Supermodular Bayesian Implementation: Learning and Incentive Design*

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Abstract

This paper examines the problem of designing mechanisms with learning properties that help guide agents to play desired equilibrium strategies. I introduce the concept of supermodular implementation where the mechanisms are constructed to induce supermodular games, i.e games with strategic complementarities. These supermodular mechanisms receive the valuable characteristics of supermodular games such as their learning properties. A social choice function (scf) is supermodular implementable if it is implementable with a supermodular mechanism. In quasilinear environments, I prove that if a scf can be implemented by a mechanism that generates bounded strategic substitutes - as opposed to strategic complementarities - then this mechanism can be converted into a supermodular mechanism that implements the scf. If the scf also satisfies some efficiency criterion, then I show that it is supermodular implementable with budget-balancing transfers. Then I address the multiple equilibrium problem. I provide general sufficient conditions for a scf to be implementable with a supermodular mechanism whose equilibria are contained in the smallest interval among all supermodular mechanisms. I also give conditions for supermodular implementability in unique equilibrium. Finally, the paper deals with general preferences by providing a Supermodular Revelation Principle.

Keywords: Implementation, mechanisms, learning dynamics, stability, strategic complementarities, supermodular games.

JEL Classification: C72, D78, D83.

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1 Introduction

The question of how an equilibrium outcome arises in a mechanism is largely open in implementation theory and mechanism design. This literature has produced numerous mechanisms that implement many social choice functions, but theoretical and experimental works reveal that many mechanisms suffer from learning and stability issues.[1] Often mechanisms do not enable boundedly rational agents to *achieve* an equilibrium outcome by learning, if used repeatedly over time. Likewise, slight perturbations in beliefs or behaviors often result in a departure from an equilibrium outcome, posing stability problems. This is particularly troublesome, because the idea behind mechanism design is usually practical in nature: Incentive design explicitly aims to construct mechanisms that *achieve* some desirable outcome in equilibrium. In reality, a static mechanism must sometimes be used repeatedly to reach an outcome. For example, the traffic authorities may set up a toll-system which in the long-run will minimize congestion and allocate users with higher benefits from driving to better roads (Sandholm [50] and [51]). A manager may design the agents' contracts to approach revenue maximization over time. A procurement department may allocate different jobs sequentially to contractors by running an auction several times. A group of scientists may create a control system for planetary exploration vehicles, so that the different units function more efficiently as the mission progresses.[2]

In this paper, I develop the theory of supermodular Bayesian implementation to improve learning and stability in mechanism design. Think of a mechanism as describing the rules of a game: It assigns feasible strategies (or messages) to the agents and specifies how these strategies map into enforceable outcomes. Since players have preferences over the different outcomes, a mechanism induces a game in the traditional sense. If this induced game is supermodular, then the mechanism is said to be supermodular. Then I define a scf to be supermodular implementable if there is a supermodular mechanism whose equilibrium strategies yield that scf as an outcome. Assuming strategies are numbers, a supermodular game is a game with strategic complementarities, i.e a game in which the marginal utility of an agent increases as other players increase their strategies. The complementarities imply that an agent wants to play a larger strategy when the others do the same. For instance, it becomes more desirable for a worker in a firm to increase her effort when others put more effort into their job.

Supermodular implementation has interesting dynamic properties. Best-replies are always increasing in supermodular games; this feature helps boundedly rational agents find their way to equilibrium, for most learning dynamics inherit some monotonicity that guides them "near" the equilibria. This theory thus contributes to fill the important gap in the literature emphasized in Jackson [28]: "Issues such as how well various mechanisms perform when players are not at equilibrium but learning or adjusting are quite important [...] and yet have not even been touched by implementation theory.

[1]Muench and Walker [44] , Cabrales [7] and Cabrales and Ponti [8] show that learning and stability may be serious issues in (resp.) the Groves-Ledyard [23], Abreu-Matsushima [2] and Sjöström [52] mechanisms. On the experimental side, Healy [24] and Chen and Tang [13] provide evidence that convergence of learning dynamics may fail in various mechanisms, such as Proportional Tax or the paired-difference mechanism.

[2]See issues related to cognitive intelligence (Parkes [47] and Tumer and Wolpert [56]).

[This topic] has not been looked at from the perspective of designing mechanisms to have nice learning or dynamic properties." For example, a principal may actually attain revenue maximization by offering the agents a contract that they will face repeatedly for a sufficiently long time. A government may reach an optimal public goods level by repeatedly applying a supermodular tax system.

Supermodular mechanisms are appealing because they receive the theoretical properties of supermodular games. Milgrom and Roberts [39] and Vives [58] show that supermodular games have a largest and a smallest equilibrium and adaptive learners end up playing profiles in between. Adaptive learners regard past play as the best predictor of their opponents' future play and best-respond to their forecast. Cournot dynamics, fictitious play and Bayesian learning are examples of adaptive learning. This convergence result extends to sophisticated learners, who react optimally to what their opponents may next best-respond (Milgrom and Roberts [40]). If a supermodular game has a unique equilibrium, then convergence to the equilibrium is ensured. Adaptive and sophisticated learning encompasses such a wide range of backward and forward-looking behaviors that supermodular mechanisms have very robust learning properties. Supermodular games are also attractive in an implementation framework because their mixed strategy equilibria are locally unstable under monotone adaptive dynamics like Cournot dynamics and fictitious play (Echenique and Edlin [20]). Ruling out mixed strategy equilibria is common in implementation theory and often arbitrary; but it is sensible in supermodular implementation. To the contrary, many pure-strategy equilibria are stable. In a parameterized supermodular game, all those equilibria that are increasing in the parameter are stable, such as the extremal equilibria (Echenique [18]).

Supermodular games and mechanisms are supported by strong experimental evidence. Healy [24] tests five public goods mechanisms in a repeated game setting and observes convergence only in those mechanisms that induce a supermodular game. Experiments on the Groves-Ledyard mechanism have shown that convergence is far better when the punishment parameter is high than when it is low (Chen and Plott [12] and Chen and Tang [13]). The Groves-Ledyard mechanism turns out to be supermodular when the punishment parameter is high. Finally, Chen and Gazzale [15] presents experiments on a game where a parameter determines the degree of complementarity. In this game, they observe that convergence is significantly better when the parameter lies in the range where the game is supermodular.

