact that they do not want to borrow results from the fact that borrowing with the optimum face value for them, that is $\phi = \pi''$ from the above remarks, would give them a return $\frac{p_{t+1}^i - \pi''}{p - \pi''}$, which is linear in p_{t+1}^i . It therefore suffices to check that the return

from borrowing and investing with these loans with face values equal to π'' always brings a lower return than lending. Therefore, to show that $r(p_{t+1}^i)$ is always higher for a lender than the return he would get from borrowing $\frac{p_{t+1}^i - \pi''}{p - \pi''}$, it remains to prove that:

$$r'(\pi) < \frac{1}{p - \pi''}.$$

This inequality obtains from noting that:

$$r(\pi) = \frac{\pi - \pi''}{p - \pi''} > \frac{\pi}{p} \Rightarrow M - pr(\pi) < M - \pi.$$

Using the assignment equation (A), together with the last inequality:

$$r'(\pi) = \frac{Mr(\pi) - pr(\pi)^2}{\pi(M - \pi)} = \frac{r(\pi)}{\pi} \frac{M - pr(\pi)}{M - \pi} < \frac{r(\pi)}{\pi} = \frac{\pi - \pi''}{\pi} \frac{1}{p - \pi''} < \frac{1}{p - \pi''}.$$

Finally, it is much easier to see why borrowers with beliefs $p_{t+1}^i \in [\pi, M]$ would never invest in cash nor lend: their expected return is higher than both that of cash investors and that of lenders, from the indifference equation $r(\pi) = \frac{\pi - \pi''}{p - \pi''}$ together with the envelope theorem on borrowers' optimization program with respect to p_{t+1}^i .

Second-order differential equation for $\Gamma(\cdot)$. From the market clearing equation for Borrowing Contracts (C):

$$r(x) = \frac{x}{p} \left[1 + \frac{f(\Gamma(x))}{f(x)} \Gamma'(x) \right].$$

Replacing in the assignment equation brings:

$$\begin{aligned} & x(\Gamma(x) - x)r'(x) - \Gamma(x)r(x) + pr(x)^2 = 0 \\ \Rightarrow & x(\Gamma(x) - x) \left[1 + \frac{f(\Gamma(x))}{f(x)} \Gamma'(x) \right] + x \left(\frac{f(\Gamma(x))}{f(x)} \Gamma'(x) \right)' x(\Gamma(x) - x) - \dots \\ & x\Gamma(x) - x \frac{f(\Gamma(x))}{f(x)} \Gamma(x)\Gamma'(x) + x^2 \left[1 + \frac{f(\Gamma(x))}{f(x)} \Gamma'(x) \right]^2 = 0 \end{aligned}$$

Expanding all the terms in brackets, and after some algebra:

$$\left(\frac{f(\Gamma(x))}{f(x)} \Gamma'(x) \right)' (\Gamma(x) - x) + \frac{f(\Gamma(x))}{f(x)} \Gamma'(x) + \left(\frac{f(\Gamma(x))}{f(x)} \right)^2 \Gamma'(x)^2 = 0.$$

B.1.2 Corollary of Equation (A) - Section 1.5

A corollary of the assignment equation (A) is that the derivative of the ex-post realized cross-sectional distribution of returns with respect to borrowers' beliefs takes a very simple form as a function of the realized price for the Real Asset p_{t+1} (denoting $x = \Gamma^{-1}(y)$ for conciseness of notation):

$$\begin{aligned} \frac{dROE^B(p_{t+1}, \Gamma^{-1}(y))}{dy} &= \frac{1}{\Gamma'(\Gamma^{-1}(y))} \frac{dROE^B(p_{t+1}, x)}{dx} \\ &= \frac{1}{\Gamma'(x)} \frac{(-r(x) + (p_{t+1} - x)r'(x))(pr(x) - x) - (pr'(x) - 1)(p_{t+1} - x)r(x)}{(pr(x) - x)^2} \\ \frac{dROE^B(p_{t+1}, \Gamma^{-1}(y))}{dy} &= \frac{1}{\Gamma'(x)} \frac{(p_{t+1} - y)(r(x) - xr'(x))}{(pr(x) - x)^2}. \end{aligned}$$

Because of prices of contracts $q(\cdot)$ are increasing in lenders' types (see Appendix B.1.1 for a proof), we have:

$$\left(\frac{x}{r(x)}\right)' = \frac{r(x) - r'(x)x}{r(x)^2} > 0.$$

Therefore $\frac{dROE^B(p_{t+1}, \Gamma^{-1}(y))}{dy}$ has the sign of $p_{t+1} - y$, which shows a non-monotone relationship between leverage and returns (see Section 1.5).

B.1.3 Proof of Lemma 1: Taylor Expansions for Cutoffs

The first Taylor expansion follows from equation (3a) which can be rewritten as:

$$1 - \frac{p_h}{M} = F\left(\frac{\pi_h'' - M}{h} + M\right).$$

Therefore, when disagreement goes to 0:

$$\frac{\pi_h'' - M}{h} + M \xrightarrow{h \rightarrow 0} m \quad \Rightarrow \quad \pi_h'' = M - (M - m)h + o(h).$$

The two following Taylor expansions follow from the market clearing equation for financial contracts (C) which states that images $\pi = \Gamma(\pi'')$ and $M = \Gamma(\pi)$ of π'' and π are such that:

$$F_h(M) - F_h(\pi_h) = \int_{\pi_h''}^{\pi_h} \frac{p_h r_h(x) - x}{x} f_h(x) dx.$$

Note for a given h , margins $\frac{p_h r_h(x) - x}{x}$ are decreasing in x because they are inversely proportional to $q_h(x)$, known to be increasing (see the proof of Proposition 1) and therefore:

$$\exists A > 0, \quad 0 \leq M - \pi_h \leq A(\pi_h - \pi_h'') \frac{p_h - \pi_h''}{\pi_h''}.$$

