

# Asset Prices, Liquidity, and Monetary Policy in an Exchange Economy

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## Abstract

I formulate a search-based asset-pricing model where equity shares and fiat money can be used as means of payment. I characterize a family of optimal stochastic monetary policies. Every policy in this family implements Friedman's prescription of zero nominal interest rates. Under an optimal policy, equity prices and returns are independent of monetary considerations. I also study a perturbation of the family of optimal policies that targets a constant but nonzero nominal interest rate. Under such policies, the average real return on equity is negatively correlated with the average inflation rate.

**Keywords:** liquidity, money, asset pricing, search

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# 1 Introduction

Many financial assets are to some degree, valued not only as claims to streams of consumption goods, but also for their *liquidity*—their usefulness in the mechanism of exchange. This observation materializes in its purest form with fiat money: an asset that is universally used in exchange and sells at a positive price, even though it represents a formal claim to nothing. In this paper I formulate a search-based model where money and an asset that represents a claim to a stochastic stream of real dividends (an equity share) can be used as means of payment, and use the theory to derive the asset-pricing implications of monetary policy.

In Section 2 I present the basic model. In Section 3 I show how liquidity considerations affect equity prices and returns in an economy with no money. I find that if the asset can help relax trading constraints in some state of the world, the equilibrium asset price is higher and its measured rate of return (dividend yield plus capital gains) is lower than they would be in an economy with no liquidity needs.

In Section 4 I introduce fiat money and define a recursive monetary equilibrium. In Section 5 I characterize a class of optimal monetary policies, and describe the behavior of asset prices, asset returns, output, inflation, and the nominal interest rate under the optimal policy. Every policy in this family implements Friedman’s prescription of zero nominal interest rates. Under an optimal policy, equity prices and returns are independent of monetary considerations.

In Section 6, I consider perturbations of the optimal monetary policy that consist of targeting a constant nominal interest rate, and discuss some of the positive implications of changes in the nominal interest rate or the inflation rate on equity prices, equity returns, and output. I find that the price of equity is increasing, and real balances are decreasing in the nominal interest rate target. The analysis also provides insights on how monetary policy must be conducted in order to support a recursive monetary equilibrium with a constant nominal interest rate (with the Pareto optimal equilibrium in which the nominal rate is zero as a special case): The growth rate of the money supply must be relatively low in states in which the real value of the equilibrium equity holdings is below average. Something similar happens with the implied inflation rate: it is relatively low between state  $x$  and a next-period state  $x'$ , if the realized real value of the equilibrium equity holdings in state  $x'$  is below its state- $x$  conditional expectation. I also find that on average, liquidity considerations can introduce a negative relationship between the nominal interest rate (and the inflation rate) and equity returns. If the average rate of inflation

is higher, real money balances are lower, and the liquidity return on equity rises, which causes its price to rise and its real measured rate of return to fall.

This paper is related to a large literature that studies how monetary considerations may help explain various features of asset prices. Some examples include Balduzzi (1996), Bansal and Coleman (1996), Bohn (1991), Boyle and Young (1988), Danthine and Donaldson (1986), Giovannini and Labadie (1991), Kiyotaki and Moore (2005), Svensson (1985), and Townsend (1987). The approach I follow is different from these previous studies. First, these papers assume that money plays a special role, either because it is the only financial asset that satisfies a cash-in-advance constraint, or because it is the only financial asset that enters the agents' utility functions. In contrast, I do not assume that money plays a special role in exchange. Second, my work builds on the literature that provides micro foundations for monetary economics based on search theory, as pioneered by Kiyotaki and Wright (1989). Specifically, the model is a version of Lagos and Wright (2005), augmented to allow for aggregate liquidity shocks, and another financial asset that can be used as means of payment the same way money can.<sup>1</sup>

## 2 The model

Time is discrete, and the horizon infinite. There is a  $[0, 1]$  continuum of infinitely lived agents. Each time period is divided into two subperiods where different activities take place. There are three nonstorable and perfectly divisible consumption goods at each date: *fruit*, *general goods*, and *special goods*.<sup>2</sup> Fruit and general goods are homogeneous goods, while special goods come in many varieties. The only durable commodity in the economy is a set of "Lucas trees." The number of trees is fixed and equal to the number of agents. Trees yield (the same amount of) a random quantity  $d_t$  of fruit in the second subperiod of every period  $t$ . The realization of  $d_t$  becomes known to all at the beginning of period  $t$  (when agents enter the first subperiod).

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<sup>1</sup>Lagos and Rocheteau (2008) was the first paper to extend Lagos and Wright (2005) to allow for another asset that competes with money as a medium of exchange. Lagos (2006) formulates a real version of Lagos and Wright (2005) with aggregate uncertainty, in which equity shares and government bonds can serve as means of payment, and uses it to study the equity premium and the risk-free rate puzzles. Ravikumar and Shao (2006) consider a related model that combines features of Lucas (1978) with features of Lagos and Wright (2005) and Shi (1997) to study the excess volatility puzzle. Geromichalos et al (2007) consider a version of Lagos and Rocheteau (2008) in which the real asset that competes with money is in fixed supply. Lester et al (2008) consider a version of Geromichalos et al (2007) in which money can be used as means of payment in all bilateral meetings, while the real asset is only accepted in some bilateral meetings.

<sup>2</sup>"Nonstorable" means that the goods cannot be carried from one subperiod to the next. This formulation with three consumption goods allows a parsimonious integration of the asset pricing model of Lucas (1978) with the model of exchange in Lagos and Wright (2005).

Production of fruit is entirely exogenous: no resources are utilized and it is not possible to affect the output at any time. The motion of  $d_t$  will be taken to follow a Markov process, defined by its transition function  $F(x', x) = \Pr(d_{t+1} \leq x' | d_t = x)$ , where  $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  is continuous. For each fixed  $x$ ,  $F(\cdot, x)$  is a distribution function with support  $\Xi \subseteq (0, \infty)$ . Assume that the process defined by  $F$  has a stationary distribution  $\psi(\cdot)$ , the unique solution to  $\psi(x') = \int F(x', x) d\psi(x)$ , and that  $F$  has the Feller property, i.e., for any continuous real-valued function  $g$  on  $\Xi$ ,  $\int g(x') dF(x', x)$  is a continuous function of  $x$ .

In each subperiod, every agent is endowed with  $\bar{n}$  units of time which can be employed as labor services. In the second subperiod, each agent has access to a linear production technology that transforms labor services into general goods. In the first subperiod, each agent has access to a linear production technology that transforms his own labor input into a particular variety of the special good that he himself does not consume. This specialization is modeled as follows. Given two agents  $i$  and  $j$  drawn at random, there are three possible events. The probability that  $i$  consumes the variety of special good that  $j$  produces but not vice-versa (a single coincidence) is denoted  $\alpha$ . Symmetrically, the probability that  $j$  consumes the special good that  $i$  produces but not vice-versa is also  $\alpha$ . In a single-coincidence meeting, the agent who wishes to consume is the *buyer*, and the agent who produces, the *seller*. The probability neither wants anything the other can produce is  $1 - 2\alpha$ , with  $\alpha \leq 1/2$ . In contrast to special goods, fruit and general goods are homogeneous, and hence consumed (and in the case of general goods, also produced) by all agents.

In the first subperiod, agents participate in a *decentralized market* where trade is bilateral (each meeting is a random draw from the set of pairwise meetings), and the terms of trade are determined by bargaining. The specialization of agents over consumption and production of the special good combined with bilateral trade, give rise to a double-coincidence-of-wants problem in the first subperiod. In the second subperiod, agents trade in a *centralized market*. Agents cannot make binding commitments, and trading histories are private in a way that precludes any borrowing and lending between people, so all trade—both in the centralized and decentralized markets—must be *quid pro quo*.

Each tree has outstanding one durable and perfectly divisible equity share that represents the bearer's ownership of a tree and confers him the right to collect the fruit dividends. I will later introduce another perfectly divisible asset—fiat money. All assets are perfectly recognizable, cannot be forged, and can be traded among agents both in the centralized and decentralized

markets. At  $t = 0$  each agent is endowed with  $a_0^s$  equity shares (and possibly also  $a_0^m$  units of fiat money).

Let the utility function for special goods,  $u : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , and the utility function for fruit,  $U : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , be continuously differentiable, bounded by  $B$  on  $\Xi$ , increasing, and strictly concave, with  $u(0) = U(0) = 0$ . Let  $-n$  be the utility from working  $n$  hours in the first subperiod. Also, suppose there exists  $q^* \in (0, \infty)$  defined by  $u'(q^*) = 1$ , with  $q^* \leq \bar{n}$ . Let both, the utility for general goods, and the disutility from working in the second subperiod, be linear. The agent's preferences are:

$$\liminf_{T \rightarrow \infty} E_0 \left\{ \sum_{t=0}^T \beta^t [u(q_t) - n_t + U(c_t) + y_t - h_t] \right\},$$

where  $\beta \in (0, 1)$ ,  $q_t$  and  $n_t$  are the quantities of special goods consumed and produced in the decentralized market,  $c_t$  denotes consumption of fruit,  $y_t$  consumption of general goods,  $h_t$  the hours worked in the second subperiod, and  $E_t$  is an expectations operator conditional on the information available to the agent at time  $t$ , defined with respect to the matching probabilities and the probability measure induced by  $F$ .<sup>3</sup>

### 3 Asset prices and liquidity in a real economy

I begin by considering a real economy where the equity share is the only asset. Let  $W(a_t, d_t)$  denote the value function of an agent who enters the centralized market holding  $a_t$  shares in a period when dividends are  $d_t$ , and let  $V(a_t, d_t)$  denote the corresponding value when he enters the decentralized market. These value functions satisfy the following Bellman equation:

$$\begin{aligned} W(a_t, d_t) &= \max_{c_t, y_t, h_t, a_{t+1}} \{U(c_t) + y_t - h_t + \beta EV(a_{t+1}, d_{t+1})\} \\ \text{s.t. } c_t + w_t y_t + \phi_t a_{t+1} &= (\phi_t + d_t) a_t + w_t h_t \\ 0 \leq c_t, 0 \leq h_t \leq \bar{n}, 0 \leq a_{t+1}. \end{aligned}$$

The agent chooses consumption of fruit ( $c_t$ ), consumption of general goods ( $y_t$ ), hours of work devoted to production of general goods ( $h_t$ ), and an end-of-period portfolio ( $a_{t+1}$ ). Dividends are paid to the bearer of the equity share after decentralized trade, but before the time- $t$

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<sup>3</sup>I follow Brock (1970) in the use of an *overtaking criterion* to rank sequences of consumption and labor because the period utility function is unbounded. The same criterion was adopted by Green and Zhou (2002).

centralized trading session. Fruit is used as numéraire:  $w_t$  is the relative price of general goods, and  $\phi_t$  is the (ex-dividend) price of a share. Substitute the first two constraints into the objective, let  $\lambda_t \equiv \frac{1}{w_t} (\phi_t + d_t)$ , and rearrange to arrive at:

$$W(a_t, d_t) = \lambda_t a_t + \max_{c_t} \left[ U(c_t) - \frac{c_t}{w_t} \right] + \max_{a_{t+1}} \left[ -\frac{\phi_t a_{t+1}}{w_t} + \beta EV(a_{t+1}, d_{t+1}) \right]. \quad (1)$$

When a buyer and a seller have share holdings  $a$  and  $\tilde{a}$  respectively, the terms at which they trade in the decentralized market are  $[q(a, \tilde{a}), p(a, \tilde{a})]$ , where  $q(a, \tilde{a}) \in \mathbb{R}_+$  is the quantity of special good traded, and  $p(a, \tilde{a}) \in \mathbb{R}_+$  represents the transfer of assets from the buyer to the seller. The value of an agent who enters the decentralized market with share holdings  $a$  in a period when the dividend realization is  $d$ , satisfies

$$\begin{aligned} V(a, d) = & \alpha \int \{u[q(a, \tilde{a})] + W[a - p(a, \tilde{a}), d]\} dG(\tilde{a}) + \\ & \alpha \int \{-q(\tilde{a}, a) + W[a + p(\tilde{a}, a), d]\} dG(\tilde{a}) + (1 - 2\alpha) W(a, d), \end{aligned} \quad (2)$$

where  $G$  denotes the distribution of share holdings among the population.