The methodology used in the paper to derive properties of a mechanism may be promising for mechanism design theory. One striking feature of the traditional design approach is how much it relies on solution concepts to reach certain objectives. For example, if the designer wants the mechanism to be robust to misspecifications of the prior, then she will likely choose implementation in dominant strategies or ex-post equilibrium. Conversely, if the designer targets full efficiency in some quasilinear environment, then she will prefer implementation in Bayesian equilibrium. Economists have attempted to solve nearly all design problems by introducing a solution concept into the implementation framework: Subgame-perfect equilibrium, undominated Nash equilibrium, coalition-proof equilibrium, etc. However, there are interesting properties for mechanisms that are attached to families of games rather than solution concepts.[3]

[3]Sandholm [50] and [51] successfully use implementation in potential games to obtain evolutionary

So why focus on the solution concept? This paper proposes an alternative approach by using a weak solution concept - Bayesian equilibrium - and by instead focusing on a class of games with nice theoretical and experimental properties.

The centerpiece of my analysis is Theorem 1. In quasilinear environments with real type spaces, I prove that if a scf can be implemented by a direct mechanism that generates bounded strategic substitutes - as opposed to strategic complementarities - then this mechanism can be turned into a direct supermodular mechanism that implements the scf. The condition of bounded substitutes is always satisfied on finite type spaces and in twice-continuously differentiable environments with compact type spaces. So, the result is fairly general. The transformation technique is constructive and simple, yet powerful. I explain it in the next section in the context of a public goods example. The transfers can be appended a piece that turns the agents' announcements into complements, and that vanishes in expectation when the opponents play truthfully; thus truthtelling remains an equilibrium after the transformation. That piece is a coordination device that rewards the agent for conforming to the direction and amplitude of her opponents' report and that punishes her for not doing so.

In quasilinear environments, the mechanism designer is often interested in that there be no transfers into or out of the system. This is known as the budget balance condition and it plays an important role in (full) efficiency. Achieving budget balancing is difficult under dominant strategy implementation (Green and Laffont [22]) but possible under Bayesian implementation (Arrow [5] and d'Aspremont and Gérard-Varet [16]). Theorem 2 shows that budget balancing is also possible under supermodular Bayesian implementation. If a scf contains an (allocation) efficient decision rule and admits a mechanism producing bounded substitutes, then it is supermodular implementable with balanced transfers. Interestingly, there are cases where dominant strategy implementation cannot balance the budget, whereas it is possible to balance the budget and induce a supermodular game with a unique equilibrium.

Complementarities help guide agents towards the equilibrium, but they are source of new equilibria with possibly bad outcomes on which agents may coordinate. Supermodular implementation relies on weak implementation, i.e only the truthful equilibrium is known to deliver the desired outcome. Yet the mechanisms here generate a largest and a smallest equilibrium. There is a multiple equilibrium problem and I deal with it by developing optimal and unique supermodular implementation. Optimal supermodular implementation involves designing a supermodular mechanism that generates the weakest complementarities among all supermodular mechanisms. I prove that the interval between the largest and the smallest equilibrium decreases with the complementarities, hence optimal implementation produces the tightest interval around the truthful equilibrium (Proposition 2). Since this interval is "small," learning leads to a profile close to truthtelling and to the desired outcome. The intuition is that agents should be rewarded or punished to adopt monotone behaviors but no more than necessary, otherwise they tend to overreact. The main result (Theorem 3) is that all twice-differentiable scf whose decision rule depends on types through an aggregate are optimally supermodular implementable. Unique supermodular implementation describes that situation where the truthful equilibrium is the unique equilibrium of the induced supermodular game. All

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properties of the mechanism.

dynamics converge to the equilibrium. Theorem 4 gives conditions for unique supermodular implementation. As a by-product, it implies coalition-proof Nash implementation by Milgrom and Roberts [43].

The theory applies to traditional models of public goods or principal multi-agent models. In a public goods example with quadratic preferences, suppose that a designer uses the expected externality mechanism to implement some decision rule (Section 2). In the induced game, many learning dynamics fail to converge to the truthful equilibrium. Nevertheless, the mechanism can be modified to induce a supermodular game where the truthful equilibrium is unique and all dynamics converge to it. In a team-production example, a principal contracts with a set of agents and monitors their contribution to maximize net profits (Section 6.1). The scf is optimally implementable and truthtelling is the unique equilibrium of the induced supermodular game. But there are also challenging applications for the theory such as binary-choice models of auctions and public goods. A possible way around this problem is approximate implementation, where the objective becomes to supermodularly implement scf that are arbitrarily *close* to a "target scf." Most bounded scf admit nearby scf that are supermodular implementable (Section 6.2). The results apply, for instance, to auctions, public goods and bargaining (Myerson and Satterthwaite [46]).

Supermodular implementation is widely applicable in quasilinear environments even though the paper limits attention to direct mechanisms. For general preferences, however, direct mechanisms may be restrictive. The Revelation Principle says that direct mechanisms cause no loss of generality under traditional weak implementation. It is particularly relevant to examine the revelation principle for supermodular implementation, because the space of mechanisms to consider is very large. The Supermodular Revelation Principle (Theorem 5) says that if there exists a mechanism that supermodularly implements a scf such that the range of the equilibrium strategies in the desired equilibrium is a lattice, then there is a direct mechanism that supermodularly implements that scf truthfully. I give an example of a supermodular implementable scf where this range is not a lattice and that cannot be supermodularly implemented by any direct mechanism. This suggests that the condition of the theorem is somewhat minimally sufficient. Although this revelation principle is not as general as the traditional one, it measures the restriction imposed by supermodular direct mechanisms and gives conditions for their use.