From there it follows that:

$$\exists B \in \mathbb{R}, \quad \pi_h = M + Bh^2 + O(h^3).$$

Therefore, π_h differs from M by an h^2 term. From equation (3b), one can write:

$$r_h(\pi_h) = \frac{\pi_h - \pi_h''}{p_h - \pi_h''}.$$

When $h \rightarrow 0$, $r_h(\cdot)$ converges uniformly to 1, because $1 \leq r_h(\cdot) \leq r_h(\pi_h) \rightarrow 1$. Therefore, the previous expression yields:

$$\pi_h - \pi_h'' = p_h - \pi_h'' + O(h^2) \quad \Rightarrow \quad \exists C \in \mathbb{R}, \quad p_h = M + Ch^2 + O(h^3).$$

To conclude the proof, it suffices to show that $B = C$. This comes from the assignment equation (A) evaluated at $x = \pi''$, which gives:

$$r_h'(\pi_h'') = \frac{\pi_h - p_h}{\pi_h''(\pi_h - \pi_h'')}.$$

Because of the previously derived Taylor expansions, this shows that:

$$r'_h(\pi_h'') = 1 + O(h) \quad \Rightarrow \quad r_h(\pi_h) = 1 + O(h^2).$$

From this and equation (3b) once again, one concludes that $B = C$, and therefore:

$$\pi_h - p_h = O(h^3).$$

From the previously derived Taylor expansions it follows immediately that:

$$L_h(M) = \frac{\pi_h p_h - \pi_h''}{\pi_h'' \pi_h - p_h} = O\left(\frac{1}{h^2}\right) \quad L_h(\pi_h) = \frac{\pi_h''}{p_h - \pi_h''} = O\left(\frac{1}{h}\right).$$

B.1.4 Proof of Proposition 2 - Section 1.4 - Limiting Pareto Distribution for Leverage

Leverage expressed in borrowers' space $L_h(\cdot)$ is a strictly increasing function of optimists' beliefs, and as such can be inverted. The inverse is going to be denoted by function $L_h^{-1}(\cdot)$:

$$l = L_h(y) = \frac{\Gamma_h^{-1}(y)}{p_h r_h(\Gamma_h^{-1}(y)) - \Gamma_h^{-1}(y)} \quad \Rightarrow \quad y = L_h^{-1}(l)$$

The countercumulative distribution function for leverage can then be expressed using $L_h^{-1}(\cdot)$:

$$\begin{aligned} 1 - G_h(l) &= \mathbb{P}(L_h(y) \geq l) = \mathbb{P}(y \geq L_h^{-1}(l) | y \geq \pi_h) \\ &= \frac{\mathbb{P}(y \geq L_h^{-1}(l) \cap y \geq \pi_h)}{\mathbb{P}(y \geq \pi_h)} \\ 1 - G_h(l) &= \mathbb{P}(L_h(y) \geq l) = \frac{1 - (F_h \circ L_h^{-1})(l)}{1 - F_h(\pi_h)}. \end{aligned}$$

Using the expression for $\Gamma_h(\cdot)$ in market clearing for financial contracts equation (C), the inverse leverage ratio function $y = L_h^{-1}(l)$ and the fact that $\Gamma_h(\pi_h) = M$ successively gives:

$$\begin{aligned} F_h(y) &= (F_h \circ \Gamma_h)(\pi_h'') + \int_{\pi_h''}^{\Gamma_h^{-1}(y)} \frac{p_h r_h(u) - u}{u} f_h(u) du \\ (F_h \circ L_h^{-1})(l) &= (F_h \circ \Gamma_h)(\pi_h'') + \int_{\pi_h''}^{\Gamma_h^{-1}(L_h^{-1}(l))} \frac{p_h r_h(u) - u}{u} f_h(u) du \\ (F_h \circ L_h^{-1})(l) &= 1 - \int_{\Gamma_h^{-1}(L_h^{-1}(l))}^{\pi_h} \frac{p_h r_h(u) - u}{u} f_h(u) du. \end{aligned}$$

Therefore, the countercumulative distribution function for leverage ratios is given by:

$$1 - G_h(l) = \frac{1}{1 - F_h(\pi_h)} \int_{\Gamma_h^{-1}(L_h^{-1}(l))}^{\pi_h} \frac{p_h r_h(u) - u}{u} f_h(u) du.$$

A second order Taylor approximation yields that for all $h \in [0, 1]$ and for all $l \in [L_h(\pi_h), L_h(M)]$, there exists $d_{hl} \in [\Gamma_h^{-1}(L_h^{-1}(l)), \pi_h]$ such that:

$$\begin{aligned} \int_{\Gamma_h^{-1}(L_h^{-1}(l))}^{\pi_h} \frac{p_h r_h(u) - u}{u} f_h(u) du &= [\pi_h - \Gamma_h^{-1}(L_h^{-1}(l))] \frac{p_h r_h(\Gamma_h^{-1}(L_h^{-1}(l))) - \Gamma_h^{-1}(L_h^{-1}(l))}{\Gamma_h^{-1}(L_h^{-1}(l))} f_h(\Gamma_h^{-1}(L_h^{-1}(l))) + \dots \\ &\quad \dots [\pi_h - \Gamma_h^{-1}(L_h^{-1}(l))]^2 \left(\frac{p_h r_h(u) - u}{u} f_h(u) \right)' (d_{hl}). \end{aligned}$$

Let us take this expression by pieces. We have, by definition of the leverage factor:

$$p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l))) - \Gamma_h^{-1}(L_h^{-1}(l)) = \frac{\Gamma_h^{-1}(L_h^{-1}(l))}{l}.$$

Therefore:

$$p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l))) - \Gamma_h^{-1}(L_h^{-1}(l)) = \frac{p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l)))}{l+1}.$$