Consider a meeting in the decentralized market between a buyer who holds  $a_t$  and a seller who holds  $\tilde{a}_t$ . The terms of trade  $(q_t, p_t)$  are determined by Nash bargaining where the buyer has all the bargaining power.<sup>4</sup> Thus,  $(q_t, p_t)$  solves

$$\max_{q_t, p_t \leq a_t} [u(q_t) + W(a_t - p_t, d_t) - W(a_t, d_t)] \text{ s.t. } -q_t + W(\tilde{a}_t + p_t, d_t) \geq W(\tilde{a}_t, d_t).$$

The constraint  $p_t \leq a_t$  indicates that the buyer cannot spend more assets than he owns. Since  $W(a_t + p_t, d_t) - W(a_t, d_t) = \lambda_t p_t$ , the bargaining problem reduces to:

$$\max_{q_t, p_t \leq a_t} [u(q_t) - \lambda_t p_t] \text{ s.t. } -q_t + \lambda_t p_t \geq 0.$$

If  $\lambda_t a_t \geq q^*$ , the buyer exchanges  $p_t = q^*/\lambda_t \leq a_t$  of his shares for  $q^*$  special goods. Else, he gives the seller all his shares, i.e.,  $p_t = a_t$ , in exchange for  $q_t = \lambda_t a_t$  special goods. Hence, the quantity of special goods traded is  $q(\lambda_t a_t)$ , where

$$q(x) = \min(x, q^*). \quad (3)$$

With the bargaining solution and the linearity of  $W(a, d)$ , the value of search (2) becomes

$$V(a, d) = \mathcal{S}(\lambda a) + W(a, d), \quad (4)$$

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<sup>4</sup>See Lagos (2006) for an analysis with generalized Nash bargaining in a related model.

where

$$\mathcal{S}(x) \equiv \alpha\{u[q(x)] - q(x)\} \quad (5)$$

is the expected gain from trading in the decentralized market. Observe that  $\mathcal{S}$  is twice differentiable almost everywhere, with  $\mathcal{S}'(x) \geq 0$  and  $\mathcal{S}''(x) \leq 0$  (both inequalities are strict for  $x < q^*$ ). Having characterized the terms of trade in decentralized exchange, I turn to the agent's individual optimization problem in the centralized market.

The agent's problem in the second subperiod is summarized by (1). Given that  $U$  is strictly concave, the optimal consumption of fruit satisfies

$$w_t U'(c_t) = 1, \quad (6)$$

and the first-order necessary and sufficient condition for the choice of  $a_{t+1}$  is

$$U'(c_t) \phi_t = \beta E_t V_1(a_{t+1}, d_{t+1}).$$

From (4),  $V_1(a_{t+1}, d_{t+1}) = [1 + \alpha(u'[q(\lambda_{t+1}a_{t+1})] - 1)]\lambda_{t+1}$ , and  $V_{11}(a_{t+1}, d_{t+1}) \leq 0$  ( $< 0$  for  $\lambda_{t+1}a_{t+1} < q^*$ ). None of the agent's choices depend on his individual asset holdings, so as in Lagos and Wright (2005),  $G$  will be degenerate (at the mean) in equilibrium.<sup>5</sup>

I will consider an equilibrium in which all prices are time-invariant functions of the aggregate state,  $d_t$ :  $w_t = w(d_t)$ ,  $\phi_t = \phi(d_t)$ , and therefore,  $\lambda_t = \frac{1}{w(d_t)}[\phi(d_t) + d_t] \equiv \lambda(d_t)$ . In words,  $\lambda(d_t)$  is the (cum-dividend) price of an equity share in state  $d_t$ , in expressed terms of general goods. With (6),

$$\lambda(x) = U'(x) [\phi(x) + x].$$

**Definition 1** *A recursive equilibrium for the economy with equity is a collection of individual decision rules  $c_t = c(d_t)$ ,  $a_{t+1} = a(d_t)$ , pricing functions  $w_t = w(d_t)$  and  $\phi_t = \phi(d_t)$ , and bilateral terms of trade  $q_t = Q(d_t)$  and  $p_t = p(d_t)$  such that: (i) given prices and the bargaining protocol, the decision rules  $c(\cdot)$ , and  $a(\cdot)$ , solve the agent's problem in the centralized market; (ii) the terms of trade in a bilateral meeting where the buyer holds  $a$ , are determined by Nash bargaining, i.e.,  $Q(d_t) = q[\lambda(d_t)a]$ , and  $p(d_t) = q[\lambda(d_t)a]/\lambda(d_t)$ ; and (iii) prices are such that the centralized market clears:  $c(d_t) = d_t$ , and  $a(d_t) = 1$  for all  $d_t$ .*

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<sup>5</sup>Also, if  $\lambda_{t+1}a_{t+1} < q^*$  for some realizations of the dividend process, the portfolio choice problem at date  $t$  has a unique solution, implying that the distribution of assets must be degenerate at the beginning of each decentralized round of trade. Regarding the constraints, the agent's maximization is subject to  $0 \leq c_t$ , which will not bind if, for example,  $U'(0) = +\infty$ . Similarly, in equilibrium, shares will be valued and somebody has to hold them, so  $0 \leq a_{t+1}$  will not bind either.

The Euler equation for equity holdings implies the pricing function for equity shares satisfies

$$U'(x) \phi(x) = \beta \int L[\phi(x')] U'(x') [\phi(x') + x'] dF(x', x), \quad (7)$$

where

$$L[\lambda(x')] \equiv 1 - \alpha + \alpha u'(q[\lambda(x')]).$$

This can be written as

$$\begin{aligned} U'(x) \phi(x) = & \beta \int_{\Omega} \{1 - \alpha + \alpha u'[\lambda(x')]\} U'(x') [\phi(x') + x'] dF(x', x) + \\ & \beta \int_{\Omega^c} U'(x') [\phi(x') + x'] dF(x', x), \end{aligned} \quad (8)$$

where

$$\Omega = \{x \in \Xi : \lambda(x) < q^*\}$$

and  $\Omega^c$  denotes its complement. The set  $\Omega$  contains the realizations of the aggregate dividend process for which the asset has value for its role as a medium of exchange, in addition to its “intrinsic” value, i.e., that which stems from the right that ownership of the asset confers to collect future dividends. So there is a sense in which  $L$  in (7) can be thought of as a stochastic *liquidity factor*. Notice that equation (8) reduces to equation (6) in Lucas (1978) if either  $\Omega = \emptyset$  (the asset has no liquidity value in any state of the world, i.e.,  $L[\lambda(x)] = 1$  for all  $x$ ), or  $\alpha = 0$  (agents have no liquidity needs). In what follows, it will often prove convenient to express (7) as a functional equation in  $\lambda$ :

$$\lambda(x) = \beta \int L[\lambda(x')] \lambda(x') dF(x', x) + x U'(x). \quad (9)$$

In applications, one will typically have to solve (9) numerically, but some useful insights regarding the properties of  $\phi(x)$  and the structure of the set  $\Omega$ , can be gained by considering some special cases that can be solved by paper-and-pencil methods.<sup>6</sup>

### 3.1 i.i.d. returns

Suppose  $\{d_t\}$  is a sequence of independent random variables:  $F(d_{t+1}, d_t) = F(d_{t+1})$ . In this case, (9) implies  $\lambda(x) - x U'(x) = \beta \Delta$ , where  $\Delta$  satisfies

$$\Delta = \int [1 - \alpha + \alpha u'(q[\beta \Delta + z U'(z)])] [\beta \Delta + z U'(z)] dF(z). \quad (10)$$

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<sup>6</sup>Notice that for the case of  $\alpha = 0$  analyzed by Lucas (1978), it is straightforward to show that (9) is a contraction. It appears that this may not be the case for  $\alpha \in (0, 1]$ , unless some more restrictive assumptions are made on  $u$ .



**Lemma 1** *There exists a unique  $\Delta$  that solves (10). This solution is positive, and strictly increasing in  $\alpha$ .*

Given the value of  $\Delta$  characterized by (10), the equity price function is

$$\phi(x) = \frac{\beta\Delta}{U'(x)}, \quad (11)$$

and the set of realizations of the dividend process for which there is a liquidity return is

$$\Omega = \{x \in \Xi : xU'(x) < q^* - \beta\Delta\}. \quad (12)$$

The following result provides a more detailed characterization of the set  $\Omega$ .

**Claim 1** *Assume  $\Xi = [\underline{x}, \infty)$ , with  $\underline{x} > 0$ , and let  $\rho(x) \equiv \frac{-xU''(x)}{U'(x)}$ .*

(i) *If  $q^* \leq \frac{\beta}{1-\beta} \int zU'(z) dF(z)$ , then  $\Omega = \emptyset$ .*

(ii) *If  $q^* > \frac{\beta}{1-\beta} \int zU'(z) dF(z)$ , and  $\rho(x) > 1$  for all  $x$ , then:*

(a)  $\Omega = \emptyset$  *if  $q^* - \beta\Delta \leq \lim_{x \rightarrow \infty} xU'(x)$*

(b)  $\Omega = \{x \in \Xi : x > x^*\}$  *if  $\lim_{x \rightarrow \infty} xU'(x) < q^* - \beta\Delta \leq \underline{x}U'(\underline{x})$ , where  $x^*$  is the unique solution to  $x^*U'(x^*) = q^* - \beta\Delta$*

(c)  $\Omega = \Xi$  *if  $\underline{x}U'(\underline{x}) < q^* - \beta\Delta$*

(iii) *If  $q^* > \frac{\beta}{1-\beta} \int zU'(z) dF(z)$ , and  $\rho(x) < 1$  for all  $x$ , then:*

(a)  $\Omega = \Xi$  *if  $\lim_{x \rightarrow \infty} xU'(x) \leq q^* - \beta\Delta$*

(b)  $\Omega = \{x \in \Xi : x < x^*\}$  *if  $\underline{x}U'(\underline{x}) < q^* - \beta\Delta < \lim_{x \rightarrow \infty} xU'(x)$ , where  $x^*$  is the unique solution to  $x^*U'(x^*) = q^* - \beta\Delta$*

(c)  $\Omega = \emptyset$  *if  $q^* - \beta\Delta \leq \underline{x}U'(\underline{x})$*

(iv) *If  $q^* > \frac{\beta}{1-\beta} \int zU'(z) dF(z)$ , and  $\rho(x) = 1$  for all  $x$ , then  $\Omega = \emptyset$  if  $q^* \leq \frac{1}{1-\beta}$ , and  $\Omega = \Xi$  if  $q^* > \frac{1}{1-\beta}$ .*

One can think of  $q^*$  as indexing the economy's liquidity needs. For instance, if  $q^* \leq \frac{\beta}{1-\beta} \int zU'(z) dF(z)$ , then  $\Omega = \emptyset$ , and  $\Delta = \frac{1}{1-\beta} \int zU'(z) dF(z)$ . That is, if  $q^*$  is relatively low, then

asset prices reduce to those in the i.i.d. example in Lucas (1978). Clearly, the same happens if one simply specifies that the asset is completely illiquid, say by setting  $\alpha = 0$ .

In general, the *gross one-period return to equity* between any two states  $x_i$  and  $x_j$  is defined as  $R^s(x_j, x_i) = \frac{\phi(x_j) + x_j}{\phi(x_i)}$ . For the i.i.d. case,

$$R^s(x_j, x_i) = \left[ 1 + \frac{x_j U'(x_j)}{\beta \Delta} \right] \frac{U'(x_i)}{U'(x_j)}. \quad (13)$$

Lemma 1 shows that  $\Delta$  is increasing in  $\alpha$ , so as the probability the asset can be used in exchange rises, (11) indicates that the price of equity rises, and (13) that its (state-by-state) return falls. Part (iv) of Claim 1 shows that the case of  $\rho(x) = 1$  for all  $x$ , is particularly simple: the asset either provides liquidity in every state or in no state, and the latter is the case if  $q^* \leq \frac{1}{1-\beta}$ , which implies  $\Delta = \frac{1}{1-\beta}$ , and therefore  $\phi(x) = \frac{\beta}{1-\beta}x$ . Conversely, if  $\frac{1}{1-\beta} < q^*$ , then  $\Omega = \Xi$ , and  $\phi(x) = \beta \Delta x$ , where  $\Delta$  solves  $u'(1 + \beta \Delta) = 1 + \frac{(1-\beta)\Delta-1}{\alpha(1+\beta\Delta)}$ . It is easy to show that the solution satisfies  $\frac{1}{1-\beta} < \Delta < \frac{q^*-1}{\beta}$ . The first inequality means that asset prices are higher in every state in the economy with liquidity needs (the economy with high  $q^*$ ). The liquidity factor is constant in all states:  $L = 1 + \alpha \{u'[\min(1 + \beta \Delta, q^*)] - 1\}$ , and  $L > 1$  since  $\Delta < \frac{q^*-1}{\beta}$ . In this case it is also possible to show that the liquidity factor,  $L$ , is increasing in  $\alpha$ .