A number of other papers are related to learning and stability in the context of implementation or mechanism design. Chen [14] deserves mention because it is one of the first papers explicitly aimed at learning and stability in mechanism design. In a complete information environment with quasilinear utilities, she constructs a mechanism that Nash implements Lindahl allocations and induces a supermodular game. My paper builds the framework of supermodular Bayesian implementation and generalizes her result in incomplete information. Abreu and Matsushima [1] establishes that for any scf f and positive ϵ, there is an ϵ-close scf f_ϵ that admits a mechanism where iterative deletion of strictly dominated strategies leads to a unique profile whose outcome is f_ϵ. Even though their result is general and strong,[4] it can be questioned on the basis of

[4]The solution concept is strong enough to predict convergence of many learning dynamics to the unique equilibrium outcome (See e.g [40]). Note that there are games where some adaptive dynamics from [39] do not converge to a uniquely rationalizable profile.

learning and stability. Following Cabrales [7], when the mechanism implements f_ϵ, it actually implements it in iteratively strictly ϵ-undominated strategies. In other words, elimination of weakly dominated strategies is the solution concept that underlies the exact-implementation problem for f (Abreu and Matsushima [2]); virtual implementation is a way of turning it into elimination of strictly dominated strategies for f_ϵ.[5] Another criticism is that it does not seem to extend to infinite sets of types, which is related to important theoretical questions (Duggan [17]). Their mechanism also employs a message space whose dimension increases to infinity as ϵ vanishes. In contrast, my paper studies exact implementation with direct mechanisms on finite or infinite type sets. Cabrales [7] and Cabrales and Serrano [9] demonstrate that there are learning dynamics that converge to desired equilibrium outcomes in a general framework of (Bayesian) Nash implementation. But those dynamics require players to strictly randomize over all improvements on past play.[6] This rules out many natural learning dynamics considered here. Finally, there are general impossibility results on the stability of equilibrium outcomes in Nash implementation (Jordan [29] and Kim [30]).

The remainder of the paper is organized as follows. Section 2 presents the leading public goods example. Section 3 gives the basic definitions of lattice theory and Section 4 lays out the framework of supermodular implementation. Section 5 contains the main results. Section 6 provides several applications of the theory to traditional models and introduces approximate supermodular implementation. Section 7 presents the supermodular revelation principle. Finally, Section 8 gives an interpretation of learning in Bayesian games and Section 9 concludes.

2 Motivation and Intuition

This section provides an economic example of a designer who uses the expected externality mechanism (Arrow [5] and d'Aspremont and Gérard-Varet [16]) to implement a scf. The environment is simple: Two agents with smooth utilities and compact real type spaces. Yet the mechanism induces a game where learning and stability fail under many dynamics.

Then I describe a new approach where the existing mechanism is modified in order to induce a supermodular game. In the example, the benefit is immediate: All learning dynamics converge to the truthful equilibrium, and the equilibrium is stable.

Consider a principal who needs to decide the level of a public good, such as the size of a bridge. Let $X = [0, 2]$ denote the possible values of the public good. There are two agents, 1 and 2, whose type spaces are $\Theta_1 = \Theta_2 \subset [0, 1]$. Types are independently uniformly distributed. The agents' preferences are quasilinear, $u_i(x, \theta_i) = V_i(x, \theta_i) + t_i$, where $x \in X$, $\theta_i \in \Theta_i$, and $t_i \in \mathbb{R}$ is the transfer from the principal to agent i. The valuation functions are $V_1(x, \theta_1) = \theta_1 x - x^2$ and $V_2(x, \theta_2) = \theta_2 x + x^2/2$.

[5] Elimination of strictly dominated strategies implies robust learning properties, but not for weakly dominated strategies because it has the perverse consequence of excluding limit points of some learning dynamics.

[6] This feature is crucial, for example, to allow play to exit an integer game after players have fallen into it.

The principal wishes to make an allocation-efficient decision, i.e she aims to maximize the sum of the valuation functions by choosing $x^*(\theta) = \theta_1 + \theta_2$. To this end, she wants the agent to reveal their true type, so she opts for the expected externality mechanism.[7] The transfers are set as follows:

$$t_1(\hat{\theta}_1, \hat{\theta}_2) = \frac{1}{2} + \frac{\hat{\theta}_1}{2} + \hat{\theta}_1^2 + \hat{\theta}_2 + \frac{\hat{\theta}_2^2}{2}, \qquad t_2(\hat{\theta}_1, \hat{\theta}_2) = -t_1(\hat{\theta}_1, \hat{\theta}_2)$$

Consider the straightforward application of learning to the (ex-ante) Bayesian game induced by this mechanism (See Section 8).[8] I will study convergence and stability of learning dynamics. Time proceeds in discrete periods $t \in \{0, 1, \ldots\}$ and agents are assumed to learn as time passes according to some rule. The strategies at time 0 are given exogenously. The agents observe the history of play from 0 to $t-1$ and then publicly play a strategy at t. More precisely, from the strategies played in the past, each agent updates her beliefs about her opponent's future strategy using some specified rule; then, given those updated beliefs, she plays the strategy which maximizes her current expected payoffs in the mechanism. In this context, a strategy is a *deception*, which is a contingent plan that specifies a type to be announced for each of an agent's possible types, and that she commits to follow after learning her type. Formally, a deception for i at period t is a function $\hat{\theta}_i^t : \Theta_i \to \Theta_i$.[9]

The questions are: Will the profile played at t converge to the truthful equilibrium as $t \to \infty$? If players were in the truthful equilibrium, will they return to this equilibrium after an exogenous perturbation? The first question asks whether the agents ever learn to play truthfully. The second one asks whether truthtelling is a stable equilibrium.

The players' payoffs determine the answers. For $i = 1, 2$, define the set of deceptions Σ_i as the set of measurable functions from Θ_i into Θ_i, and let $\mathcal{P}(\Sigma_i)$ be the set of (Borel) probability measures over Σ_i. Let $\mu_i^t \in P(\Sigma_j)$ be player i's beliefs about player j's deceptions at time t. A learning model is defined by a rule that takes the history of play as input and that generates beliefs μ_i^t as output.

Letting $cplt(i) = 2(-1)^i/i$, player i's expected utility in the mechanism is

$$E[u_i|\mu_i^t] = -\frac{\hat{\theta}_i^2}{2} + \big(\theta_i + cplt(i) E\big[\, E_{\theta_j}[\hat{\theta}_j^t(\theta_j)] \,|\, \mu_i^t\,\big] - (-1)^i/i\,\big)\hat{\theta}_i \tag{1}$$

up to a constant, where $E[.|\,\mu_i^t]$ is i's expectation over Σ_j (j's deceptions) given her beliefs μ_i^t.

In (1), *cplt* determines how players' strategies depend on one another. Since $cplt(1) < 0$ and $cplt(2) > 0$, if player 1 believes player 2's strategy has increased on average, then 1 decreases her strategy and vice-versa; whereas 2 tries to match any average-variation in 1's strategy. Players essentially chase one another, and so this game has a flavor of "matching-pennies" that will be the source of instability and learning deficiency.