Moreover:

$$\begin{aligned} \pi_h - \Gamma_h^{-1}(L_h^{-1}(l)) &= \pi_h - p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l))) + p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l))) - \Gamma_h^{-1}(L_h^{-1}(l)) \\ \pi_h - \Gamma_h^{-1}(L_h^{-1}(l)) &= \frac{p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l)))}{l+1} - p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l))) + \pi_h. \end{aligned}$$

Therefore:

$$\begin{aligned} &\frac{l^2}{1 - F_h(\pi_h)} [\pi_h - \Gamma_h^{-1}(L_h^{-1}(l))] \frac{p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l))) - \Gamma_h^{-1}(L_h^{-1}(l))}{\Gamma_h^{-1}(L_h^{-1}(l))} f_h(\Gamma_h^{-1}(L_h^{-1}(l))) \\ &= \frac{1}{1 - F_h(\pi_h)} [\Gamma_h^{-1}(L_h^{-1}(l)) + l (\pi_h - p_h r_h (\Gamma_h^{-1}(L_h^{-1}(l))))] f_h(\Gamma_h^{-1}(L_h^{-1}(l))) \end{aligned}$$

Finally, the last part is given by:

$$\left(\frac{p_h r_h(u) - u}{u} f_h(u) \right)' (d_{hl}) = \frac{p_h r_h(d_{hl}) - d_{hl}}{d_{hl}} f_h'(d_{hl}) + \frac{p_h r_h'(d_{hl})}{d_{hl}} f_h(d_{hl}) - \frac{p_h}{d_{hl}^2} r_h(d_{hl}) f_h(d_{hl}).$$

Let us look at the upper tail of the distribution, that is assume that $l \in [L_h(M)/2, L_h(M)]$. From Lemma 1, we have that:

$$\frac{1}{2} O\left(\frac{1}{h^2}\right) = \frac{L_h(M)}{2} \leq l \leq L_h(M) = O\left(\frac{1}{h^2}\right) \Rightarrow l = O\left(\frac{1}{h^2}\right).$$

Using the previous derivations:

$$\exists A_h, \quad \|l^2(1 - G_h(l)) - A_h\|_{\infty}^{[L_h(M)/2, L_h(M)]} \xrightarrow{h \rightarrow 0} 0,$$

B.1.5 Proof of Corollary 1 - Equilibrium with Flat Priors in Economy \mathcal{E}^B

In the case of flat priors, one can calculate in closed form the assignment function, return function, return on equity function, leverage distribution function. The main reason is that the second-order differential equation $\Gamma''(\Gamma - x) + \Gamma'^2 + \Gamma' = 0$ is shown to have a very simple closed form expression.

Assignment Function. Using equation (C) and the fact that beliefs are flat with $f(x) = 1/H$, one can write:

$$r(x) = \frac{x}{p} (1 + \Gamma'(x)).$$

Replacing this expression of $r(\cdot)$ in equation (A) gives:

$$\begin{aligned} x(\Gamma(x) - x) [\Gamma'(x) + x\Gamma''(x) + 1] - x\Gamma(x)\Gamma'(x) - x\Gamma(x) + x^2 (\Gamma'(x) + 1)^2 &= 0 \\ \Rightarrow x^2 \Gamma''(x) (\Gamma(x) - x) + x^2 \Gamma'(x) + x^2 \Gamma'(x)^2 &= 0. \end{aligned}$$

A necessary condition is therefore, using that $x \neq 0$ (no lender thinks the asset will be worth nothing, something we guess here and will verify later):

$$\Gamma''(x) (\Gamma(x) - x) + \Gamma'(x) + \Gamma'(x)^2 = 0.$$

We are now going to prove that this second-order non-linear differential equation admits a closed-form solution. Using that for all x , $\Gamma'(x) \neq 0$ allows to write that this implies:

$$\frac{(\Gamma'(x)^2 - 1) (\Gamma''(x) (\Gamma(x) - x) + \Gamma'(x) + \Gamma'(x)^2)}{\Gamma'(x)^2} = 0.$$

Using that $\Gamma'(x)^2 - 1 = (\Gamma'(x) - 1) (\Gamma'(x) + 1) = (2\Gamma'(x) - (1 + \Gamma'(x))) (\Gamma'(x) + 1)$:

$$\frac{(\Gamma'(x)^2 - 1) (\Gamma'(x) + 1) \Gamma'(x) + 2\Gamma''(x) (1 + \Gamma'(x)) \Gamma'(x) (\Gamma(x) - x) - \Gamma''(x) (\Gamma(x) - x) (1 + \Gamma'(x))^2}{\Gamma'(x)^2} = 0.$$

Recognizing the derivative of a fraction:

$$\left(\frac{(\Gamma(x) - x) (1 + \Gamma'(x))^2}{\Gamma'(x)} \right)' = 0.$$

Using the initial condition on $\Gamma(\cdot)$, $\Gamma(\pi'') = \pi$, and that on $r(\cdot)$ which is equivalent to one on $\Gamma'(\cdot)$ from (C):

$$r(\pi'') = 1 \quad \Rightarrow \quad \Gamma'(\pi'') = \frac{p}{\pi''} r(\pi'') - 1 = \frac{p - \pi''}{\pi''},$$

One can integrate the previous differential equation as:

$$\frac{(\Gamma(x) - x) (1 + \Gamma'(x))^2}{\Gamma'(x)} = \frac{\pi - \pi''}{p - \pi''} \frac{p^2}{\pi''} \equiv \hat{\pi}.$$

Note that through this step the degree of the differential equation has been reduced by one level of differentiation. Therefore:

$$\frac{1}{4} \left(-2 - 2\Gamma'(x) + \frac{4\hat{\pi} - 4\hat{\pi}\Gamma'(x)}{2\sqrt{\hat{\pi}^2 + 4\hat{\pi}x - 4\hat{\pi}\Gamma(x)}} \right) = 0 \quad \Rightarrow \quad \left(-2x - 2\Gamma(x) + \sqrt{\hat{\pi}^2 + 4\hat{\pi}x - 4\hat{\pi}\Gamma(x)} \right)' = 0.$$