### 3.2 Correlated returns with log preferences over special goods

Next, generalize the dividend process by allowing it to be serially correlated over time, but specialize preferences over special goods by assuming  $u(q) = \log q$ .<sup>7</sup> In this case,  $q^* = 1$ , so  $u'\{q[\lambda(x)]\} = \max[1, \lambda(x)^{-1}]$  and (9) becomes

$$\lambda(x) = \beta \int \{(1 - \alpha) \lambda(x') + \alpha \max[\lambda(x'), 1]\} dF(x', x) + x U'(x). \quad (14)$$

**Lemma 2** *There exists a unique continuous and bounded function,  $\lambda$ , that solves (14). Moreover,  $\lambda(x) > 0$  for all  $x$ .*

In general, the liquidity constraint  $\lambda(x) \leq 1$  may bind in some states and not in others, but to illustrate, consider two special cases. First, if the constraint never binds, i.e.,  $\lambda(x) \geq 1$  for all  $x \in \Xi$ , then (14) reduces to

$$\lambda(x) = \beta \int \lambda(x') dF(x', x) + x U'(x), \quad (15)$$

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<sup>7</sup>Strictly speaking, standard CRRA preferences do not satisfy the maintained assumption  $u(0) = 0$ . But as in Lagos and Wright (2005), similar results would obtain by adopting  $u(q) = \frac{(q+b)^{1-\sigma} - b^{1-\sigma}}{1-\sigma}$  with  $\sigma > 0$ , and  $b > 0$  but small. Note that  $\frac{-qu''(q)}{u'(q)} = \frac{\sigma}{1+b/q}$ , and as  $\sigma \rightarrow 1$ ,  $u(q) \rightarrow \ln(q+b) - \ln(b)$ , and  $\frac{-qu''(q)}{u'(q)} \rightarrow \frac{1}{1+b/q}$ .

which is identical to equation (6) in Lucas (1978), after substituting  $\lambda(x) = U'(x)[\phi(x) + x]$ . Alternatively, if the constraint binds in every state of the world, i.e.,  $\lambda(x) < 1$  for all  $x \in \Xi$ , then (14) becomes

$$\lambda(x) = \beta(1 - \alpha) \int \lambda(x') dF(x', x) + \beta\alpha + xU'(x). \quad (16)$$

Let  $x = x_t$ ,  $x' = x_{t+1}$ ,  $\phi(x) = \phi_t$ , revert to a sequential formulation and iterate on (16), to arrive at

$$\phi_t = \frac{\alpha\beta}{1 - \beta(1 - \alpha)} \frac{1}{U'(x_t)} + E_t \sum_{j=1}^{\infty} \frac{[\beta(1 - \alpha)]^j U'(x_{t+j})}{U'(x_t)} x_{t+j}. \quad (17)$$

If one shuts down the decentralized market, say by setting  $\alpha = 0$ , then (17) reduces to a standard textbook asset pricing equation (e.g., equation (3.11) in Sargent (1987), p. 96). Note that (17) was derived under the assumption that  $\lambda(x_t) < 1$  for all  $x_t$ , or equivalently, that  $U'(x_t)(\phi_t + x_t) < 1$  for all  $x_t$ . This is indeed the case in equilibrium if

$$\frac{\alpha\beta}{1 - \beta(1 - \alpha)} + E_t \sum_{j=1}^{\infty} [\beta(1 - \alpha)]^j U'(x_{t+j}) x_{t+j} + U'(x_t) x_t < 1, \quad \text{for all } x_t. \quad (18)$$

For a particular specification of preferences, the following result provides a more explicit characterization of the set  $\Omega$  under correlated returns.

**Claim 2** *Suppose  $u(c) = \log c$  and  $U(c) = \varepsilon u(c)$ . If  $\varepsilon < 1 - \beta$ , then  $\Omega = \Xi$ , and*

$$\phi(x) = \frac{\beta[\alpha + (1 - \alpha)\varepsilon]}{\varepsilon[1 - \beta(1 - \alpha)]} x. \quad (19)$$

*Alternatively, if  $\varepsilon \geq 1 - \beta$ , then  $\Omega \subset \Xi$  (the asset provides no liquidity in some state).*

## 4 Asset prices and liquidity in a monetary economy

Consider the economy analyzed in the previous section, but suppose there is a second asset: *money*. Money is intrinsically useless (it is not an argument of any utility or production function), and unlike equity, ownership of money does not constitute a right to collect any resources. Let  $\mathbf{s}_t = (d_t, M_t)$  denote the aggregate state of the economy at time  $t$ , where  $M_t$  is the money supply at time  $t$ . The money supply is set by a government that injects or withdraws money via lump-sum transfers or taxes in the second subperiod of every period, i.e.,  $M_{t+1} = M_t + T_t$ , where  $T_t$  is the lump-sum transfer (or tax, if negative). Let  $\mathbf{a} = (a^s, a^m)$  denote the portfolio of

an agent who holds  $a^s$  shares and  $a^m$  dollars. Let  $W(\mathbf{a}, \mathbf{s})$  and  $V(\mathbf{a}, \mathbf{s})$  be the values from entering the centralized, and decentralized market, respectively, with portfolio  $\mathbf{a}$  when the aggregate state is  $\mathbf{s}$ . These value functions satisfy the following Bellman equation:

$$\begin{aligned} W(\mathbf{a}_t, \mathbf{s}_t) &= \max_{c_t, y_t, h_t, \mathbf{a}_{t+1}} \{U(c_t) + y_t - h_t + \beta EV(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})\} \\ \text{s.t. } c_t + w_t y_t + \phi_t \mathbf{a}_{t+1} &= (\phi_t^s + d_t) a_t^s + \phi_t^m (a_t^m + T_t) + w_t h_t \\ 0 \leq c_t, 0 \leq h_t \leq \bar{n}, 0 \leq \mathbf{a}_{t+1}, \end{aligned}$$

where  $\phi_t \equiv (\phi_t^s, \phi_t^m)$ . The agent chooses consumption of fruit ( $c_t$ ), consumption of general goods ( $y_t$ ), labor supply ( $h_t$ ), and an end-of-period portfolio ( $\mathbf{a}_{t+1}$ ). Again fruit is used as numéraire:  $w_t$  is the relative price of the general good,  $\phi_t^s$  is the (ex-dividend) price of a share, and  $1/\phi_t^m$  the dollar price of fruit. Substitute the budget constraint into the objective and rearrange to arrive at:

$$W(\mathbf{a}_t, \mathbf{s}_t) = \lambda_t \mathbf{a}_t + \tau_t + \max_{c_t} \left[ U(c_t) - \frac{c_t}{w_t} \right] + \max_{\mathbf{a}_{t+1}} \left[ -\frac{\phi_t \mathbf{a}_{t+1}}{w_t} + \beta EV(\mathbf{a}_{t+1}, \mathbf{s}_{t+1}) \right], \quad (20)$$

where  $\tau_t = \frac{1}{w_t} \phi_t^m T_t$ , and  $\lambda_t = (\lambda_t^s, \lambda_t^m)$ , with  $\lambda_t^s \equiv \frac{1}{w_t} (\phi_t^s + d_t)$  and  $\lambda_t^m \equiv \frac{1}{w_t} \phi_t^m$ .

Let  $[q(\mathbf{a}, \tilde{\mathbf{a}}), \mathbf{p}(\mathbf{a}, \tilde{\mathbf{a}})]$  denote the terms at which a buyer who owns portfolio  $\mathbf{a}$  trades with a seller who owns portfolio  $\tilde{\mathbf{a}}$ , where  $q(\mathbf{a}, \tilde{\mathbf{a}}) \in \mathbb{R}^+$  is the quantity of special good traded, and  $\mathbf{p}(\mathbf{a}, \tilde{\mathbf{a}}) = [p^s(\mathbf{a}, \tilde{\mathbf{a}}), p^m(\mathbf{a}, \tilde{\mathbf{a}})] \in \mathbb{R}^+ \times \mathbb{R}^+$  represents the transfer of assets from the buyer to the seller (the first argument is the transfer of equity). The value of search in the decentralized market satisfies

$$\begin{aligned} V(\mathbf{a}, \mathbf{s}) &= \alpha \int \{u[q(\mathbf{a}, \tilde{\mathbf{a}})] + W[\mathbf{a} - \mathbf{p}(\mathbf{a}, \tilde{\mathbf{a}}), \mathbf{s}]\} d\mathbf{G}(\tilde{\mathbf{a}}) \\ &\quad + \alpha \int \{W[\mathbf{a} + \mathbf{p}(\tilde{\mathbf{a}}, \mathbf{a}), \mathbf{s}] - q(\tilde{\mathbf{a}}, \mathbf{a})\} d\mathbf{G}(\tilde{\mathbf{a}}) + (1 - 2\alpha) W(\mathbf{a}, \mathbf{s}), \end{aligned}$$

where  $\mathbf{G}$  denotes the distribution of portfolios.

Consider a meeting in the decentralized market between a buyer with portfolio  $\mathbf{a}_t$  and a seller with portfolio  $\tilde{\mathbf{a}}_t$ . The terms of trade,  $(q_t, \mathbf{p}_t)$ , are determined by Nash bargaining where the buyer has all the bargaining power:

$$\max_{q_t, \mathbf{p}_t \leq \mathbf{a}_t} [u(q_t) + W(\mathbf{a}_t - \mathbf{p}_t, \mathbf{s}_t) - W(\mathbf{a}_t, \mathbf{s}_t)] \text{ s.t. } W(\tilde{\mathbf{a}}_t + \mathbf{p}_t, \mathbf{s}_t) - q_t \geq W(\tilde{\mathbf{a}}_t, \mathbf{s}_t).$$

The constraint  $\mathbf{p}_t \leq \mathbf{a}_t$  indicates that the buyer in a bilateral meeting cannot spend more than the assets he owns. Since  $W(\mathbf{a}_t + \mathbf{p}_t, \mathbf{s}_t) - W(\mathbf{a}_t, \mathbf{s}_t) = \lambda_t \mathbf{p}_t$ , the bargaining problem is

$$\max_{q_t, \mathbf{p}_t \leq \mathbf{a}_t} [u(q_t) - \lambda_t \mathbf{p}_t] \text{ s.t. } \lambda_t \mathbf{p}_t - q_t \geq 0.$$

If  $\lambda_t \mathbf{a}_t \geq q^*$ , the buyer gets  $q_t = q^*$  special goods in exchange for a vector  $\mathbf{p}_t$  of assets with real value  $\lambda_t \mathbf{p}_t = q^* \leq \lambda_t \mathbf{a}_t$ . Else, the buyer gives the seller  $\mathbf{p}_t = \mathbf{a}_t$ , in exchange for  $q_t = \lambda_t \mathbf{a}_t$  special goods. Hence, the quantity of output exchanged is  $q(\lambda_t \mathbf{a}_t)$ , with  $q(\cdot)$  given by (3). Note that,  $\frac{\partial q(\lambda_t \mathbf{a}_t)}{\partial a_t^s} = \lambda_t^s$  if  $\lambda_t \mathbf{a}_t < q^*$ , and  $\frac{\partial q(\lambda_t \mathbf{a}_t)}{\partial a_t^s} = 0$  if  $\lambda_t \mathbf{a}_t \geq q^*$ , and  $\frac{\partial q(\lambda_t \mathbf{a}_t)}{\partial a_t^m} = \frac{\partial q(\lambda_t \mathbf{a}_t)}{\partial a_t^s} \frac{\lambda_t^m}{\lambda_t^s}$ . With the bargaining solution, the value function in the decentralized market becomes

$$V(\mathbf{a}_t, \mathbf{s}_t) = \mathcal{S}(\lambda_t \mathbf{a}_t) + W(\mathbf{a}_t, \mathbf{s}_t), \quad (21)$$

with  $\mathcal{S}(\cdot)$  given by (5).

The agent's problem in the second subperiod is summarized by (20). The optimal consumption of fruit still satisfies (6). The necessary and sufficient first-order conditions for the choices of  $a_{t+1}^s$  and  $a_{t+1}^m$  are<sup>8</sup>

$$\begin{aligned} U'(c_t) \phi_t^s &= \beta E_t \frac{\partial V(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})}{\partial a_{t+1}^s} \\ U'(c_t) \phi_t^m &\geq \beta E_t \frac{\partial V(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})}{\partial a_{t+1}^m}, \text{ " = " if } a_{t+1}^m > 0. \end{aligned}$$

Again, the agent's choices do not depend on his individual asset holdings, and the distribution of assets,  $\mathbf{G}$ , will be degenerate (at the mean) in equilibrium. From (21),  $\frac{1}{\lambda_t^s} \frac{\partial V(\mathbf{a}_t, \mathbf{s}_t)}{\partial a_t^s} = \frac{1}{\lambda_t^m} \frac{\partial V(\mathbf{a}_t, \mathbf{s}_t)}{\partial a_t^m} = 1 - \alpha + \alpha u'[q(\lambda_t \mathbf{a}_t)]$ , so these first-order conditions can be written as

$$\begin{aligned} U'(c_t) \phi_t^s &= \beta \int \{1 - \alpha + \alpha u'[q(\lambda_{t+1} \mathbf{a}_{t+1})]\} \lambda_{t+1}^s dF(d_{t+1}, d_t) \\ U'(c_t) \phi_t^m &\geq \beta \int \{1 - \alpha + \alpha u'[q(\lambda_{t+1} \mathbf{a}_{t+1})]\} \lambda_{t+1}^m dF(d_{t+1}, d_t), \text{ " = " if } m_{t+1} > 0. \end{aligned}$$

Let  $\mu : \Xi \rightarrow \mathbb{R}^+$ , and suppose that the government follows a stationary monetary policy,  $M_{t+1} = \mu(d_t) M_t$ . For the positive analysis I will focus on *admissible* monetary policies, i.e.,  $\mu \in \mathcal{C}^+$ , where  $\mathcal{C}^+$  is the space of continuous, bounded, and nonnegative real-valued functions