[7] This mechanism allows truthful implementation of allocation-efficient decision rules (See [5], [16] or Section 23.D in Mas-Colell et al. [36]) i.e truthtelling is a Bayesian equilibrium of the mechanism.

[8] See Chapter 1 of Fudenberg and Levine [21] for a justification and discussion of myopic learning.

[9] Announcing a deception in the Bayesian game might seem more realistic when type sets are finite (the example has similar conclusions in the finite case), but here it will come down to choosing an intercept between -1 and 1.

Learning often fails to occur in this example. There are many learning dynamics for which, not only do the agents not converge to truth-revealing but the play cycles forever. Consider first weighted fictitious play (See e.g Ho [25]) in the case where, for simplicity, types are in $\{0, .5, 1\}$. So Σ_i is finite. Deceptions are initially assigned arbitrary weights and the beliefs are given by the frequencies of the different deceptions in the total weight. Given $0 < \pi < 1$, beliefs are updated each period by multiplying all weights by $1-\pi$ and by adding one to the weight of the opponent's deception played at the last period. If players use an identical rule π, the profile converges to the truthful equilibrium unless π is too high ($\pi > .8$), in which case cycling occurs. But there is no reason a priori for both players to use the same learning rule. For asymmetric rules, learning becomes more uncertain. The player with the highest π often outweighs the other one in a non-linear fashion and prevents learning.[10]

Consider now the model with continuous types in $[0, 1]$. A dynamics is said to be Cournot if each player believes that her opponents will play at t what they played at $t-1$. In the example, Cournot dynamics cycles and this conclusion holds wherever the dynamics starts (except truthtelling). Besides, if the agents were to play the truthful equilibrium, the slightest belief perturbation would destabilize it under Cournot adjustment.

Cournot dynamics is prone to cycling, because the past only matters through the last period. But cycling prevails for many families of dynamics with a larger memory size, where for example players remember the last T periods and believe that a probability distribution over their opponents' past strategies best describe their future behavior.[11]

Learning also fails for other forms of learning dynamics than adaptive dynamics, such as the sophisticated learning dynamics à la Milgrom-Roberts [40].

Although strategic complementarities are not necessary for convergence, their absence clearly causes the learning failures in the example.

The theory I develop suggests to transform an existing mechanism into one which induces a supermodular game. The main insight is to use transfers to create complementarities between agents' announcements. The general transformation technique is simple and efficient. After transforming the mechanism, all adaptive and sophisticated dynamics converge to the truthful equilibrium, and the equilibrium is stable.

Consider the above two-agent environment and recall that truthtelling is a Bayesian equilibrium in the expected externality mechanism. Now player 1 could be subsidized if she accepts to change the value of the public good as 2 wishes, and taxed otherwise. From 1's point of view, 2 prefers large values of the public good when 2 reports large types on average, i.e $E_{\theta_2}[\hat{\theta}_2(.)] \geq E_{\theta_2}[\theta_2]$. If 2 prefers small values, then the inequality

[10]If 1 learned according to a fictitious play rule with π_1 while 2 used π_2, then the sequence would enter a cycle for many values of $\pi_1 \geq .9, \pi_2 \geq .55$

[11]Consider dynamics where players remember the last T periods. They assign a probability π to the deception played at $t-1$ and $(1-\pi)\delta^k/C$ to that played at $t-k$ where C is normalized so that the probabilities add up to one. Simulations reveal that learning fails under many values of the parameters. Let $(\hat{\theta}_1^0(.), \hat{\theta}_2^0(.))$ be the pair of zero-functions. For $T \in \{2, 3\}, \delta = .9$ and $\pi \geq .5$, the process enters a cycle even though the last few periods are weighted almost equally. This suggests that increasing the memory size may improve learning. For $T = 4, \delta = .8$ and $\pi \leq .65$, the profile converges to the truthful equilibrium, but it cycles for $\pi \geq .7$. But a larger memory does not necessarily improve learning, as cycling reappears when $T = \{5, 6\}, \delta = .8$ for values of π below .65.

is reversed. The new tax system could subsidize 1 if 1 reports large types when 2 does so, and tax 1 if 1 still reports large types when 2 does not. Possible transfers $t_1^{SM}(.)$ accomplishing this task are constructed by appending $\rho_1\hat{\theta}_1(\hat{\theta}_2 - E_{\theta_2}[\theta_2])$ to the current transfers, where ρ_1 is an arbitrary parameter capturing the punishment or reward intensity:

$$t_1^{SM}(\hat{\theta}) = E_{\theta_2}[t_1(\hat{\theta}_1, \theta_2)] + \rho_1\hat{\theta}_1(\hat{\theta}_2 - E_{\theta_2}[\theta_2])$$

Agent 2's transfers are modified similarly with parameter ρ_2. The intuition is that there should be ρ_1 large enough such that, regardless of 1's original incentives, the reward (punishment) for (not) following 2 now is so high that 1 becomes willing to follow 2 along any learning dynamics. But by doing so, we actually created a supermodular mechanism. Note $\partial^2 t_1^{SM}(\hat{\theta})/\partial\hat{\theta}_1\partial\hat{\theta}_2 = \rho_1$. Thus, if $\partial^2 V_1(x_1(\hat{\theta}), \theta_1)/\partial\hat{\theta}_1\partial\hat{\theta}_2$ is bounded below, a condition called *bounded substitutes*,[12] then there is ρ_1 large enough such that

$$\frac{\partial^2 V_1(x_1(\hat{\theta}_1, \hat{\theta}_2), \theta_1)}{\partial\hat{\theta}_1\partial\hat{\theta}_2} + \frac{\partial^2 t_1^{SM}(\hat{\theta}_1, \hat{\theta}_2)}{\partial\hat{\theta}_1\partial\hat{\theta}_2} \geq 0, \text{ for all } \hat{\theta}, \theta_1. \quad (2)$$

A similar equation holds for agent 2, which implies that the Bayesian game induced by the mechanism is supermodular.[13] Further, t_1^{SM} and t_1 have the same expectation when the opponents play truthfully: $E_{\theta_2}[t_1^{SM}(., \theta_2)] = E_{\theta_2}[t_1(., \theta_2)]$. Thus if 1's best-reply under t_1 was to tell the truth when 2 played truthfully, then it must be the case under t_1^{SM}. So truthtelling is an equilibrium after modifying the transfers.