Using the initial condition once again $\Gamma(\pi'') = \pi$ and integrating gives:

$$-2x - 2\Gamma(x) + \sqrt{\hat{\pi}^2 + 4\hat{\pi}x - 4\hat{\pi}\Gamma(x)} = -2\pi'' - 2\pi + \sqrt{\hat{\pi}^2 + 4\hat{\pi}\pi'' - 4\hat{\pi}\pi} \equiv \tilde{\pi}.$$

Finally, solving for $\Gamma(\cdot)$ gives:

$$\Gamma(x) = \frac{1}{2} \left(-\hat{\pi} - 2x - \tilde{\pi} + \sqrt{2} \sqrt{\hat{\pi} (4x + \tilde{\pi} + \hat{\pi})} \right).$$

Return function. The return obtained by lenders on their loan contracts is immediately obtained from equation (C):

$$r(x) = \frac{x}{p} (1 + \Gamma'(x)) = \frac{x}{p} \sqrt{\frac{2\hat{\pi}}{4x + \hat{\pi} + \tilde{\pi}}}.$$

Cutoffs. In order to calculate the cutoffs, one must then make use of equations (3a), (3b) and (3c). From market clearing equation (3a) together with $M = w/S$, and normalizing all prices by $M = 1$ (it is straightforward to show that everything is homogeneous), one arrives at:

$$\pi'' = 1 - hp.$$

From equation (3a), and using the market clearing equation for financial contracts (C) at $x = \pi$:

$$\begin{aligned} r(\pi) &= \frac{\pi - \pi''}{p - \pi''} \quad \text{and} \quad 1 + \Gamma'(\pi) = \frac{p}{\pi} r(\pi) \\ \Rightarrow \quad \Gamma'(\pi) &= \frac{p}{\pi} \frac{\pi - \pi''}{p - \pi''} - 1. \end{aligned}$$

Finally equation (3c) gives a third equation $\Gamma(\pi) = 1$ which overall, gives a system of three equations and three unknowns to determine the price of the real asset p , and cutoffs π and π'' . First use $\pi'' = 1 - hp$ to eliminate π'' in equation (3c) or $\Gamma(\pi) = 1$ which yields, after some simplifications, to:

$$\sqrt{-\frac{p^2((h+2)p-1)(hp+\pi-1)^2}{(hp-1)(hp+p-1)^2}} - \frac{\pi(h+1)^2p + h(h+2)(hp+p-1)^2 - 1}{(h+1)^2(hp+p-1)} + \frac{2}{(h+1)^2} = 1.$$

This allows can be rearranged into a second-order polynomial in π , which can therefore be expressed as a function of h and p as:

$$\begin{aligned} \pi &= \frac{1}{2} \left(2 + h - h(2+h)p - \sqrt{h^2(1-hp)(-1+(2+h)p)} \right) \\ \text{or } \pi &= \frac{1}{2} \left(2 + h - h(2+h)p + \sqrt{h^2(1-hp)(-1+(2+h)p)} \right) \end{aligned}$$

One could continue the calculations with those two possibilities and conclude at the end when arriving at a contradiction on the implied price of the Real Asset, but it is easier to remark that from the derived Taylor expansions in the general case (see Lemma 1), namely that $\pi = 1 - O(h^2)$, it is possible to eliminate the first solution which has a first order Taylor expansion with a negative coefficient on the first order term h . Therefore:

$$\pi = \frac{1}{2} \left(2 + h - h(2+h)p + \sqrt{h^2(1-hp)(-1+(2+h)p)} \right)$$

Plugging this expression for π as well as using $\pi'' = 1 - hp$ to eliminate π'' once again in equation (3b) gives, after some simple but lengthy algebra:

$$\begin{aligned} & - \left(h + \sqrt{-h^2(-1+hp)(-1+(2+h)p)} \right) \sqrt{\frac{hp^2(-1+(2+h)p) \left(hp + \sqrt{-h^2(-1+hp)(-1+(2+h)p)} \right)}{(-1+p+hp)^2}} + \dots \\ & \dots p \left(\sqrt{2} \sqrt{-h^2(-1+hp)(-1+(2+h)p)} + \sqrt{2}h \left(1 + \sqrt{-h^2(-1+hp)(-1+(2+h)p)} \right) \right) + \dots \\ & \dots p \left(h^2 \sqrt{\frac{hp^2(-1+(2+h)p) \left(hp + \sqrt{-h^2(-1+hp)(-1+(2+h)p)} \right)}{(-1+p+hp)^2}} \right) = 0 \end{aligned}$$

$$\begin{aligned}
&\Rightarrow h(-1 + (2 + h)p) \left(hp + \sqrt{-h^2(-1 + hp)(-1 + (2 + h)p)} \right) \left(h - h^2p + \sqrt{-h^2(-1 + hp)(-1 + (2 + h)p)} \right)^2 - \dots \\
&\dots 2(-1 + p + hp)^2 \left(h + \sqrt{-h^2(-1 + hp)(-1 + (2 + h)p)} + h\sqrt{-h^2(-1 + hp)(-1 + (2 + h)p)} \right)^2 = 0 \\
&\Rightarrow 2h^2 + 2h^3 - 2hp - 10h^2p - 16h^3p - 8h^4p + 4hp^2 + 20h^2p^2 + 37h^3p^2 + 36h^4p^2 + 12h^5p^2 - 2hp^3 - \dots \\
&\dots 14h^2p^3 - 32h^3p^3 - 44h^4p^3 - 32h^5p^3 - 8h^6p^3 + 2h^2p^4 + 9h^3p^4 + 14h^4p^4 + 17h^5p^4 + 10h^6p^4 + 2h^7p^4 - \dots \\
&\dots 2\sqrt{-h^2(-1 + hp)(-1 + (2 + h)p)} (1 + h - (2 + h(4 + h))p + (1 + h(3 + h - h^2))p^2 + h^3(2 + h)p^3) = 0
\end{aligned}$$