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<sup>8</sup>Note that  $\frac{\partial^2 V(\lambda_t \mathbf{a}_t, \mathbf{s}_t)}{\partial (a_t^s)^2} = \alpha u''[q(\lambda_t \mathbf{a}_t)] (\lambda_t^s)^2$  if  $\lambda_t \mathbf{a}_t < q^*$ , and  $\frac{\partial^2 V(\lambda_t \mathbf{a}_t, \mathbf{s}_t)}{\partial (a_t^s)^2} = 0$  if  $\lambda_t \mathbf{a}_t \geq q^*$ . So  $\frac{\partial^2 V(\lambda_t \mathbf{a}_t, \mathbf{s}_t)}{\partial (a_t^s)^2} \leq 0$  (and  $< 0$  if  $\lambda_t \mathbf{a}_t < q^*$ ). Also,  $\frac{\partial^2 V(\lambda_t \mathbf{a}_t, \mathbf{s}_t)}{\partial (a_t^m)^2} = \frac{\partial^2 V(\lambda_t \mathbf{a}_t, \mathbf{s}_t)}{\partial (a_t^s)^2} (\lambda_t^m / \lambda_t^s)^2$ , and  $\frac{\partial^2 V(\lambda_t \mathbf{a}_t, \mathbf{s}_t)}{\partial a_t^s \partial a_t^m} = \frac{\partial^2 V(\lambda_t \mathbf{a}_t, \mathbf{s}_t)}{\partial (a_t^s)^2} \frac{\lambda_t^m}{\lambda_t^s}$ , so  $V(\lambda_t \mathbf{a}_t, d_t)$  is concave in  $\mathbf{a}_t$  (strictly concave for  $\lambda_t \mathbf{a}_t < q^*$ ).

on  $\Xi$ . Let  $\mathbf{s}_t = (x_t, M_t)$  denote the state of the economy at time  $t$ . The transition function  $F$  together with the policy function  $\mu$ , induces a transition function for  $\mathbf{s}_t$ , i.e., if  $\mathbf{s} = (x, M)$ , and  $\mathbf{s}' = (x', M')$ , then  $\Pr(\mathbf{s}_{t+1} \leq \mathbf{s}' | \mathbf{s}_t = \mathbf{s}) = \mathbb{I}_{\{\mu(x)M \leq M'\}} F(x', x) \equiv \mathbf{F}(\mathbf{s}', \mathbf{s})$ . Let  $\Psi$  be the associated stationary distribution, i.e., let  $\Psi$  be the unique solution to  $\Psi(\mathbf{s}') = \int \mathbf{F}(\mathbf{s}', \mathbf{s}) d\Psi(\mathbf{s})$ . I will consider a recursive equilibrium in which all prices are time-invariant functions of the aggregate state,  $\mathbf{s}_t = (x_t, M_t)$ , i.e.,  $w_t = w(\mathbf{s}_t)$ ,  $\phi_t^s = \phi^s(\mathbf{s}_t)$ ,  $\phi_t^m = \phi^m(\mathbf{s}_t)$ , and  $\lambda_t = \lambda(\mathbf{s}_t) = [\lambda^s(\mathbf{s}_t), \lambda^m(\mathbf{s}_t)]$ , where  $\lambda^s(\mathbf{s}_t) \equiv \frac{1}{w(\mathbf{s}_t)} [\phi^s(\mathbf{s}_t) + x_t]$  and  $\lambda^m(\mathbf{s}_t) = \frac{1}{w(\mathbf{s}_t)} \phi^m(\mathbf{s}_t)$ .

**Definition 2** *Given a monetary policy rule  $\mu$ , a recursive equilibrium is a collection of individual decision rules  $c_t = c(\mathbf{s}_t)$ ,  $\mathbf{a}_{t+1} = \mathbf{a}(\mathbf{s}_t) = [a^s(\mathbf{s}_t), a^m(\mathbf{s}_t)]$ , pricing functions  $w_t = w(\mathbf{s}_t)$ ,  $\phi_t^s = \phi^s(\mathbf{s}_t)$ ,  $\phi_t^m = \phi^m(\mathbf{s}_t)$ , and bilateral terms of trade  $q_t = Q(\mathbf{s}_t)$  and  $\mathbf{p}_t = \mathbf{p}(\mathbf{s}_t)$  such that: (i) given prices and the bargaining protocol, the decision rules  $c(\cdot)$ , and  $\mathbf{a}(\cdot)$ , solve the agent's problem in the centralized market; (ii) the terms of trade are determined by Nash bargaining, i.e.,  $Q(\mathbf{s}_t) = q(\lambda(\mathbf{s}_t) \mathbf{a}(\mathbf{s}_t))$  and  $\lambda(\mathbf{s}_t) \mathbf{p}(\mathbf{s}_t) = \min[\lambda(\mathbf{s}_t) \mathbf{a}(\mathbf{s}_t), q^*]$ ; and (iii) prices are such that the centralized market clears, i.e.,  $c(\mathbf{s}_t) = d_t$ ,  $a^s(\mathbf{s}_t) = 1$ . The equilibrium is monetary if  $\phi^m(\mathbf{s}_t) > 0$  for all  $\mathbf{s}_t$ , and in this case the money-market clearing condition is  $a^m(\mathbf{s}_t) = \mu(d_t) M_t$ .*

From (6), if the current state is  $\mathbf{s} = (x, M)$ , then

$$\lambda^s(\mathbf{s}) = U'(x) [\phi^s(\mathbf{s}) + x] \quad \text{and} \quad \lambda^m(\mathbf{s}) = U'(x) \phi^m(\mathbf{s}).$$

In words,  $\lambda^s(\mathbf{s})$  is the real value (in terms of general goods) of an agent's equilibrium equity holding in the search market, and  $\lambda^m(\mathbf{s})$  is the real value of a unit of money (also in terms of marginal utility of fruit). Let  $z(\mathbf{s})$  represent the value of the equilibrium money holdings in state  $\mathbf{s} = (x, M)$ , i.e.,

$$z(\mathbf{s}) \equiv \lambda^m(\mathbf{s}) M, \tag{22}$$

and let  $\Lambda(\mathbf{s})$  be the value (in terms of general goods) of the equilibrium portfolio that an agent carries into the search market in state  $\mathbf{s}$ , i.e.,

$$\Lambda(\mathbf{s}) \equiv \lambda^s(\mathbf{s}) + z(\mathbf{s}).$$

In equilibrium, the Euler equations for equity and money holdings imply

$$\lambda^s(\mathbf{s}_t) = \beta \int L[\Lambda(\mathbf{s}_{t+1})] \lambda^s(\mathbf{s}_{t+1}) d\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t) + x_t U'(x_t) \tag{23}$$

$$z(\mathbf{s}_t) \geq \frac{\beta}{\mu(x_t)} \int L[\Lambda(\mathbf{s}_{t+1})] z(\mathbf{s}_{t+1}) d\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t) \tag{24}$$

respectively, where (24) holds with “=” if  $a^m(\mathbf{s}_t) > 0$ , and

$$L[\Lambda(\mathbf{s}_{t+1})] \equiv 1 - \alpha + \alpha u'(\min\{\Lambda(\mathbf{s}_{t+1}), q^*\}).$$

Note that  $L(\mathbf{s}_{t+1}) \geq 1$  for all dividend realizations,  $x_{t+1} \in \Xi$ , with  $L(\mathbf{s}_{t+1}) > 1$  for  $x_{t+1} \in \Omega^m(\mathbf{s}_t)$ , where

$$\Omega^m(\mathbf{s}_t) = \{x_{t+1} \in \Xi : \lambda^s[x_{t+1}, \mu(x_t) M_t] + \lambda^m[x_{t+1}, \mu(x_t) M_t] \mu(x_t) M_t < q^*\}.$$

Notice that  $z(\mathbf{s}_t) = 0$  for all  $\mathbf{s}_t$  solves (24), and also, that  $z(\mathbf{s}_t) = 0$  for all  $\mathbf{s}_t$  implies  $\Lambda(\mathbf{s}_t) = \lambda^s(\mathbf{s}_t)$  for all  $\mathbf{s}_t$ , so a nonmonetary equilibrium exists provided the functional equation (9) has a solution.

#### 4.1 Observables

In this section I derive expressions for the nominal interest rate, the rate of increase in nominal prices, the return on equity, and other empirically observable functions of the equilibrium allocations and prices. In Section 6, I will discuss the relationships between these variables to highlight some positive predictions of the theory.

**Relative prices.** There are three goods in this economy: special goods, general goods, and fruit. Recall that  $\phi^m(\mathbf{s})$ ,  $\phi^s(\mathbf{s})$ , and  $w(x)$  (which equals  $1/U'(x)$ ) are expressed in terms of fruit. The bargaining solution implies that in every bilateral trade, the buyer hands over a portfolio of assets that is worth  $\min(\Lambda(\mathbf{s}), q^*)$  general goods, in exchange for  $\min(\Lambda(\mathbf{s}), q^*)$  special goods. Hence, the relative price of special goods in terms of general goods equals 1. The following table summarizes the full set of relative prices.

Price of ↓ in terms of →	special good	general good	fruit	money
special good	1	1	$\frac{1}{U'(x)}$	$\frac{1}{U'(x)\phi^m(\mathbf{s})}$
general good	1	1	$\frac{1}{U'(x)}$	$\frac{1}{U'(x)\phi^m(\mathbf{s})}$
fruit	$U'(x)$	$U'(x)$	1	$\frac{1}{\phi^m(\mathbf{s})}$
money	$U'(x)\phi^m(\mathbf{s})$	$U'(x)\phi^m(\mathbf{s})$	$\phi^m(\mathbf{s})$	1

Table 1: Relative prices in state  $\mathbf{s} = (x, M)$

**Nominal interest rate.** In order to derive an expression for the “shadow” nominal interest rate, imagine there existed an additional asset in this economy, a one-period risk-free bond

that pays a unit of money in the centralized market, which cannot be used in the decentralized exchange. Let  $\phi^n(\mathbf{s}_t)$  denote the state- $\mathbf{s}_t$  price of this nominal bond. In equilibrium, this price must satisfy

$$U'(x_t) \phi^n(\mathbf{s}_t) = \beta \int U'(x_{t+1}) \phi^m(\mathbf{s}_{t+1}) d\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t). \quad (25)$$

Notice that  $\frac{\phi^n(\mathbf{s}_t)}{\phi^m(\mathbf{s}_t)}$  is the dollar price of a nominal bond in state  $\mathbf{s}_t$ , so  $1 + i(\mathbf{s}_t) = \frac{\phi^m(\mathbf{s}_t)}{\phi^n(\mathbf{s}_t)}$  is the gross nominal interest rate in state  $\mathbf{s}_t$ . Hence, in a monetary equilibrium,

$$1 + i(\mathbf{s}_t) = \frac{\int L[\Lambda(\mathbf{s}_{t+1})] z(\mathbf{s}_{t+1}) d\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t)}{\int z(\mathbf{s}_{t+1}) d\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t)}. \quad (26)$$

**Inflation.** The price of money,  $\phi^m(\mathbf{s})$ , is quoted in terms of fruit. Since the relative price of fruit in terms of general goods in state  $\mathbf{s} = (x, M)$  is  $U'(x)$ , the price of money in terms of general goods is  $U'(x) \phi^m(\mathbf{s}) = \lambda^m(\mathbf{s})$ . Let

$$\pi_f(\mathbf{s}', \mathbf{s}) = \frac{\phi^m(\mathbf{s})}{\phi^m(\mathbf{s}')} - 1 \quad (27)$$

denote the change in the dollar price of fruit between state  $\mathbf{s} = (x, M)$  and a next-period state  $\mathbf{s}' = (x', \mu(x) M)$  that follows from  $\mathbf{s}$  under a monetary policy  $\mu$ . Similarly, let

$$\pi_g(\mathbf{s}', \mathbf{s}) = \frac{\lambda^m(\mathbf{s})}{\lambda^m(\mathbf{s}')} - 1 \quad (28)$$

denote the change in the dollar price of the general good. Expected (gross) inflation as measured by the dollar price of the general good, conditional on the information available in state  $\mathbf{s} = (x, M)$ , under the monetary policy  $\mu$ , is  $1 + \tilde{\pi}_g(\mathbf{s}) = \int [1 + \pi_g(\mathbf{s}', \mathbf{s})] d\mathbf{F}(\mathbf{s}', \mathbf{s})$ , where  $\mathbf{s}' = (x', \mu(x) M)$ . Expected inflation measured by the dollar price of fruit,  $\tilde{\pi}_f(x)$ , is defined analogously. The average (long-run) inflation rate, measured by the dollar price of good  $i = f, g$ , is  $\bar{\pi}_i = \int \tilde{\pi}_i(\mathbf{s}) d\Psi(\mathbf{s})$ .