In addition to its intuitive appeal, this technique can be powerful. Theorem 4 of Section 5.3 implies that there are values ρ_1 and ρ_2 for which truthtelling is the unique equilibrium of the supermodular mechanism in this example. All adaptive dynamics now converge to the truthful equilibrium, and the equilibrium is stable.

3 Lattice-theoretic Definitions and Supermodular Games

The basic definitions of lattice theory in this section are discussed in Milgrom-Roberts [39] and Topkis [54].

A set M with a transitive, reflexive, antisymmetric binary relation $\succeq$ is a *lattice* if for any $x, y \in M$, $x \vee y \equiv \sup_M\{x, y\}$ and $x \wedge y \equiv \inf_M\{x, y\}$ exist. It is *complete* if for every non-empty subset A of M, $\inf_M A$ and $\sup_M A$ exist. A nonempty subset A of M is a *sublattice* if for all $x, y \in A$, $x \vee y, x \wedge y \in A$. A *closed interval* $[x, y]$ in M is the set of $m \in M$ such that $y \succeq m \succeq x$. The *order-interval* topology on a lattice is the topology whose subbasis for the closed sets is the set of closed intervals. All lattices in the paper are endowed with their order-interval topology. In Euclidean spaces the order-interval topology coincides with the usual topology.

Let T be a partially ordered set; $g : M \to \mathbb{R}$ is *supermodular* if, for all $m, m' \in M$, $g(m) + g(m') \leq g(m \wedge m') + g(m \vee m')$; $g : M \times T \to \mathbb{R}$ has *increasing (decreasing)*

[12] This condition is satisfied in the present public goods example.

[13] If the complete information payoffs define a supermodular game for each $\theta \in \Theta$, then the (ex-ante) Bayesian game is supermodular. Loosely speaking, supermodular games are characterized by utility functions whose cross-partial derivatives are positive.

differences in (m,t) if, whenever $m \succeq m'$ and $t \succeq t'$, $g(m,t) - g(m',t) \geq (\leq) g(m,t') - g(m',t')$; $g : M \times T \to \mathbb{R}$ satisfies the *single-crossing property* in (m,t) if, whenever $m \succeq m'$ and $t \succeq t'$, $g(m'',t') \geq g(m',t')$ implies $g(m'',t'') \geq g(m',t'')$ and $g(m'',t') > g(m',t')$ implies $g(m'',t'') > g(m',t'')$. If g has decreasing differences in (m,t), then variables m and t are said to be substitutes. If g has increasing differences or satisfies the single-crossing property in (m,t), then m and t are said to be complements.

A game is described by a tuple $(N, \{(M_i, \succeq_i)\}, u)$, where N is a finite set of players; each $i \in N$ has a strategy space M_i with an order $\succeq_i$ and a payoff function $u_i : \prod_{i \in N} M_i \to \mathbb{R}$; and $u = (u_i)$.

DEFINITION 1 *A game $\mathcal{G} = (N, \{(M_i, \succeq_i)\}, u)$ is supermodular if for all $i \in N$,*

1. *$(M_i, \succeq_i)$ is a complete lattice;*
2. *u_i is bounded, supermodular in m_i for each m_{-i} and has increasing differences in (m_i, m_{-i});*
3. *u_i is upper-semicontinuous in m_i for each m_{-i}, and continuous in m_{-i} for each m_i.*

4 Supermodular Implementation: The Framework

Let $N = \{1, \ldots n\}$ denote a collection of agents. A planner faces a measurable set Y of alternatives with generic element $y \in Y$. For each agent $i \in N$, let Θ_i be the measurable space of i's possible types. Let $\Theta_{-i} = \prod_{j \neq i} \Theta_j$. Agents have a common prior ϕ on Θ known to the planner. The planner's desired outcomes are represented by a measurable social choice function $f : \Theta \to Y$.

A mechanism is a tuple $\Gamma = (\{(M_i, \succeq_i)\}, g)$ where each agent i's message space M_i is endowed with an order $\succeq_i$ and is a measurable space; $g : M \to Y$ is a measurable outcome function. A strategy for agent i is a measurable function $m_i : \Theta_i \to M_i$. Denote by $\Sigma_i(M_i)$ the set of equivalence classes of measurable functions from $(\Theta_i, \mathcal{F}_i)$ to M_i. This set is endowed with the pointwise order, also denoted $\succeq_i$. A direct mechanism is one for which each $M_i = \Theta_i$ and $g = f$. In this case, $\Sigma_i(\Theta_i)$ is called the set of i's deceptions and its elements are denoted $\hat{\theta}_i(.)$. Direct mechanisms vary by the order on type spaces.

Each agent i's preferences over alternatives are given by a measurable utility function $u_i : Y \times \Theta_i \to \mathbb{R}$. These utility functions are uniformly bounded by some $\overline{u}$. For $m_{-i} \in \prod_{j \neq i} M_j$, agent i's preferences over messages in M_i are given by her ex-post payoffs $u_i(g(m_i, m_{-i}), \theta_i)$. Agent i's ex-ante payoffs are defined as $u_i^g(m_i(.), m_{-i}(.)) = E_\theta[u_i(g(m_i(\theta_i), m_{-i}(\theta_{-i}), \theta_i)]$ for any profile $m(.)$, where $E_\theta[.]$ is the expectation with respect to ϕ.

There are three stages at which it is relevant to formulate the game induced by mechanism Γ: Ex-ante, interim and ex-post (complete information). The paper mostly adopts an ex-ante perspective, as the objective is that the ex-ante induced game $\mathcal{G} = (N, \{(\Sigma_i(M_i), \succeq_i)\}, u^g)$ be supermodular (See Section 8). However, if message sets are compact sublattices of some Euclidean space, then a sufficient condition for $\mathcal{G}$ to be supermodular is that the complete information game induced by Γ be supermodular for every profile of true types. This explains the next definitions. If a scf is Bayesian implementable with a mechanism that always induces an ex-post supermodular game, then it is supermodular implementable.