Finally, one arrives at a polynomial of degree eight, separable in polynomials of maximum degree two in p :

$$\begin{aligned}
&4h^4(-1 + p + hp)^4 [2(1 + h^2) - 2(1 + h)(1 + 2h^2)p + h(2 + h)(1 + 2h^2)p^2] \dots \\
&\dots [2(1 + h)^2 - 2(1 + h)(1 + 2h(2 + h))p + h(2 + h)(1 + 2h(2 + h))p^2] = 0
\end{aligned}$$

The price of the Real Asset is therefore necessarily given by the following alternatives:

$$\begin{aligned}
p &= \frac{1}{1 + h} = 1 - h + O(h^2) \quad \text{or} \quad p = \frac{1 - \sqrt{(-1 + h)^2(1 + 2h^2)} + h(1 + 2h(1 + h))}{h(2 + h)(1 + 2h^2)} = 1 - \frac{h^2}{2} + O(h^3) \\
\text{or} \quad p &= \frac{1 + \sqrt{(-1 + h)^2(1 + 2h^2)} + h(1 + 2h(1 + h))}{h(2 + h)(1 + 2h^2)} = \frac{1}{h} - \frac{1}{2} + O(h) \\
\text{or} \quad p &= \frac{2(1 + h)^2}{1 + \sqrt{(1 + h)^2(1 + 2h(2 + h))} + h(5 + 2h(3 + h))} = 1 - 2h + O(h^2) \\
\text{or} \quad p &= \frac{2(1 + h)^2}{1 - \sqrt{(1 + h)^2(1 + 2h(2 + h))} + h(5 + 2h(3 + h))} = \frac{1}{h} - \frac{1}{2} + O(h).
\end{aligned}$$

Note that the only solution having a Taylor expansion compatible with Lemma 1 (proved in Appendix B.1.3) is therefore the second one¹⁸:

$$p = \frac{1 + h(1 + 2h(1 + h)) - \sqrt{(-1 + h)^2(1 + 2h^2)}}{h(2 + h)(1 + 2h^2)}$$

The cutoff π in the economy with flat priors then results after some algebra:

$$\begin{aligned}
\pi &= \frac{1}{2} \left(2 + h - h(2 + h)p + \sqrt{h^2(1 - hp)(-1 + (2 + h)p)} \right) \\
\pi &= \frac{1}{2 + 4h^2} \left[1 + \sqrt{\frac{h^2}{1 + 2h^2}} + \sqrt{(-1 + h)^2(1 + 2h^2)} + 2h^2 \left(1 + \sqrt{\frac{h^2}{1 + 2h^2}} \right) \right].
\end{aligned}$$

And cutoff π'' obtains immediately:

$$\pi'' = 1 - ph = \frac{1 + 2h^2 + \sqrt{(-1 + h)^2(1 + 2h^2)}}{2 + h + 4h^2 + 2h^3}.$$

Gathering those results, and generalizing to any maximum value of beliefs M :

$$\begin{aligned}
p &= \frac{1 + h + 2h^2 + 2h^3 - \sqrt{(-1 + h)^2(1 + 2h^2)}}{2h + h^2 + 4h^3 + 2h^4} M & \pi'' &= \frac{1 + 2h^2 + \sqrt{(-1 + h)^2(1 + 2h^2)}}{2 + h + 4h^2 + 2h^3} M \\
\pi &= \frac{1}{2 + 4h^2} \left[1 + \sqrt{\frac{h^2}{1 + 2h^2}} + \sqrt{(-1 + h)^2(1 + 2h^2)} + 2h^2 \left(1 + \sqrt{\frac{h^2}{1 + 2h^2}} \right) \right] M.
\end{aligned}$$

¹⁸The reason why other solutions were found at this stage is because of the constructive nature of the proof. In particular, squares of both sides of an equation were taken multiple times. But any solution must necessarily satisfy restrictions imposed in Lemma 1.

The annex cutoffs $\hat{\pi}$ and $\tilde{\pi}$ used to define assignment function $\Gamma(\cdot)$ and assignment interest rates function $r(\cdot)$ can therefore also be expressed in closed form as a function of h and M , albeit in a sometimes lengthy way. For example, $\hat{\pi}$ is given by:

$$\hat{\pi} = \frac{\left(1 + h + 2h^2 + 2h^3 - \sqrt{(-1 + h)^2(1 + 2h^2)}\right)^2 \left(2\sqrt{\frac{h^2}{1+2h^2}} + 4h^2\sqrt{\frac{h^2}{1+2h^2}} + 2h^3 \left(1 + \sqrt{\frac{h^2}{1+2h^2}}\right)\right)}{2h(2 + h)(1 + 2h^2) \left(1 + 2h^2 + \sqrt{(-1 + h)^2(1 + 2h^2)}\right) \left(1 + 2h^2 - (1 + h)\sqrt{(-1 + h)^2(1 + 2h^2)}\right)} M + \dots$$

$$\dots \frac{\left(1 + h + 2h^2 + 2h^3 - \sqrt{(-1 + h)^2(1 + 2h^2)}\right)^2 \left(h \left(1 + \sqrt{\frac{h^2}{1+2h^2}} + \sqrt{(-1 + h)^2(1 + 2h^2)}\right)\right)}{2h(2 + h)(1 + 2h^2) \left(1 + 2h^2 + \sqrt{(-1 + h)^2(1 + 2h^2)}\right) \left(1 + 2h^2 - (1 + h)\sqrt{(-1 + h)^2(1 + 2h^2)}\right)} M$$