**Real interest rate.** In order to derive an expression for the real interest rate, imagine there existed an additional asset in this economy, a one-period risk-free bond that pays a unit of fruit in the centralized market, which cannot be used in the decentralized exchange. Let  $\phi^r(\cdot)$  denote the price function corresponding to this real bond. In equilibrium, it must satisfy

$$U'(x_t) \phi^r(x_t) = \beta \int U'(x_{t+1}) dF(x_{t+1}, x_t). \quad (29)$$



Notice that  $\phi^r(x_t)$  is the relative price of a sure claim to fruit in period  $t + 1$  in terms of fruit in state  $\mathbf{s}_t = (x_t, M_t)$ , so

$$1 + r(x_t) = \frac{1}{\phi^r(x_t)} \quad (30)$$

is the gross real interest rate in state  $\mathbf{s}_t$ . In a monetary equilibrium, we can combine (26) with (22), (24), (29), and (30) to arrive at

$$\frac{1 + r(x_t)}{1 + i(\mathbf{s}_t)} = \int \omega(x_{t+1}, x_t) [1 + \pi_f(\mathbf{s}_{t+1}, \mathbf{s}_t)]^{-1} d\mathbf{F}(\mathbf{s}_{t+1}, \mathbf{s}_t), \quad (31)$$

where  $\omega(x_{t+1}, x_t) \equiv U'(x_{t+1}) [\int U'(x_{t+1}) dF(x_{t+1}, x_t)]^{-1}$ . Condition (31) is a stochastic generalization of the simple Fisher equation: it equates the real-to-nominal gross interest rate ratio, to the conditional expectation of the marginal-utility-of-fruit-weighted reciprocal of the inflation rate (in this case, as measured by the price of fruit).<sup>9</sup>

**Output.** In state  $\mathbf{s} = (x, M)$ , the quantity of fruit equals the endowment,  $x$ . The quantity of special goods produced equals  $\alpha \min(\Lambda(\mathbf{s}), q^*)$ . Production of general goods is carried out by agents who acted as buyers in the previous round of decentralized trade. From the budget constraint that each agent faces in the centralized market, we see that in order to replenish his asset holdings, each agent who was a buyer in the previous decentralized market needs to produce  $\min(\Lambda(\mathbf{s}), q^*)$  general goods. Hence the total output of general goods in state  $\mathbf{s}$  is  $\alpha \min(\Lambda(\mathbf{s}), q^*)$ . Aggregate output, expressed in terms of general (or special) goods, is  $Y_g(\mathbf{s}) = xU'(x) + 2\alpha \min(\Lambda(\mathbf{s}), q^*)$ . Aggregate output expressed in terms of fruit is  $Y_f(\mathbf{s}) = Y_g(\mathbf{s}) / U'(x)$ . Nominal aggregate output in state  $\mathbf{s} = (x, M)$  is  $Y_n(\mathbf{s}) = Y_f(\mathbf{s}) / \phi^m(\mathbf{s})$ .

**Real return on equity.** The real (in terms of fruit) gross return on equity between state  $\mathbf{s} = (x, M)$  and a next-period state  $\mathbf{s}' = (x', M')$  is  $R^s(\mathbf{s}', \mathbf{s}) = \frac{\phi^s(\mathbf{s}') + x'}{\phi^s(\mathbf{s})}$ . The expected return on equity, conditional on the information available in state  $\mathbf{s}$ , under monetary policy  $\mu$  is  $\tilde{R}^s(\mathbf{s}) = \int R^s(\mathbf{s}', \mathbf{s}) d\mathbf{F}(\mathbf{s}', \mathbf{s})$ . The average (long-run) equity return is  $\bar{R}^s = \int \tilde{R}^s(\mathbf{s}) d\Psi(\mathbf{s})$ .

## 5 Optimal monetary policy

The Pareto optimal allocation can be found by solving the problem of a planner who wishes to maximize average (equally-weighted) expected utility. Given the initial condition  $d_0 \in \Xi$ , the

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<sup>9</sup>Svensson (1985) derives an expression like (31) for his cash-in-advance economy.

planner's problem consists of finding the sequence  $\{c_t, q_t\}_{t=0}^{\infty}$  that achieves

$$\begin{aligned} V^* &= \max_{\{c_t, q_t\}_{t=0}^{\infty}} \mathcal{W}(\{c_t, q_t\}_{t=0}^{\infty}) \\ \text{s.t. } & 0 \leq q_t, 0 \leq c_t \leq d_t, \end{aligned} \quad (32)$$

where

$$\mathcal{W}(\{c_t, q_t\}_{t=0}^{\infty}) = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \{ \alpha [u(q_t) - q_t] + U(c_t) \} \right\}.$$

The conditional expectation  $E_0$  is defined with respect to the transition probability  $F(d_{t+1}, d_t)$ . The solution is to set  $\{c_t, q_t\}_{t=0}^{\infty} = \{d_t, q^*\}_{t=0}^{\infty}$ . From Definition 2, it is clear that the competitive allocation in the centralized market always coincides with the efficient allocation. However, the equilibrium allocation may have  $q_t < q^*$  in some states. That is, in general, consumption and production in the decentralized market may be too low in a monetary equilibrium.

**Proposition 1** *Let  $k$  be an arbitrary constant with  $k \geq 1$ , and  $\bar{z} : \mathbb{X} \rightarrow \mathbb{R}^{++}$  be an arbitrary bounded function. Define*

$$z^*(x) = \begin{cases} kq^* - \lambda^s(x) & \text{if } x \in \Omega \\ \bar{z}(x) & \text{if } x \in \Omega^c, \end{cases} \quad (33)$$

where  $\lambda^s(x)$  is the unique continuous, bounded, and strictly positive solution to (15).

(i) *There exists a recursive monetary equilibrium under the monetary policy*

$$\mu^*(x) = \beta \frac{\int z^*(x') dF(x', x)}{z^*(x)}, \quad (34)$$

and the equilibrium prices of equity and money are

$$\phi^s(x) = \frac{\lambda^s(x) - xU'(x)}{U'(x)} \quad (35)$$

$$\phi^m(\mathbf{s}) = \frac{z^*(x)}{U'(x)M} \quad (36)$$

for  $x \in \Xi$ , and  $\mathbf{s} = (x, M) \in \Xi \times \mathbb{R}^+$ .

(ii) *The monetary policy  $\mu^*$  is optimal.*

(iii) *The nominal interest rate is constant and equal to 0 under the policy (34).*

(iv) *The equilibrium state-by-state gross inflation rate is*

$$1 + \pi_f^*(x', x) = \beta \frac{\int z^*(x') dF(x', x)}{z^*(x')} \frac{U'(x')}{U'(x)} \quad (37)$$

if measured by the price of fruit, or

$$1 + \pi_g^*(x', x) = \beta \frac{\int z^*(x') dF(x', x)}{z^*(x')} \quad (38)$$

if measured by the price of general goods. Let  $\bar{\pi}_g^*$  denote the average (long-run) inflation in the price of general goods. Then,  $\bar{\pi}_g^* \geq \beta - 1$ , with strict inequality unless  $z^*$  is a degenerate random variable.

Part (i) of Proposition 1 characterizes a family of optimal stochastic monetary policies indexed by the number  $k$  and the function  $\bar{z}$ . A monetary policy in this class induces  $q_t = q^*$  in every bilateral trade for every realization of the aggregate state. Under (34), the pricing function (35) is identical to the one derived for the nonmonetary economy with  $\Omega = \emptyset$ , which is the pricing function in Lucas (1978). An optimal monetary policy ensures that the marginal return to the agent from carrying an additional dollar into the decentralized market is zero, i.e., equal to the government's (marginal) cost of providing real balances. An optimal policy induces agents to hold just enough money so that  $L[\Lambda(\mathbf{s})] = 1$  with probability 1 in the equilibrium. Part (iii) confirms that a monetary policy in the family characterized in part (i) implements the Friedman rule: it induces a monetary equilibrium with the nominal interest rate equal to zero in every state.<sup>10</sup> In deterministic environments, the Friedman rule can often be described by the simple deterministic monetary policy rule of deflating at the rate of time preference (see, e.g., Lagos and Wright, 2005). In this environment there are stochastic liquidity needs, and the family of optimal policies in (34), is stochastic.<sup>11</sup> According to part (iv), neither (37) nor (38) need to equal  $\beta$ , and in fact, the long-run average inflation rate,  $\bar{\pi}_g^*$ , exceeds  $\beta - 1$  in general.

One way to think of the family defined in (34), is that it is constructed so that the real money balances  $z^*$  as defined in (33) can be part of a monetary recursive equilibrium. Since the constant  $k \geq 1$  and the bounded function  $\bar{z}$  that define the real balances  $z^*$  are arbitrary, it is apparent that there is a large class of monetary policies for which there exists a monetary recursive equilibrium with zero nominal rate in every state. For example, Proposition 1 goes through if we replace  $z^*$  in (33) with any strictly positive, bounded function  $\hat{z}$  with the property

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<sup>10</sup>Even with no illiquid nominal bonds of the kind used to derive the nominal interest rate, (26), this marginal cost of holding nominal money balances (which in equilibrium equals the marginal benefit of holding money) has a natural interpretation as the price that the agent is willing to pay to own a dollar at the beginning of period  $t$  (and be able to use it in exchange) rather than at the end of period  $t$ , after the round of decentralized trade.

<sup>11</sup>In Lagos (2008) I characterize a large family of deterministic monetary policies that implement the Friedman rule in a generalization of this stochastic environment.

that  $\hat{z}(x) + \lambda^s(x) \geq q^*$  for all  $x \in \Xi$ . The class of policies described by  $z^*$  is of interest because as shown below (Proposition 3), it can be obtained as the limit of the class of policies that implement a constant (but possibly nonzero) nominal interest rate, as this target nominal interest rate approaches zero. Proposition 2, to which I turn next, provides another reason why the class of policies described in Proposition 1 is of interest.

Define the *allocation rule*  $Q(\mathbf{s}, \mu) : \Xi \times \mathbb{R}^+ \times \mathcal{C}^+ \rightarrow \mathbb{R}^+$ , and a *price rule*  $\Phi^m(\mathbf{s}, \mu) : \Xi \times \mathbb{R}^+ \times \mathcal{C}^+ \rightarrow \mathbb{R}^+$ . The allocation rule  $Q(\mathbf{s}, \mu)$  specifies the quantity of special goods traded in every bilateral meeting of a monetary recursive equilibrium under the policy rule  $\mu$ , in a period when the aggregate state is  $\mathbf{s}$ . The price rule  $\Phi^m(\mathbf{s}, \mu)$  specifies the value of money (in terms of fruit) in a monetary recursive equilibrium under the policy rule  $\mu$ , in a period when the aggregate state is  $\mathbf{s}$ .

**Proposition 2** *Assume  $B \leq (1 - \beta)q^*$ , and let  $\lambda^s(x)$  be the unique continuous, bounded, and strictly positive solution to (15). Let  $\mu^*$  be as in (34), but with  $z^*(x) = q^* - \lambda^s(x)$  for all  $x \in \Xi$ , and let*

$$\mathcal{M} = \{\mu \in \mathcal{C}^+ : Q(\mathbf{s}, \mu) = q^* \text{ for all } \mathbf{s} \in \Xi \times \mathbb{R}^+\}.$$

*Then  $\mu^* \in \mathcal{M}$ , and for all  $\mu \in \mathcal{M}$ ,  $\Phi^m(\mathbf{s}, \mu^*) \leq \Phi^m(\mathbf{s}, \mu)$  for all  $\mathbf{s} \in \Xi \times \mathbb{R}^+$ .*

Proposition 2 corresponds to a parametrization for which, if the equity was priced as in an economy with no liquidity needs, agents would in fact experience a shortage of liquidity in every state. More formally, the assumption  $B \leq (1 - \beta)q^*$  implies  $\lambda^s(x) \leq q^*$  for all  $x \in \Xi$ , or equivalently,  $\Omega = \Xi$ . The proposition identifies, among the whole class of optimal monetary policies, the monetary policy that minimizes the value of money, and shows that this policy lies within the particular class of optimal policies described in Proposition 1.

## 6 Asset prices, liquidity, and monetary policy

In this section I consider a class of (possibly non-optimal) policies to study the relationship between asset prices, liquidity returns, and monetary policy from a positive standpoint. The class of optimal policies described in Proposition 1 can be characterized by the fact that they induce a nominal interest rate that is: (a) constant, and (b) equal to zero. The following proposition studies equilibrium under policies that induce a constant nominal interest rate, which is possibly higher than zero.

**Proposition 3** Let  $L(\delta) = 1 - \alpha + \alpha u'(\delta q^*)$ , and  $\underline{\delta}$  be defined by  $L(\underline{\delta}) = 1/\beta$ . Let  $\delta_0 \in (\underline{\delta}, 1)$  be given, and suppose that  $B \leq [1 - \beta L(\delta_0)] \delta_0 q^*$ . For any  $\delta \in [\delta_0, 1]$  define

$$z(x; \delta) = \delta q^* - \lambda(x; \delta) \quad (39)$$

where  $\lambda(x; \delta)$  is the unique continuous, bounded, and strictly positive solution to (15), but with  $\beta$  replaced by  $\beta L(\delta)$ .

(i) There exists a recursive monetary equilibrium under the monetary policy

$$\mu(x; \delta) = \beta L(\delta) \frac{\int z(x'; \delta) dF(x', x)}{z(x; \delta)}, \quad (40)$$

and the equilibrium prices of equity and money are

$$\phi^s(x; \delta) = \frac{\lambda(x; \delta) - xU'(x)}{U'(x)} \quad (41)$$

$$\phi^m(\mathbf{s}; \delta) = \frac{z(x; \delta)}{U'(x)M} \quad (42)$$

for  $\mathbf{s} = (x, M) \in \Xi \times \mathbb{R}^+$ .

(ii) The gross nominal interest rate in state  $\mathbf{s}$ ,  $1 + i(\mathbf{s}; \delta)$ , is constant and equal to  $L(\delta)$  under the policy (40).

(iii) The equilibrium state-by-state gross inflation rate is

$$1 + \pi_f(x', x; \delta) = \beta L(\delta) \frac{\int z(x'; \delta) dF(x', x)}{z(x'; \delta)} \frac{U'(x')}{U'(x)} \quad (43)$$

if measured by the price of fruit, or

$$1 + \pi_g(x', x; \delta) = \beta L(\delta) \frac{\int z(x'; \delta) dF(x', x)}{z(x'; \delta)} \quad (44)$$

if measured by the price of general goods.