Definition 2 *The mechanism* Γ *supermodularly implements the scf* $f(.)$ *if there exists a Bayesian equilibrium* $m^*(.)$ *such that* $g(m^*(\theta)) = f(\theta)$ *for all* $\theta \in \Theta$*, and if the induced game* $\mathcal{G}(\theta) = (N, \{(M_i, \succeq_i)\}, u(g(.), \theta))$ *is supermodular for all* $\theta \in \Theta$*. The scf* f *is said to be supermodular implementable.*[14]

Definition 3 *A scf is truthfully supermodular implementable if there exists a direct mechanism that supermodularly implements the scf* $f(.)$ *such that* $\hat{\theta}(\theta) = \theta$ *for all* $\theta \in \Theta$ *is a Bayesian equilibrium.*

Since the paper is mostly concerned with direct Bayesian mechanisms, I often omit the qualifications of "truthful," "truthfully" and "Bayesian."

5 Supermodular Implementation on Quasilinear Domains

This section deals with supermodular implementation when agents have quasilinear utility functions. The objective is to give general conditions under which a scf is supermodular implementable and the mechanism satisfies some further requirements. There are four main results. The first provides general conditions for supermodular implementability. The second answers the question of supermodular implementation and budget balancing. The third gives sufficient conditions for a scf to be supermodular implementable with a game whose interval between extremal equilibria is the smallest possible. The fourth offers sufficient conditions for supermodular implementability in unique equilibrium.

5.1 Environment and Definitions

An alternative y is a vector $(x, t_1, \ldots, t_n)$ where x is an element of a compact set $X \subset \mathbb{R}^m$ and $t_i \in \mathbb{R}$ for all i. Each agent i has a type space $\Theta_i \subset \mathbb{R}$ (finite or infinite). Endow Θ_i with the usual order. Notice that $\Sigma_i(\Theta_i)$ is a complete lattice with the pointwise order.[15]

Let X_i be a compact subset of $\mathbb{R}^{m_i}$ such that $X_i = X$ or $\prod_{i \in N} X_i = X$. For all $i \in N$, preferences are quasilinear with utility function $u_i(x, \theta_i) = V_i(x_i, \theta_i) + t_i$ where $x_i \in X_i$. The function $V_i : X_i \times \Theta_i \to \mathbb{R}$ is called i's valuation function and the vector of those valuations is denoted V.

In this environment, a scf $f = (x, t)$ is composed of a decision rule $x : \Theta \mapsto (x_i(\theta))$ where $x_i : \Theta \to X_i$, and transfer functions $t_i : \Theta \to \mathbb{R}$.

[14]Definitions 2 and 3 are also simplifying definitions. It is sufficient but not necessary that $\mathcal{G}(\theta)$ be supermodular for each θ in order for the ex-ante Bayesian game to be supermodular. For example, if the prior is mostly concentrated on some subset Θ' of Θ, it may not be necessary to make the ex-post payoffs supermodular for types in $\Theta \backslash \Theta'$. Of course, the possibility of neglecting $\Theta \backslash \Theta'$ depends on how unlikely that set is compared to how submodular the utility function may be for types in that set.

[15]See Lemma 1 in Van Zandt [57].

Say that the valuation functions and the decision rule are twice-continuously differentiable if for all i, there exist open sets $O_i \supset \Theta_i$ and $U_i \supset X_i$, such that $V_i : U_i \times O_i \to \mathbb{R}$ and $x_i : \prod_{i \in N} O_i \to U_i$ are twice-continuously differentiable.

The valuation functions and the decision rule form a *continuous family* if for all i, V_i is bounded, $V_i(x_i(\hat{\theta}), \theta_i)$ is continuous in $\hat{\theta}_{-i}$ for fixed $\hat{\theta}_i$ and θ_i, and $V_i(x_i(\hat{\theta}), \theta_i)$ is upper-semicontinuous in $\hat{\theta}_i$ for fixed $\hat{\theta}_{-i}$ and θ_i.

Agents' types are assumed to be independently distributed. For all i, the distribution of i's types admits a bounded density with full support.

Here a scf f is *(truthfully) supermodular implementable* if truthtelling is a Bayesian equilibrium of the supermodular game induced by the direct mechanism.

The next definitions describe conditions on the composition of the valuation functions and the decision rule.

For any $\theta'_i, \theta''_i \in \Theta_i$, let $\Delta V_i((\theta''_i, \theta'_i), \hat{\theta}_{-i}, \theta_i) = V_i(x_i(\theta''_i, \hat{\theta}_{-i}), \theta_i) - V_i(x_i(\theta'_i, \hat{\theta}_{-i}), \theta_i)$.

Say that the valuation functions and the decision rule (V, x) produce *bounded substitutes*, if for all $i \in N$, there is $T_i \in \mathbb{R}$ such that, for all $\theta''_i \geq \theta'_i$ and $\theta''_{-i} \geq \theta'_{-i}$, $\Delta V_i((\theta''_i, \theta'_i), \theta''_{-i}, \theta_i) - \Delta V_i((\theta''_i, \theta'_i), \theta'_{-i}, \theta_i) \geq T_i(\theta''_i - \theta'_i) \sum_{j \neq i}(\theta''_j - \theta'_j)$ for all $\theta_i \in \Theta_i$. Equivalently, substitutes are said to be bounded by T_i. The condition requires the difference quotient of any player's marginal valuation to be uniformly bounded below.[16] In twice-differentiable environments, this is equivalent to the existence of a uniform lower bound on the cross-partial derivatives. In other words, if agents' announcements are strategic substitutes in the game with no transfers,[17] so $\partial^2 V_i(x_i(\hat{\theta}), \theta_i)/\partial \hat{\theta}_i \partial \hat{\theta}_j < 0$, then at least there is a bound on the negative magnitude of these cross-partial derivatives. Notice that this assumption is always satisfied when type sets are finite. Moreover, it is also satisfied whenever the decision rule and the valuation functions are twice-continuously differentiable functions on compact type sets.

Say that the composition of the valuation functions and the decision rule is *ω-Lipschitz* if for each $i \in N$, there exists $\omega_i > 0$ such that for all $\hat{\theta}_{-i}$ and θ_i, $\Delta V_i((\theta''_i, \theta'_i), \hat{\theta}_{-i}, \theta_i) \leq \omega_i(\theta''_i - \theta'_i)$, for all $\theta''_i \geq \theta'_i$. The same definition applies to transfer functions. In differentiable environments, it simply means that the corresponding first-derivatives are uniformly bounded above.