Similarly, $\tilde{\pi}$ can be given as a lengthy but simple function of h and M . Finally, we can verify that the Taylor expansions of p , π , and π'' verify Lemma 1. More precisely:

$$p = M - \frac{h^2}{2}M - \frac{h^3}{2}M + O(h^4) \quad \pi = M - \frac{h^2}{2}M + \frac{3h^4}{4}M + O(h^5)$$

$$\pi'' = M - hM + \frac{h^3}{2}M + O(h^4) \quad \pi - p = \frac{h^3}{2}M + O(h^4).$$

Finally, the Taylor expansions of maximum and minimum leverage ratios are given by:

$$L(M) = \frac{\pi}{\pi''} \frac{p - \pi''}{\pi - p} = \frac{2}{h^2} + \frac{2}{h} + 2 + O(h) \quad L(\pi) = \frac{\pi''}{p - \pi''} = \frac{1}{h} - \frac{1}{2} + O(h).$$

B.1.6 Equilibrium with Flat Priors in Economy \mathcal{E}^B - Other Quantities

Return on Equity function. Return on equity at agents' expectations is given by:

$$ROE^e(\Gamma^{-1}(y), y) = \frac{(y - \Gamma^{-1}(y)) r(\Gamma^{-1}(y))}{pr(\Gamma^{-1}(y)) - \Gamma^{-1}(y)}.$$

The inverse of the assignment function is easy to calculate:

$$\Gamma(x) = \frac{1}{2} \left(-\hat{\pi} - 2x - \tilde{\pi} + \sqrt{2}\sqrt{\hat{\pi}(4x + \tilde{\pi} + \hat{\pi})} \right)$$

$$\Rightarrow \Gamma^{-1}(y) = \frac{1}{2} \left(\hat{\pi} - \tilde{\pi} - 2y + \sqrt{2}\sqrt{\hat{\pi}^2 - \hat{\pi}\tilde{\pi} - 4\hat{\pi}y} \right).$$

After some lines of algebra, one gets an expression for $ROE^e(\Gamma^{-1}(y), y)$:

$$ROE^e(\Gamma^{-1}(y), y) = \frac{\hat{\pi} \left(\sqrt{2}\sqrt{\hat{\pi}(-\tilde{\pi} + \hat{\pi} - 4y)} - \tilde{\pi} + \hat{\pi} - 4y \right)}{p \left(\sqrt{2}\sqrt{\hat{\pi} \left(2\sqrt{2}\sqrt{\hat{\pi}(-\tilde{\pi} + \hat{\pi} - 4y)} - \tilde{\pi} + 3\hat{\pi} - 4y \right)} - 2\hat{\pi} \right)}$$

Replacing the cutoffs by the expressions above gives an expression for $ROE^e(\Gamma^{-1}(y), y)$ as a function of the primitives of the model h and M .

Leverage Distribution Function. As in the general case, the derivation of the leverage function is given by:

$$l = L(y) = \frac{\Gamma^{-1}(y)}{pr(\Gamma^{-1}(y)) - \Gamma^{-1}(y)}$$

$$\Rightarrow L(y) = \frac{\sqrt{\hat{\pi} \left(2\sqrt{2}\sqrt{\hat{\pi}(-\tilde{\pi} + \hat{\pi} - 4y)} - \tilde{\pi} + 3\hat{\pi} - 4y \right)}}{\sqrt{2}\hat{\pi} - \sqrt{\hat{\pi} \left(2\sqrt{2}\sqrt{\hat{\pi}(-\tilde{\pi} + \hat{\pi} - 4y)} - \tilde{\pi} + 3\hat{\pi} - 4y \right)}}$$

The inverse of this leverage function is then:

$$L^{-1}(l) = \frac{-l^2\tilde{\pi} - 2l\tilde{\pi} - \tilde{\pi} + l^2\hat{\pi} + 2l\hat{\pi} - \hat{\pi}}{4(l+1)^2}.$$

Therefore, the cumulative distribution function for leverage of optimists can be calculated:

$$\begin{aligned} G(l) &= \mathbb{P}(L(y) \leq l) = \mathbb{P}(y \leq L^{-1}(l) | y \geq \pi) \\ &= \frac{\mathbb{P}(y \leq L^{-1}(l) \cap y \geq \pi)}{\mathbb{P}(y \geq \pi)} \\ G(l) &= \mathbb{P}(L(y) \leq l) = \frac{(F \circ L^{-1})(l) - F(\pi)}{1 - F(\pi)}. \end{aligned}$$

In the case of flat priors, replacing $F(\cdot)$ and $\Gamma(\cdot)$ by their expressions gives:

$$\begin{aligned} (F \circ L^{-1})(l) &= 1 - \frac{M - L^{-1}(l)}{H} \\ &= 1 - \frac{M}{H} + \frac{1}{H} \left(\frac{-l^2\tilde{\pi} - 2l\tilde{\pi} - \tilde{\pi} + l^2\hat{\pi} + 2l\hat{\pi} - \hat{\pi}}{4(l+1)^2} \right). \end{aligned}$$

Hence:

$$\begin{aligned} G(l) &= \frac{(F \circ L^{-1})(l) - \frac{\pi - (M(1-h))}{H}}{1 - \frac{\pi - (M(1-h))}{H}} = 1 - \frac{1 - (F \circ L^{-1})(l)}{1 - \frac{\pi - (M(1-h))}{H}} \\ G(l) &= 1 - \frac{M}{M - \pi} + \frac{1}{M - \pi} \left(\frac{-l^2\tilde{\pi} - 2l\tilde{\pi} - \tilde{\pi} + l^2\hat{\pi} + 2l\hat{\pi} - \hat{\pi}}{4(l+1)^2} \right). \end{aligned}$$

Similarly, replacing the cutoffs by the expressions above gives an expression for $G(l)$ as a function of the disagreement parameter h .