(iv) Consider the recursive monetary equilibrium induced by the monetary policy (40) with  $\delta \in (\delta_0, 1]$ , and the recursive monetary equilibrium induced by (40) with  $\delta' \in [\delta_0, \delta)$ . Then: (a)  $\lambda(\cdot; \delta) < \lambda(\cdot; \delta')$ , (b)  $z(x; \delta') < z(x; \delta)$ , (c)  $\phi^s(\cdot; \delta) < \phi^s(\cdot; \delta')$ , (d)  $\phi^m(\cdot; \delta') < \phi^m(\cdot; \delta)$ , (e)  $i(\cdot; \delta) < i(\cdot; \delta')$ . Let  $\tilde{\pi}_g(x; \delta) \equiv \int \pi_g(x', x; \delta) dF(x', x)$ , then: (f)  $1 + \tilde{\pi}_g(x; \delta) \geq \beta L(\delta)$  (with strict inequality unless  $\lambda(x; \delta)$  is a degenerate random variable).

(v)  $\lim_{\delta \rightarrow 1} \mu(x; \delta) = \mu(x; 1) = \mu^*(x)$ , with  $\mu^*(x)$  as given in Proposition 2.

The class of monetary policies described in part (i) is indexed by the parameter  $\delta$ , which according to part (ii), effectively determines the level of the constant nominal interest rate that the policy targets. Part (iv) shows that under the proposed policy, the price of equity is increasing in the nominal interest rate target (decreasing in  $\delta$ ), while real balances and the value of money are decreasing in the nominal interest rate implemented by the policy (increasing in  $\delta$ ). Expected inflation as measured by the dollar price of general goods, conditional on the information available in state  $\mathbf{s} = (x, M)$ , and  $\bar{\pi}_g(\delta) \equiv \int \tilde{\pi}_g(x; \delta) d\psi(x)$ , are bounded below by  $\beta L(\delta)$ . Part (v) shows that as  $\delta \rightarrow 1$  the policy  $\mu(x; \delta)$  approaches the optimal policy described in Proposition 2, and therefore the monetary equilibrium characterized by (39)–(42) converges to the efficient equilibrium of Proposition 2.

Proposition 3 provides insights on how the monetary policy (40) can support a recursive monetary equilibrium with a constant nominal interest rate, with the optimal equilibrium in which the nominal rate is constant and zero as a special case. According to (40), the money growth rate should be relatively low in states in which the real value of the equilibrium equity holdings, is below average. For example, with  $\delta = 1$  (the optimal policy of Proposition 2),  $\mu^*(x) < \beta$  if and only if  $\lambda(x) < \int \lambda(x') dF(x', x)$ , and  $\mu^*(x) = \beta$  if  $\lambda(x) = \int \lambda(x') dF(x', x)$ . Something similar happens with the implied inflation rate. From (44), for example, the inflation rate between state  $x$  and a next-period state  $x'$  is relatively low if the realized real value of the equilibrium equity holdings in state  $x'$  is below the conditional expectation held in state  $x$ .

**Corollary 1** *Consider the economy described in Proposition 3, with  $dF(x', x) = dF(x)$ . Then for any  $\delta \in [\delta_0, 1]$ , there exists a recursive monetary equilibrium under the monetary policy*

$$\mu(x; \delta) = \beta L(\delta) \frac{\delta q^* - \frac{1}{1-\beta L(\delta)} \int x' U'(x') dF(x')}{\delta q^* - \frac{\beta L(\delta)}{1-\beta L(\delta)} \int x' U'(x') dF(x') - x U'(x)}, \quad (45)$$

*and the equilibrium prices of equity and money are*

$$\phi^s(x; \delta) = \frac{\beta L(\delta)}{1 - \beta L(\delta)} \frac{\int x' U'(x') dF(x')}{U'(x)} \quad (46)$$

$$\phi^m(\mathbf{s}; \delta) = \frac{\delta q^* - \frac{\beta L(\delta)}{1-\beta L(\delta)} \int x' U'(x') dF(x') - x U'(x)}{U'(x) M} \quad (47)$$

*for  $\mathbf{s} = (x, M) \in \Xi \times \mathbb{R}^+$ . The state-by-state inflation rate measured by the price of fruit is*

$$\pi_f(x', x; \delta) = \frac{U'(x')}{U'(x)} \mu(x'; \delta) - 1. \quad (48)$$

The state-by-state inflation rate measured by the price of general goods is

$$\pi_g(x', x; \delta) = \mu(x'; \delta) - 1. \quad (49)$$

As long as  $B \leq (1 - \beta)q^*$ , the monetary policy and the equilibrium obtained in Corollary 1 with  $\delta = 1$ , coincide with the optimal monetary policy and the efficient equilibrium we would obtain if we assumed  $dF(x', x) = dF(x)$  in Proposition 2. From (45), for every  $\delta \in [\delta_0, 1]$ ,  $\partial\mu(x; \delta)/\partial x$  is proportional to  $1 - \rho(x)$ , i.e., the rate of money creation is procyclical if  $\rho(x) < 1$ . In particular, notice that this is also true for the optimal policy,  $\mu(x; 1)$ . From (46), the level of asset prices is increasing in the nominal interest rate,  $L(\delta)$ , so the state-by-state real return on equity,

$$R^s(x', x; \delta) = \left[ 1 + \frac{1 - \beta L(\delta)}{\beta L(\delta)} \frac{U'(x')x'}{\int x'U'(x')dF(x')} \right] \frac{U'(x)}{U'(x')}, \quad (50)$$

is decreasing in the nominal interest rate. A higher nominal interest rate implies that buyers are on average short of liquidity, so equity becomes more valuable as it is used by buyers to relax their trading constraints. This additional liquidity value means that the real financial return on equity, e.g., (50), will be lower, on average, at a higher interest rate.

**Corollary 2** *Consider the economy described in Proposition 3, with  $U(c) = \log c$ , and assume  $1 < [1 - \beta L(\delta_0)]\delta_0 q^*$ . For any  $\delta \in [\delta_0, 1]$ , there exists a recursive monetary equilibrium under the monetary policy*

$$\mu(x; \delta) = \beta L(\delta), \quad (51)$$

and the equilibrium prices of equity and money are

$$\phi^s(x; \delta) = \frac{\beta L(\delta)}{1 - \beta L(\delta)} x \quad (52)$$

$$\phi^m(\mathbf{s}; \delta) = \frac{\delta q^* - \frac{1}{1 - \beta L(\delta)}}{M} x \quad (53)$$

for  $x \in \Xi$ , and  $\mathbf{s} = (x, M) \in \Xi \times \mathbb{R}^+$ . The gross nominal interest rate in state  $\mathbf{s}$ , i.e.,  $1 + i(\mathbf{s}; \delta)$ , is constant and equal to  $L(\delta)$  under the policy (51). The equilibrium gross state-by-state gross inflation rate is

$$1 + \pi_f(x', x; \delta) = \beta L(\delta) \frac{x}{x'} \quad (54)$$

if measured by the price of fruit, or

$$1 + \pi_g(x', x; \delta) = \beta L(\delta). \quad (55)$$

As long as  $B \leq (1 - \beta) q^*$ , the monetary policy and the equilibrium obtained in Corollary 1 with  $\delta = 1$ , coincide with the optimal monetary policy and the efficient equilibrium we would obtain if we assumed  $U(c) = \log c$  in Proposition 2.

## 7 Conclusion

In this paper I have formulated a model in which agents hold assets not only for their intrinsic value, but also to use them as means of payment. In particular, I have considered a financial structure with two assets, equity shares of a stochastic real exogenous dividend, and fiat money. In this context, I characterized a family of optimal, stochastic monetary policy rules. Every policy in this family implements Friedman’s prescription of zero nominal interest rates. Under an optimal policy, equity prices and returns are independent of monetary considerations.

I have also studied a class monetary policies that target a constant, but nonzero nominal interest rate. For this perturbation of the family of optimal policies, I have found that the model articulates the idea that, to the extent that a financial asset may be used to facilitate transactions, this liquidity service will be priced in the asset and reflected in its measured financial return. In addition, even if the asset is real, as would be the case with an equity share that represents a claim to a real exogenous dividend stream, whenever it serves this liquidity role alongside a monetary asset, the equity price and rate of return will depend on monetary considerations. Since agents are free to use any combination of assets for exchange purposes, even if the equity is real and its dividend exogenous, part of its return will be linked to its liquidity return, and this liquidity return in turn depends on the quantity of real money balances—which is a function of the inflation rate. On average, if the rate of inflation is higher, real money balances are lower, and the liquidity return on equity rises, which causes its price to rise and its real measured rate of return (dividend yield plus capital gains) to fall. This type of logic could help to rationalize the fact that historically, real stock returns and inflation have been negatively correlated—an observation long considered anomalous in the finance literature.<sup>12</sup>

The model has a number of implications for the time-paths of output, inflation, interest rates, equity prices, and equity returns, and it would be interesting to explore these implications further. For example, even though variations in aggregate output are effectively exogenous under the types of monetary policies that were considered, the theory can produce a negative

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<sup>12</sup>See, e.g., Bodie (1976), Bordo et al (2008), Boudoukh and Richardson (1993), Fama (1981), Fama and Schwert (1977), Gultekin (1983), Jaffe and Mandelker (1976), Kaul (1987), Marshall (1992), Nelson (1976).



correlation between the inflation rate and the growth rate of output—a short-run “Phillips curve”—but one that is entirely generated by a monetary policy designed to target a constant nominal interest rate in an economy with stochastic liquidity needs.

## A Proofs

**Proof of Lemma 1.** Let

$$\Upsilon(\Delta) \equiv \int \{1 - \alpha + \alpha u'(\min\{\beta\Delta + xU'(x), q^*\})\} [\beta\Delta + xU'(x)] dF(x) - \Delta, \quad (56)$$

then  $\Upsilon(\Delta) = 0$  is equivalent to (10). Note that  $\Upsilon(0) > 0$ . Also,  $\Upsilon(\Delta) = \int xU'(x) dF(x) - (1 - \beta)\Delta$  for all  $\Delta \geq q^*/\beta$ , so  $\lim_{\Delta \rightarrow \infty} \Upsilon(\Delta) = -\infty$ . Since  $\Upsilon$  is continuous, there exists a  $\Delta > 0$  such that  $\Upsilon(\Delta) = 0$ . Differentiate  $\Upsilon(\Delta)$  to get

$$\Upsilon'(\Delta) = -(1 - \beta) + \alpha\beta \int_{\Omega} [\beta\Delta + xU'(x)] u'' dF(x) + \alpha\beta \int_{\Omega} (u' - 1) dF(x)$$

where  $\Omega = \{x \in \Xi : \beta\Delta + xU'(x) < q^*\}$ , and with  $u''$  and  $u'$  evaluated at  $\beta\Delta + xU'(x)$ . Note that  $\Upsilon(\Delta) = 0$  implies

$$\alpha\beta \int_{\Omega} (u' - 1) dF(x) = 1 - \beta - \varsigma,$$

where  $\varsigma = \frac{1}{\Delta} \int (1 - \alpha + \alpha u') xU'(x) dF(x) > 0$ . Therefore

$$\Upsilon'(\Delta)|_{\Upsilon(\Delta)=0} = \alpha\beta \int_{\Omega} u'' \lambda(z) dF(z) - \varsigma < 0,$$

so  $\Upsilon(\Delta) = 0$  has a unique solution. Finally,

$$\frac{\partial \Upsilon(\Delta)}{\partial \alpha} = \int_{\Omega} (u' - 1) [\beta\Delta + xU'(x)] dF(x) \geq 0, \text{ “} > \text{” if } \Omega \neq \emptyset,$$

so

$$\left. \frac{d\Delta}{d\alpha} \right|_{\Upsilon(\Delta)=0} = \frac{\partial \Upsilon(\Delta)/\partial \alpha}{-\Upsilon'(\Delta)|_{\Upsilon(\Delta)=0}} \geq 0, \text{ “} > \text{” if } \Omega \neq \emptyset. \blacksquare$$

**Proof of Claim 1.** (i) Note that  $q^* - \beta\Delta \leq 0$  if and only if  $\Upsilon(q^*/\beta) \geq 0$ , where the function  $\Upsilon$  is defined in (56). It is straightforward to verify that  $\Upsilon(q^*/\beta) \geq 0$  if and only if  $q^* \leq \frac{\beta}{1-\beta} \int zU'(z) dF(z)$ . But if this is the case, then  $\Omega = \emptyset$  since  $xU'(x) \geq 0$  for all  $x$ .

(ii)  $\rho(x) > 1$  implies that  $xU'(x)$  is strictly decreasing. The condition in part (ii)(a) implies that  $xU'(x) \geq q^* - \Delta$  for all  $x$ . The condition in part (ii)(c) implies that  $xU'(x) < q^* - \Delta$  for all  $x$ . Since  $xU'(x)$  is continuous and strictly decreasing, the condition in part (ii)(b) implies there exists a unique  $x^* \in [\underline{x}, \infty)$  characterized by  $x^*U'(x^*) = q^* - \beta\Delta$  such that  $xU'(x) < q^* - \Delta$  if and only if  $x > x^*$ .