Say that the composition of the valuation functions and the decision rule has *γ-increasing differences* if for each $i \in N$, there is $\gamma_i > 0$ such that for all $\hat{\theta}''_i \geq \hat{\theta}'_i$ and $\theta''_i \geq \theta'_i$, $E_{\theta_{-i}}[\Delta V_i((\hat{\theta}''_i, \hat{\theta}'_i), \theta_{-i}, \theta''_i)] - E_{\theta_{-i}}[\Delta V_i((\hat{\theta}''_i, \hat{\theta}'_i), \theta_{-i}, \theta'_i)] \geq \gamma_i(\hat{\theta}''_i - \hat{\theta}'_i)(\theta''_i - \theta'_i)$. This condition requires the expected marginal valuation to be sufficiently increasing in a player's true type.

Note that the conditions of bounded substitutes, ω-Lipschitz and γ-increasing differences are simple bounds on derivatives, generalized to hold in non-differentiable environments.

5.2 General Result and Implementation with Budget Balance

This subsection contains two main results. According to the first theorem, if the scf and the utility functions are relatively well-behaved, in the sense of continuous families

[16] Recall Section 2.

[17] See Section 3

and bounded substitutes, then a decision rule is implementable with transfers if and only if it is supermodular implementable with transfers. The second theorem provides sufficient conditions to satisfy budget balancing.

THEOREM 1 *Let decision rule $x(.)$ and the valuation functions form a continuous family with bounded substitutes. There exist transfers t such that $f=(x,t)$ is implementable and $E_{\theta_{-i}}[t_i(.,\theta_{-i})]$ is upper-semicontinuous (usc), if and only if, there are transfers t^{SM} such that (x,t^{SM}) is supermodular implementable and $E_{\theta_{-i}}[t_i^{SM}(.,\theta_{-i})]$ is usc. Moreover, transfers t_i and t_i^{SM} have the same expected value at the truthful equilibrium.*

PROOF: Sufficiency is immediate. So suppose that $f=(x,t)$ is Bayesian implementable and transfers t are such that $E_{\theta_{-i}}[t_i(.,\theta_{-i})]$ is usc for all i. Then,

$$E_{\theta_{-i}}[V_i(x_i(\theta_i,\theta_{-i}),\theta_i)]+E_{\theta_{-i}}[t_i(\theta_i,\theta_{-i})]\geq E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i,\theta_{-i}),\theta_i)]+E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})] \quad (3)$$

for all $\hat{\theta}_i$. For $\rho_i\in\mathbb{R}$, let

$$\delta_i(\hat{\theta}_i,\hat{\theta}_{-i})=\sum_{j\neq i}\rho_i\hat{\theta}_i\hat{\theta}_j, \quad (4)$$

and define

$$t_i^{SM}(\hat{\theta}_i,\hat{\theta}_{-i})=\delta_i(\hat{\theta}_i,\hat{\theta}_{-i})+E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})]-E_{\theta_{-i}}[\delta_i(\hat{\theta}_i,\theta_{-i})]. \quad (5)$$

Note that transfers t_i and t_i^{SM} have the same expected value: $E_{\theta_{-i}}[t_i^{SM}(.,\theta_{-i})]=E_{\theta_{-i}}[t_i(.,\theta_{-i})]$. Thus (x,t^{SM}) is Bayesian implementable by (3). Moreover, $\delta_i:\Theta\rightarrow\mathbb{R}$ is continuous and bounded. So it follows from the Bounded Convergence Theorem that $E_\theta[\delta_i(\hat{\theta}_i(\theta_i),\hat{\theta}_{-i}\ (\theta_{-i}))-E_{\theta_{-i}}[\delta_i(\hat{\theta}_i(\theta_i),\theta_{-i})]]$ is continuous in $\hat{\theta}(.)$. Since transfers t are such that $E_{\theta_{-i}}[t_i(.,\theta_{-i})]$ is usc, Fatou's Lemma implies that $E_\theta[t_i^{SM}(\hat{\theta}_i(\theta_i),\hat{\theta}_{-i}(\theta_{-i}))]$ is usc in $\hat{\theta}_i(.)$ for each $\hat{\theta}_{-i}(.)$. Therefore, payoffs u_i^f satisfy the continuity requirements for supermodular games. Next I show that it is possible to choose ρ_i so that u_i^f has increasing differences in $(\hat{\theta}_i(.),\hat{\theta}_{-i}(.))$. By bounded substitutes, there exists T_i such that, for all $\theta_i''\geq\theta_i'$ and $\theta_{-i}''\geq\theta_{-i}'$, $\Delta V_i((\theta_i'',\theta_i'),\theta_{-i}'',\theta_i)-\Delta V_i((\theta_i'',\theta_i'),\theta_{-i}',\theta_i)\geq T_i(\theta_i''-\theta_i')\sum(\theta_j''-\theta_j')$ for all $\theta_i\in\Theta_i$. Set $\rho_i>-T_i$. Choose any $\theta_i''\geq_i\theta_i'$ and $\theta_{-i}''\geq_{-i}\theta_{-i}'$. The function $u_i(x_i(\hat{\theta}_i,\hat{\theta}_{-i}),\theta_i)$ has increasing differences in $(\hat{\theta}_i,\hat{\theta}_{-i})$ for each θ_i, if the following expression is positive for all θ_i,

$$\Delta V_i((\theta_i'',\theta_i'),\theta_{-i}'',\theta_i)-\Delta V_i((\theta_i'',\theta_i'),\theta_{-i}',\theta_i)+\sum_{j\neq i}\rho_i\big(\theta_i''\theta_j''+\theta_i'\theta_j'-\theta_i''\theta_j'-\theta_i'\theta_j''\big). \quad (6)$$

Given $\rho_i>-T_i$, (6) is greater than

$$\Delta V_i((\theta_i'',\theta_i'),\theta_{-i}'',\theta_i)-\Delta V_i((\theta_i'',\theta_i'),\theta_{-i}',\theta_i)-T_i\sum_{j\neq i}(\theta_i''-\theta_i')(\theta_j''-\theta_j'). \quad (7)$$

Bounded substitutes immediately imply that (7) is positive for all θ_i, hence so is (6). By Lemma 1, the utility function u_i^f has increasing differences in $(\hat{\theta}_i(.),\hat{\theta}_{-i}(.))$. Finally, since Θ_i is a chain, Lemma 1 implies u_i^f is supermodular in $\hat{\theta}_i(.)$. Q.E.D

Theorem 1 shows that the class of implementable scf that can be supermodularly implemented in Bayesian equilibrium is large, as there are only mild boundedness and continuity conditions on the utility functions and the scf. The transfers are at the heart of the result: It is always possible to add complementarities into the transfers without affecting the incentives that appear in the expected value.