B.2 Proofs for Proposition 4 - Short-Sales Economy \mathcal{E}^S

Denote by $ROE^S(p_{t+1}^i, \gamma)$ the return on equity for a short-seller with beliefs p_{t+1}^i selling a Short-Sales Contract of type (γ) , according to the convention in Definition 3: it is a promise to return one unit of Real Asset collateralized by γ units of Cash. Denoting by $q_s(\gamma)$ the price of such a contract, selling one Short-Sales Contract of type (γ) allows to raise $q_s(\gamma)$ in Cash, which must be complemented by an amount $\gamma - q_s(\gamma)$ in Cash collateral to arrive at γ in total. Therefore, the amount of Cash that one unit of equity allows to invest in is given by the number of Short-Sales contracts a short-seller can sell $1/(\gamma - q_s(\gamma))$ multiplied by the price of one such contract:

$$l_s(\gamma) = \frac{q_s(\gamma)}{\gamma - q_s(\gamma)}.$$

Then $ROE^S(p_{t+1}^i, \gamma)$ is given by the mean of the return on Cash, with weight the size of the investment in Cash for each unit of equity $1 + l_s(\gamma)$, and the total return given to lenders $p_{t+1}^i/q_s(\gamma) = p_{t+1}^i/pr_s(\gamma)$ with weight the size of the borrowed funds $-l_s(\gamma)$:

$$ROE^S(p_{t+1}^i, \gamma) = 1 + l_s(\gamma) - \frac{p_{t+1}^i}{p} r_s(\gamma) l_s(\gamma) = \frac{\gamma - p_{t+1}^i}{\gamma - q_s(\gamma)}. \quad (R_s)$$

Denote by $ROE^A(p_{t+1}^i, \gamma)$ the return on equity for an asset lender with beliefs p_{t+1}^i . With one unit of wealth, he can purchase $1/q_s(\gamma)$ units of Short-Sales Contracts of type (γ) , each of which provides him with an expected payoff equal to $\min\{p_{t+1}^i, \gamma\}$. Therefore:

$$ROE^A(p_{t+1}^i, \gamma) = \frac{1}{q_s(\gamma)} \min\{p_{t+1}^i, \gamma\}. \quad (R_a)$$

The return on equity for an unlevered buyer, denoted by $ROE^U(p_{t+1}^i)$ is:

$$ROE^U(p_{t+1}^i) = \frac{p_{t+1}^i}{p}. \quad (R_u)$$

As in Borrowing Economy \mathcal{E}^B (see Appendix B.1.1), the return on equity for a borrower with beliefs p_{t+1}^i selling a contract of type (ϕ) , whose price is denoted by $q(\phi)$ and associated implicit interest rate $r(\phi)$ is given by:

$$ROE^B(p_{t+1}^i, \phi) = \frac{p_{t+1}^i}{p} (1 + l(\phi)) - r(\phi)l(\phi) = \frac{p_{t+1}^i - \phi}{p - q(\phi)}. \quad (R_b)$$

Where the leverage associated to a contract of type (ϕ) is given by:

$$l(\phi) = \frac{q(\phi)}{p - q(\phi)}.$$

Similarly, the return on equity for a lender with beliefs p_{t+1}^i buying a contract of type (ϕ) is given by:

$$ROE^L(p_{t+1}^i, \phi) = \frac{1}{q(\phi)} \min\{p_{t+1}^i, \phi\}. \quad (R_l)$$

Given his beliefs p_{t+1}^i , an agent chooses to be a short-seller, an asset lender, an unlevered buyer, a borrower or a Cash lender to maximize his return on equity. He chooses his type and conditional on being a borrower, a lender, a short-seller or an asset lender, the type of contract he buys or sells (that is, the value of ϕ or γ).

To avoid repetition, I shall not solve lenders' and borrowers' problems which are exactly the same as in Borrowing Economy \mathcal{E}^B (see Appendix B.1.1) - the small differences (initial conditions) being discussed at length in the main text (Section 2.1.2). I shall now turn to short-sellers' and asset lenders' problems, which are qualitatively similar to borrowers and lenders in Economies \mathcal{E}^B and \mathcal{E}^S .

Solution to Asset Lender's problem. A first remark, as in Borrowing Economy \mathcal{E}^B is that $r_s(\cdot)$ is a decreasing function of Cash collateral, as short-sellers would never choose contracts with both higher Cash collateral (allowing to achieve a lower leverage ratio) and a higher return, or selling with a lower price. Moreover, it will prove useful to show that $\gamma/q_s(\gamma) = \gamma r_s(\gamma)/p$ is increasing in γ . The reason why this is the case is that the program of a short-seller can be written as:

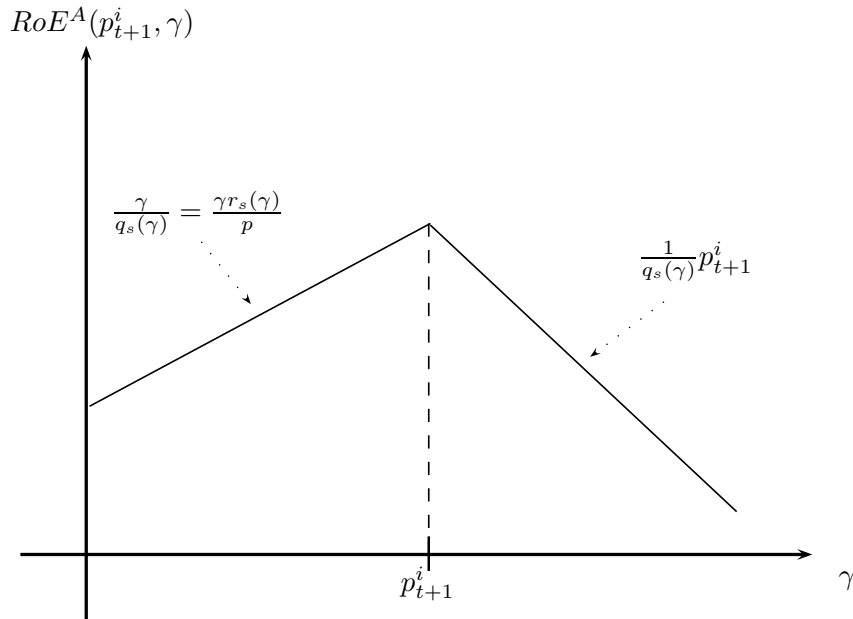
$$ROE^S(p_{t+1}^i, \gamma) = \frac{\gamma - p_{t+1}^i}{\gamma - q_s(\gamma)} = 1 + \frac{1 - \frac{p_{t+1}^i}{q_s(\gamma)}}{\frac{\gamma}{q_s(\gamma)} - 1}.$$