(iii)  $\rho(x) < 1$  implies that  $xU'(x)$  is strictly increasing. The condition in part (iii)(a) implies that  $xU'(x) < q^* - \Delta$  for all  $x$ . The condition in part (iii)(c) implies that  $xU'(x) \geq$

$q^* - \Delta$  for all  $x$ . Since  $xU'(x)$  is continuous and strictly increasing, the condition in part (iii)(b) implies there exists a unique  $x^* \in (\underline{x}, \infty)$  characterized by  $x^*U'(x^*) = q^* - \beta\Delta$ , such that  $xU'(x) < q^* - \Delta$  if and only if  $x < x^*$ .

(iv)  $\rho(x) = 1$  implies  $\Omega = \{x \in \Xi : 1 + \beta\Delta < q^*\}$ , so either  $\Omega = \emptyset$  or  $\Omega = \Xi$ . Note that  $1 + \beta\Delta < q^*$ , and hence  $\Omega = \Xi$ , if and only if  $\Upsilon((q^* - 1)/\beta) < 0$ . It is straightforward to verify that  $\Upsilon((q^* - 1)/\beta) < 0$  if and only if  $q^* > \frac{1}{1-\beta}$ . Hence,  $q^* > \frac{1}{1-\beta}$  implies  $\Omega = \Xi$ , and  $q^* \leq \frac{1}{1-\beta}$  implies  $\Omega = \emptyset$ . ■

**Proof of Lemma 2.** Let  $\mathcal{C}$  denote the space of continuous and bounded real-valued functions defined on  $\mathbb{R}^+$ . The right side of (14) defines a mapping  $T$ , i.e., for any  $g \in \mathcal{C}$ ,

$$(Tg)(x) = xU'(x) + \beta \int \{(1-\alpha)g(x') + \alpha \max[g(x'), 1]\} dF(x', x).$$

$U$  is continuously differentiable, concave, and bounded, with  $U(0) = 0$ , so  $xU'(x) \in \mathcal{C}$ . Also,  $(1-\alpha)g(x) + \alpha \max[g(x), 1] \in \mathcal{C}$ , and since  $F$  has the Feller property,  $Tg \in \mathcal{C}$ . Hence  $T : \mathcal{C} \rightarrow \mathcal{C}$ . Notice that for all  $f, g \in \mathcal{C}$  such that  $f(x) \leq g(x)$  for all  $x \in \mathbb{R}^+$ ,  $(Tf)(x) \leq (Tg)(x)$  for all  $x \in \mathbb{R}^+$ . Let  $\Xi^+ = \{x \in \Xi : g(x) > 1\}$ , and  $\Xi^- = \Xi \setminus \Xi^+$ . Then for all  $k, x \in \mathbb{R}^+$ ,

$$\begin{aligned} [T(g+k)](x) - (Tg)(x) &= \beta \int \{(1-\alpha)[g(x') + k] + \alpha \max[g(x') + k, 1]\} dF(x', x) \\ &\quad - \beta \int \{(1-\alpha)g(x') + \alpha \max[g(x'), 1]\} dF(x', x) \\ &= \beta(1-\alpha)k + \beta\alpha \int \max[g(x') + k, 1] dF(x', x) \\ &\quad - \beta\alpha \int \max[g(x'), 1] dF(x', x) \\ &= \beta(1-\alpha)k + \beta\alpha \int_{\Xi^+} dF(x', x) k \\ &\quad + \beta\alpha \int_{\Xi^-} \max[g(x') - 1 + k, 0] dF(x', x) \\ &\leq \beta k. \end{aligned}$$

Hence, if we let  $\|\cdot\|$  denote the sup norm, i.e.,  $\|f - g\| = \sup_{x \in \mathbb{R}^+} |f(x) - g(x)|$  for any  $f, g \in \mathcal{C}$ ,  $T$  satisfies Blackwell's sufficient conditions (Theorem 3.3 in Stokey and Lucas, 1989), so  $T$  is a contraction mapping on the complete metric space  $(\mathcal{C}, \|\cdot\|)$ . By the Contraction Mapping Theorem (Theorem 3.2 in Stokey and Lucas, 1989), there exists a unique  $\lambda \in \mathcal{C}$  that satisfies  $\lambda = T\lambda$ . To show that  $\lambda > 0$ , for any  $g \in \mathcal{C}$ , define the mapping  $T^+$  as

$$(T^+g)(x) = xU'(x) + \beta \int \{(1-\alpha) \max[g(x'), 0] + \alpha \max[g(x'), 1]\} dF(x', x).$$

Clearly,  $T^+ : \mathcal{C} \rightarrow \mathcal{C}$ , and since  $T^+$  is a contraction on  $(\mathcal{C}, \|\cdot\|)$ , there exists a unique  $\lambda^+ \in \mathcal{C}$  such that  $\lambda^+ = T^+\lambda^+$ . For any  $g \in \mathcal{C}$ ,  $T^+g > 0$ , so  $T^+\lambda^+ = \lambda^+ > 0$ . But notice that for any  $g \in \mathcal{C}$  with  $g > 0$ ,  $Tg = T^+g$ . Therefore  $T\lambda^+ = T^+\lambda^+ = \lambda^+$ , so  $\lambda = \lambda^+ > 0$ . ■

**Proof of Claim 2.** Guess  $\Omega = \Xi$ , and substitute  $U(c) = \varepsilon \log c$  in (17) to arrive at (19). Then note that  $\lambda(x) = \frac{\beta\alpha + \varepsilon}{1 - \beta(1 - \alpha)} < 1$  if and only if  $\varepsilon < 1 - \beta$ , which verifies the guess. Clearly, there cannot exist an equilibrium with  $\Omega = \Xi$  if  $\varepsilon \geq 1 - \beta$ , so in this case the equilibrium must have  $\Omega \subset \Xi$ . ■

**Proof of Proposition 1.** The functional equation (15) is a special case of (14), so by Lemma 2, it has a unique solution  $\lambda^s$  in the space of continuous and bounded functions,  $\mathcal{C}$ , and this solution is strictly positive. (i) To show that there exists a monetary equilibrium under (34), construct one as follows. Let  $z^*$  and  $\lambda^s$  be defined as in the statement of the proposition. Let  $\mathbf{s} = (x, M)$  denote a state, and let  $\mu(x)$  be some monetary policy. Consider the price functions  $\phi^s(x)$  in (35),  $\phi^m(\mathbf{s})$  in (35), and  $w(x) = 1/U'(x)$ , together with the allocation functions  $c(\mathbf{s}) = x$ ,  $a^s(\mathbf{s}) = 1$ ,  $a^m(\mathbf{s}) = \mu(x)M$ , and  $Q(\mathbf{s}) = q(\Lambda(x))$ , with  $\Lambda(x) = U'(x)[\phi^s(x) + x] + z^*(x)$ . The claim to be established is that these allocation and price functions constitute a monetary recursive equilibrium if  $\mu = \mu^*$ . The allocation functions  $c(\mathbf{s})$ ,  $a^s(\mathbf{s})$ , and  $Q(\mathbf{s})$  clearly satisfy the equilibrium conditions. Moreover,  $z^*(x) > 0$  for all  $x$ , so  $\mu^*(x) > 0$  for all  $x$ , which implies that  $a^m(\mathbf{s}) = \mu^*(x)M > 0$  for all  $\mathbf{s}$ , so  $a^m(\mathbf{s})$  is consistent with a monetary equilibrium. Next, we show that the proposed price functions indeed support the proposed allocations as a monetary recursive equilibrium under the monetary policy  $\mu^*$ . If  $\mu = \mu^*$ , in the conjectured equilibrium,

$$\Lambda(x) = \lambda^s(x) + \{\mathbb{I}_{\{x \in \Omega\}}[kq^* - \lambda^s(x)] + \mathbb{I}_{\{x \in \Omega^c\}}\bar{z}(x)\} \geq q^* \quad \text{for all } x \in \Xi, \quad (57)$$

with strict inequality for  $x \in \Omega^c$  (since  $\bar{z}(x) > 0$ ), and strict inequality for  $x \in \Omega$  only if  $k > 1$ . Next, verify that (35) and (36) satisfy the Euler equations (23) and (24) under the monetary policy specified in (34). From (57),  $L[\Lambda(\mathbf{s})] = 1$  for all  $\mathbf{s} \in \Xi \times \mathbb{R}^+$ , so the Euler equation (24) reduces to

$$z(x) \geq \frac{\beta}{\mu(x)} \int z(x') dF(x', x). \quad (58)$$

(This condition would have to hold with “=” if we want to support a monetary equilibrium.) Notice that  $z(x) = z^*(x)$  (simply substitute (36) into the definition (22)), so (58) holds with equality given that  $\mu(x) = \mu^*(x)$ . The fact that under  $\mu^*$ ,  $L[\Lambda(\mathbf{s})] = 1$  for all  $\mathbf{s} \in \Xi \times \mathbb{R}^+$ , also

implies that the Euler equation (35) reduces to (15), and  $\lambda^s$  is the solution to (15), so (35) is satisfied in a monetary equilibrium under (34). Since  $\lambda^s$  solves (15), we have  $\lambda^s(x) - xU'(x) = \beta \int \lambda^s(x') dF(x', x)$ , and since  $\lambda^s(x) > 0$  for all  $x$ ,  $\phi^s(x) > 0$  for all  $x$ . Also, since  $z^*(x) > 0$  for all  $x \in \Xi$ ,  $\phi^m(\mathbf{s}) > 0$  for all  $\mathbf{s} \in \Xi \times \mathbb{R}^+$ , so the equilibrium constructed is indeed monetary. In Lagos (2008) (Proposition 1), I show that the following transversality conditions must be satisfied in any equilibrium

$$\liminf_{t \rightarrow \infty} E_0 \left\{ \beta^t \frac{1}{w_t} \phi_t^s a_{t+1}^s \right\} = 0 \quad (59)$$

$$\liminf_{t \rightarrow \infty} E_0 \left\{ \beta^t \frac{1}{w_t} \phi_t^m a_{t+1}^m \right\} = 0. \quad (60)$$

To conclude, notice that evaluated at the prices and allocations of the proposed equilibrium, the left side of (59) becomes

$$\liminf_{t \rightarrow \infty} E_0 \left\{ \beta^t U'(x_t) \phi^s(x_t) \right\}, \quad (61)$$

and the left side of (60) becomes

$$\liminf_{t \rightarrow \infty} E_0 \left\{ \beta^t U'(x_t) \phi^m(\mathbf{s}_t) \mu^*(x_t) M_t \right\}. \quad (62)$$

With (35), (61) can be written as  $\liminf_{t \rightarrow \infty} E_0 \left\{ \beta^t [\lambda^s(x) - xU'(x)] \right\}$ , and since  $\lambda^s(x) - xU'(x)$  is bounded (see the proof of Lemma 2),

$$\lim_{t \rightarrow \infty} E_0 \left\{ \beta^t U'(x_t) \phi^s(x_t) \right\} = 0.$$

With (34) and (36), (62) can be written as  $\lim_{t \rightarrow \infty} E_0 \left\{ \beta^{t+1} \int z^*(x') dF(x', x_t) \right\}$ , and since  $z^*(x)$  is bounded,

$$\lim_{t \rightarrow \infty} E_0 \left\{ \beta^t U'(x_t) \phi^m(\mathbf{s}_t) \mu^*(x_t) M_t \right\} = 0.$$

Therefore, the proposed allocation and price functions constitute a monetary recursive equilibrium under the monetary policy (34).

(ii) The monetary equilibrium constructed in part (i) has  $c(\mathbf{s}) = x$ , and from (57), also  $Q(\mathbf{s}) = q^*$  for all  $\mathbf{s} = (x, M)$ , so it implements the optimal allocation that solves (32).

(iii) Immediate from (26), and the fact that  $L[\Lambda(\mathbf{s})] = 1$  for all  $\mathbf{s} \in \Xi \times \mathbb{R}^+$  in the proposed equilibrium.