Since $q_s(\gamma) = p/r_s(\gamma)$ is increasing in γ because $r_s(\cdot)$ is an increasing function, the numerator is increasing in γ and therefore $\gamma/q_s(\gamma)$ cannot be decreasing for two traded contracts. It follows that $\gamma r_s(\gamma)/p$ is increasing. All this allows to solve Asset Lender's problem because:

$$\begin{aligned} \frac{\partial ROE^A(p_{t+1}^i, \gamma)}{\partial \gamma} &= \left(\frac{\gamma}{q_s(\gamma)} \right)' = \left(\frac{\gamma r_s(\gamma)}{p} \right)' > 0 \quad \text{for } \gamma \leq p_{t+1}^i \\ \frac{\partial ROE^A(p_{t+1}^i, \gamma)}{\partial \gamma} &= p_{t+1}^i \frac{-q_s'(\gamma)}{q_s(\gamma)^2} < 0 \quad \text{for } \gamma \geq p_{t+1}^i. \end{aligned}$$

Therefore, an asset lender with beliefs p_{t+1}^i necessarily buys a Short-Sales Contract with a cash-collateral corresponding to his expectation of future period's price $\gamma = p_{t+1}^i$ ((see Figure 19). This is in fact quite intuitive. If the cash-collateral is lower than his beliefs then the lender always chooses the contract with the highest cash-collateral because the contract. If the cash-collateral of the loan is higher than his beliefs about the future price of the asset, then the lender expects to always get the asset back. There if no point in that case in having a too high cash-collateral, because such contract will necessarily come with a higher price in equilibrium.

Figure 19: CHOICE OF A SHORT-SALES CONTRACT (γ) FOR ASSET LENDERS WITH BELIEFS p_{t+1}^i



Solution to Short-Seller's problem. Conditional on being a short-seller, an agent with beliefs p_{t+1}^i chooses a Short-Sales Contract to sell $(\gamma)_s$ to maximize:

$$ROE^S(p_{t+1}^i, \gamma) = \frac{\gamma - p_{t+1}^i}{\gamma - q_s(\gamma)} = \frac{\gamma - p_{t+1}^i}{\gamma r_s(\gamma) - p} r_s(\gamma).$$

The first order condition for this maximization program is that:

$$\frac{\partial ROE^S(p_{t+1}^i, \gamma)}{\partial \gamma} = \frac{(r_s(\gamma) + r'_s(\gamma)(\gamma - p_{t+1}^i))(\gamma r_s(\gamma) - p) - (r_s(\gamma) + \gamma r'_s(\gamma)(\gamma - p_{t+1}^i)r_s(\gamma))}{(\gamma r_s(\gamma) - p)^2}.$$

This optimality condition for a short-seller with beliefs p_{t+1}^i simplifies into:

$$p(\gamma - p_{t+1}^i)r'_s(\gamma) + pr_s(\gamma) - p_{t+1}^i r_s(\gamma)^2 = 0.$$

In equilibrium, a short-seller with beliefs $p_{t+1}^i = x$ must be incentivized to choose a contract $(\gamma)_s$ such that he is matched to asset lender y with $x = \Gamma_s(y)$, therefore a short-seller $\Gamma_s(y)$ must be incentivized to choose contract $(\gamma)_s = (y)_s$ that asset lender y buys. This gives equation (A'_s) , the assignment equation for short-sellers:

$$\forall y \in [\pi, \pi'], \quad p(y - \Gamma_s(y))r'_s(y) + pr_s(y) - \Gamma_s(y)r_s(y)^2 = 0.$$

B.3 Proofs for Proposition 6 - Securitization Economies \mathcal{E}_n^B

For brevity, I shall only focus on the details where the reasoning is different from that used to prove Proposition 1. Mainly, it is both lenders' and lenders of type 2's problems which are now different, which I'll turn to next. The reader should refer to Appendix B.1.1 for borrowers' problem and more detailed derivations.

Lenders' Problem. By a similar argument as in Appendix B.1.1, lenders only buy Borrowing Contracts that are neither overcollateralized nor undercollateralized, whose face value is therefore equal to their beliefs. Denote by $r_1(p_{t+1}^i)$ the interest rate they implicitly earned on those Borrowing Contracts. Selling a Type-2 Borrowing Contract with face value ϕ allows to raise $q_2(\phi)$ from the sale, and to buy a Borrowing Contract at price $q_1(p_{t+1}^i)$, which means that:

$$l_2(\phi) = \frac{q_2(\phi)}{q_1(x) - q_2(\phi)} = \frac{\phi}{x \frac{r_2(\phi)}{r_1(x)} - \phi}.$$

Then the expected Return on Equity $ROE_2^L(p_{t+1}^i, \phi)$ of lenders when they sell a Type-2 Borrowing Contract $(\phi)_2$ is given by:

$$ROE_2^L(p_{t+1}^i, \phi) = r_1(x)(1 + l_2(\phi)) - r_2(\phi)l_2(\phi) = \frac{(x - \phi)r_2(\phi)}{x \frac{r_2(\phi)}{r_1(x)} - \phi}.$$

Writing that a lender with beliefs $p_{t+1}^i = x$ must be incentivized to choose ϕ such that he is matched with the corresponding type 2 lender z such that $x = \Gamma_2(z)$ allows to conclude.