(iv) Combine (36) and (34) with (27) to arrive at (37). Combine (36) and (34) with (28) to arrive at (38). Let  $\tilde{\pi}_g^*(x) = \int \pi_g^*(x', x) dF(x', x)$ , and  $\bar{\pi}_g^* = \int \tilde{\pi}_g^*(x) d\psi(x)$ . With (38) and

Jensen's Inequality,

$$1 + \tilde{\pi}_g^*(x) = \beta \int \frac{z^*(x') dF(x', x)}{z^*(x')} dF(x', x) \geq \beta,$$

for all  $x$ , with strict inequality unless  $z^*(x)$  is a degenerate random variable. This implies  $\bar{\pi}_g^* = \int \tilde{\pi}_g^*(x) d\psi(x) \geq \beta - 1$ . ■

**Proof of Proposition 2.** The functional equation (15) is a special case of (14), so by Lemma 2, it has a unique solution  $\lambda^s$  in the space of continuous and bounded functions,  $\mathcal{C}$ , and this solution is strictly positive. Let  $\mathcal{C}'$  denote the space of continuous real-valued functions bounded by  $q^*$  in the sup norm, and let  $\mathcal{C}''$  be the space of continuous real-valued functions,  $g$ , that satisfy  $\sup_{x \in \Xi} |g(x)| < q^*$ . Let  $T$  be the mapping defined by  $(Tg)(x) = \beta \int g(x') dF(x', x) + xU'(x)$ , so that  $\lambda^s$  satisfies  $\lambda^s = T\lambda^s$ . Suppose  $g \in \mathcal{C}'$ , then

$$|(Tg)(x)| = \left| \beta \int g(x') dF(x', x) + xU'(x) \right| \leq \beta \sup_{x \in \Xi} |g(x)| + \sup_{x \in \Xi} |xU'(x)| < q^*,$$

where the last inequality follows from the fact that  $g \in \mathcal{C}'$ , and

$$\sup_{x \in \Xi} |xU'(x)| < \sup_{x \in \Xi} |U(x)| \leq B,$$

together with the hypothesis  $B \leq (1 - \beta)q^*$ . Thus,  $T(\mathcal{C}') \subseteq \mathcal{C}'' \subseteq \mathcal{C}'$ , and since  $\mathcal{C}'$  is a closed subset of  $\mathcal{C}$ , it follows (Corollary 1 in Stokey and Lucas (1989), p. 52) that  $\lambda^s \in \mathcal{C}''$ . That is,  $0 < \lambda^s(x) < q^*$ , and therefore  $z^*(x) > 0$  for all  $x \in \Xi = \Omega$ .

The monetary equilibrium induced by  $\mu^*$  given  $z^*(x) = q^* - \lambda^s(x)$  for all  $x \in \Xi$ , is the same monetary equilibrium induced by  $\mu^*$  in Proposition 1 given (33), but for the special case with  $\Omega^c = \emptyset$  and  $k = 1$ . Hence,  $Q(\mathbf{s}, \mu^*) = q^*$  for all  $\mathbf{s} \in \Xi \times \mathbb{R}^+$ , so  $\mu^* \in \mathcal{M}$ . Next, define the *allocation rule*  $Z(\mathbf{s}, \mu) : \Xi \times \mathbb{R}^+ \times \mathcal{C}^+ \rightarrow \mathbb{R}^+$ , with the interpretation that  $Z(\mathbf{s}, \cdot)$  represents the value of the equilibrium money holdings in state  $\mathbf{s} = (x, M)$ , as defined in (22). Define the price rule  $\Phi^s(\mathbf{s}, \mu)$ , i.e., the equity price in a monetary recursive equilibrium under the policy rule  $\mu$ , in a period when the aggregate state is  $\mathbf{s}$ . Consider some  $\mu \in \mathcal{M}$ , such that  $\mu \neq \mu^*$ . Since  $\mu \in \mathcal{M}$ ,  $\Phi^s(\mathbf{s}, \mu) = \Phi^s(\mathbf{s}, \mu^*) = \phi^s(x)$ , with  $\phi^s(x)$  as given in (35). Also,  $\mu \in \mathcal{M}$  implies

$$U'(x) [\phi^s(x) + x] + Z(\mathbf{s}, \mu^*) = q^* \leq U'(x) [\phi^s(x) + x] + Z(\mathbf{s}, \mu)$$

for all  $\mathbf{s} \in \Xi \times \mathbb{R}^+$ . Since  $Z(\mathbf{s}, \mu) \equiv U'(x) \Phi^m(\mathbf{s}, \mu) M$ , this inequality implies  $\Phi^m(\mathbf{s}, \mu^*) \leq \Phi^m(\mathbf{s}, \mu)$  for all  $\mathbf{s} \in \Xi \times \mathbb{R}^+$ . ■

**Proof of Proposition 3.** (i) For any given  $\delta \in (\underline{\delta}, 1]$ , the function  $\lambda(x; \delta)$  is the solution to

$$\lambda(x) = \beta L(\delta) \int \lambda(x') dF(x', x) + xU'(x), \quad (63)$$

which is the same as (15), but with discount factor  $\beta L(\delta) \in [\beta, 1]$ , instead of  $\beta \in (0, 1)$ . This functional equation is a special case of (14), so by Lemma 2, for any given  $\delta \in (\underline{\delta}, 1]$ , it has a unique solution  $\lambda(\cdot; \delta)$  in the space of continuous and bounded functions,  $\mathcal{C}$ , and this solution is strictly positive. Let  $\mathcal{C}'$  denote the space of continuous real-valued functions bounded by  $\delta_0 q^*$  in the sup norm, and let  $\mathcal{C}''$  be the space of continuous real-valued functions,  $g$ , that satisfy  $\sup_{x \in \Xi} |g(x)| < \delta_0 q^*$ . Let  $T_\delta$  denote the mapping defined by  $(T_\delta g)(x) = \beta L(\delta) \int g(x') dF(x', x) + xU'(x)$ , so that  $\lambda(\cdot; \delta)$  satisfies  $\lambda(\cdot; \delta) = T_\delta \lambda(\cdot; \delta)$ . If  $g \in \mathcal{C}'$ ,

$$|(T_{\delta_0} g)(x)| = \left| \beta L(\delta_0) \int g(x') dF(x', x) + xU'(x) \right| \leq \beta L(\delta_0) \sup_{x \in \Xi} |g(x)| + \sup_{x \in \Xi} |xU'(x)| < \delta_0 q^*,$$

where the last inequality follows from the fact that  $g \in \mathcal{C}'$ , and

$$\sup_{x \in \Xi} |xU'(x)| < \sup_{x \in \Xi} |U(x)| \leq B,$$

together with the hypothesis  $B \leq [1 - \beta L(\delta_0)] \delta_0 q^*$ . Thus,  $T_{\delta_0}(\mathcal{C}') \subseteq \mathcal{C}'' \subseteq \mathcal{C}'$ , and since  $\mathcal{C}'$  is a closed subset of  $\mathcal{C}$ , it follows (Corollary 1 in Stokey and Lucas (1989), p. 52) that  $\lambda(\cdot; \delta_0) \in \mathcal{C}''$ . Hence

$$0 < \lambda(x; \delta_0) < \delta_0 q^* \quad (64)$$

and therefore  $z(x; \delta_0) > 0$  for all  $x \in \Xi$ .

Next, we establish that if  $\lambda(\cdot; \delta) = T_\delta \lambda(\cdot; \delta)$ , and  $\lambda(\cdot; \delta') = T_{\delta'} \lambda(\cdot; \delta')$ , for  $\delta, \delta' \in [\delta_0, 1]$ , then

$$\delta' < \delta \Rightarrow \lambda(\cdot; \delta) < \lambda(\cdot; \delta'). \quad (65)$$

Let  $h(x) \equiv \lambda(x; \delta') - \lambda(x; \delta)$ , then

$$h(x) = \hat{\beta} \int h(x') dF(x', x) + v(x) \quad (66)$$

where  $\hat{\beta} = \beta L(\delta') \in (0, 1)$ , and  $v(x) \equiv \beta [L(\delta') - L(\delta)] \int \lambda(x'; \delta) dF(x', x)$ , with  $L(\delta') - L(\delta) > 0$ . Notice that  $v \in \mathcal{C}$  and  $v(x) > 0$  for all  $x$ , since  $\lambda(\cdot; \delta) \in \mathcal{C}$ , and  $\lambda(x; \delta) > 0$  for all  $x$  (the properties of  $\lambda(\cdot; \delta)$  follow from Lemma 2, since  $\lambda(\cdot; \delta)$  is the fixed point of (63)). Therefore, by Lemma 2, the fixed point of (66) is strictly positive, i.e.,  $h(x) > 0$  for all  $x$ , which implies (65).

Combined with (64), (65) implies that for every  $\delta \in [\delta_0, 1]$ ,  $\lambda(x; \delta) \leq \lambda(x; \delta_0) < \delta_0 q^* \leq \delta q^*$ , so  $z(x; \delta) > 0$  for all  $x \in \Xi$  and all  $\delta \in [\delta_0, 1]$ .

For a fixed  $\delta \in [\delta_0, 1]$ , the equilibrium is constructed in a similar manner as in the proof of Proposition 2. Let  $z(x; \delta)$  as given in (39) be the value of the equilibrium money holdings (expressed in terms of marginal utility of fruit), then  $\Lambda(\mathbf{s}) = \lambda(x; \delta) + z(x; \delta) = \delta q^*$  for all  $\mathbf{s} = (x, M) \in \Xi \times \mathbb{R}^+$ , which implies  $L[\Lambda(\mathbf{s})] = L(\delta)$  for all  $\mathbf{s}$ . With this, (23) becomes (63), and as stated in the proposition,  $\lambda(\cdot; \delta)$  is the unique fixed point. The definition  $\lambda(x; \delta) \equiv U'(x)[\phi^s(x) + x]$  then gives the equilibrium price function for equity, (41). Under the proposed monetary policy, (40), the Euler equation (24), holds with equality. Finally, (42) is obtained from the definition  $z(x; \delta) \equiv U'(x)\phi^m(\mathbf{s})M$ . The equilibrium is monetary, since  $z(x; \delta) > 0$  for all  $x \in \Xi$ .

(ii) Immediate from (26), and the fact that  $L[\Lambda(\mathbf{s})] = L(\delta)$  for all  $\mathbf{s}$  in the proposed equilibrium.

(iii) The equilibrium state-by-state gross rate of change in fruit price is  $1 + \pi(\mathbf{s}', \mathbf{s}) = \frac{\phi^m(\mathbf{s})}{\phi^m(\mathbf{s}')}$ , which becomes (43) after substituting (42). The equilibrium state-by-state gross rate of change in the price of general goods is  $1 + \pi_g(\mathbf{s}', \mathbf{s}) = \frac{\lambda^m(\mathbf{s})}{\lambda^m(\mathbf{s}')}$ , where  $\lambda^m(\mathbf{s}) = U'(x)\phi^m(\mathbf{s})$ , which gives (44).

(iv) (a) This was shown (i).

(b) From (39) and part (a),

$$z(x; \delta') = \delta' q^* - \lambda(x; \delta') < \delta q^* - \lambda(x; \delta) = z(x; \delta).$$

(c) Immediate from (41) and part (a).

(d) Immediate from (42) and part (b).

(e) Immediate from (ii) and  $L'(\delta) = \alpha u''(\delta q^*) q^* < 0$ .

(f) With (44), and Jensen's Inequality,

$$\int [1 + \pi_g(x', x; \delta)] dF(x', x) = \beta L(\delta) \int \frac{\int z(x'; \delta) dF(x', x)}{z(x'; \delta)} dF(x', x) \geq \beta L(\delta),$$

with strict inequality unless  $z(x; \delta)$  is a degenerate random variable.

(v)  $\lim_{\delta \rightarrow 1} z(x; \delta) = q^* - \lambda(x; 1)$ , where  $\lambda(x; 1)$  is the unique continuous, bounded, and strictly positive solution to (15), i.e.,  $\lambda^s(x)$  in Proposition 2. Thus,  $z(x; 1)$  is the same function as  $z^*(x)$  in Proposition 2. Finally, from (40),

$$\mu(x; 1) = \beta L(1) \frac{\int z(x'; 1) dF(x', x)}{z(x; 1)} = \beta \frac{\int z^*(x') dF(x', x)}{z^*(x)},$$



is the same as  $\mu^*(x)$  described in Proposition 2. ■

**Proof of Corollary 1.** The expressions (45)–(49) are obtained from setting  $F(x', x) = F(x)$  in Proposition 3. Notice that the assumption  $B \leq [1 - \beta L(\delta_0)] \delta_0 q^*$  implies

$$\sup_{x \in \Xi} |x U'(x)| < \sup_{x \in \Xi} |U(x)| \leq B \leq [1 - \beta L(\delta_0)] \delta_0 q^* \leq [1 - \beta L(\delta)] \delta q^*$$

for all  $\delta \in [\delta_0, 1]$ , so just as in Proposition 3,  $\mu(x; \delta) > 0$ , for all  $x \in \Xi$  and  $\phi^m(\mathbf{s}; \delta) > 0$  for all  $\mathbf{s} = (x, M) \in \Xi \times \mathbb{R}^+$ . The value of the equilibrium money holdings in state  $\mathbf{s} = (x, M)$ , under the policy  $\mu(x; \delta)$  in the equilibrium described in the statement, is

$$z(x; \delta) = U'(x) \phi^m(\mathbf{s}; \delta) M = \delta q^* - \frac{\beta L(\delta)}{1 - \beta L(\delta)} \int x' U'(x') dF(x') - x U'(x) > 0$$

for all  $x \in \Xi$ . ■

**Proof of Corollary 2.** The expressions (51)–(55) are obtained from (40)–(44) by substituting  $U'(x) = 1/x$ . For any  $\delta \in [\delta_0, 1]$ , the assumption  $1 < [1 - \beta L(\delta_0)] \delta_0 q^*$  guarantees the equilibrium is indeed monetary, since the assumption implies that  $\phi^m(\mathbf{s}; \delta) > 0$  for all  $\mathbf{s} = (x, M) \in \Xi \times \mathbb{R}^+$  (money is valued), and that  $z(x; \delta) = \delta q^* - \frac{1}{1 - \beta L(\delta)} > 0$  for all  $x \in \Xi$  (the value of the equilibrium money holdings under the policy  $\mu(x; \delta)$  in the equilibrium described in the statement is always positive). ■

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