REMARK. Since players receive the same expected utility in equilibrium from (x, t) and (x, t^{SM}), if (x, t) satisfies some ex-ante or interim participation constraints, then so does (x, t^{SM}).

Recall that, if type spaces are finite or if the valuations and the decision rule are twice-continuously differentiable on compact type sets, then the assumptions of bounded substitutes and continuity are satisfied. This leads to the following important corollaries which cover many cases of interest.

COROLLARY **1** *Let type spaces Θ_i be finite subsets of $\mathbb{R}$. For any valuation functions, if the scf $f = (x, t)$ is implementable, then there exist transfers t^{SM} such that (x, t^{SM}) is supermodular implementable.*

COROLLARY **2** *Let type spaces Θ_i be compact subsets of $\mathbb{R}$ and let $f = (x, t)$ be an implementable scf such that $E_{\theta_{-i}}[t_i(., \theta_{-i})]$ is usc. If the decision rule and the valuation functions are twice-continuously differentiable, then there exist transfers t^{SM} such that (x, t^{SM}) is supermodular implementable.*

The previous results state conditions that apply to Bayesian implementable scf. In some instances it may not be obvious whether the decision rule admits implementing transfers whose expected value is usc. Standard implementation results in differentiable environments demonstrate that the expected value of the transfers in an implementable scf takes an explicit form.[18] This leads to the next proposition.

PROPOSITION **1** *Let $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$ for $i \in N$. If decision rule $x(.)$ and the valuation functions form a continuous family with bounded substitutes such that $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$ and $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]/\partial\theta_i$ is increasing in $\hat{\theta}_i$, then there are transfers t^{SM} such that (x, t^{SM}) is supermodular implementable.*

To identify supermodular implementable decision rules, Proposition 1 suggests to choose those rules that lead each agent i's expected marginal valuation to be nondecreasing. By Theorem 1 and Proposition 7 in Appendix B, any such rule is supermodular implementable with transfers t^{SM}, combining (5) and (13).

The rest of this section investigates supermodular implementation under the budget balance condition. In some design problems, the planner should not realize a net gain from the mechanism. While the planner cannot sustain deficits, full efficiency requires there be no waste of numéraire. A scf is *fully efficient* if it maximizes the sum of the utility functions (not only the valuation functions) subject to the feasibility constraint $\sum t_i \leq 0$. So the transfers must add up to zero for each vector of true types. However, complementarities between agents' announcements may be irreconcilable with budget balancing, as shown in the next example.

[18]See e.g Proposition 23.D.2 in Mas Colell et al. [36] for linear utility functions.

Example 1 Consider the public goods example of Section 2. In this example, if there exist transfers $\{t_i^{SM}(.)\}_{i=1,2}$ such that the resulting scf (x, t^{SM}) is supermodular implementable, then inequality (2) must hold for both agents. That is, the cross-partial derivatives of $t_1(\hat{\theta})$ must be greater than 2 and the cross-partial derivatives of $t_2(\hat{\theta})$ must be greater than -1; hence their sum will be strictly greater than 0. The budget balance condition requires $\sum_{i=1,2} t_i(\hat{\theta}) = 0$, so the sum of the cross-partial derivatives of the transfers must be null. As a result, budget balancing must be violated in this example if there is supermodular implementation.

This example points to the difficulty of balancing budget in some situations with two players. The next theorem provides sufficient conditions for a scf to be supermodular implementable using balanced transfers. Say that a decision rule x is *allocation-efficient*, if $x(\theta) \in \text{argmax}_{x \in X} \sum_{i \in N} V_i(x_i, \theta_i)$ for all $\theta \in \Theta$. Basically, if substitutes are bounded, any allocation-efficient decision rule can be paired with a transfer scheme to give a fully-efficient supermodular-implementable scf.

Theorem 2 *Let $n \geq 3$. Consider an allocation-efficient decision rule $x(.)$. If the valuation functions and the decision rule form a continuous family with bounded substitutes, then there exist balanced transfers t^{BB} such that (x, t^{BB}) is supermodular implementable.*

The proof appears in Appendix B and it is constructive. Transfers t^{BB} correspond to a transformation of the transfers in the expected externality mechanism, and they rely on two observations. First, any player's transfer in the expected externality mechanism displays no complementarities or substitutes, because transfers are separable in announcements. Second, there is a transformation of these transfers similar to that in Theorem 1 that enables to add complementarities while preserving incentives and budget balancing. The key observation is that the transfers from Theorem 1 add complementarity between agents' announcements in a pairwise fashion. As soon as there is a third agent, it is possible to subtract from each individual's transfer those complementarities that come from the other agents' transfers and that do not concern that individual, thus balancing the whole system.

Theorem 2 can be modified to apply to situations where, for every realization of types, enough taxes need to be raised to pay the cost of x. This constraint takes the form $\sum_{i \in N} t_i(\theta) \geq C(x(\theta))$ for all θ, where C is the cost function mapping X into $\mathbb{R}^+$.[19]

5.3 Optimal and Unique Supermodular Implementation

This subsection deals with the multiple equilibrium problem in supermodular implementation. Even if a mechanism has an equilibrium outcome with some desirable property, it may have other equilibrium outcomes that are undesirable. The concept of supermodular implementation relies on weak implementation: For direct mechanisms, only the truthful equilibrium is known to have the desired outcome. It follows from

[19] An additional sufficient condition to apply the theorem is that $C(.)$ and $x(.)$ produce bounded substitutes. See e.g Lemma 2 in Ledyard and Palfrey [33] for transfers satisfying this budget balance condition. Note that these transfers are separable in types except (possibly) for $C(x(\theta))$, so there are no complementarities or substitutes beyond those contained in $C(x(\theta))$.

[39] that adaptive dynamics lead to play between the greatest and the least equilibrium, so players may learn to play an untruthful equilibrium associated with a bad outcome. Therefore, it is important to minimize the size of the interval between the extremal equilibria, called the interval prediction, and to take the number of equilibria into consideration. If the interval prediction is small, then learning leads to a profile close to truthtelling and to the desired outcome. For these reasons, supermodular implementation is particularly powerful when truth-revealing is the unique equilibrium.

5.3.1 Optimal Implementation

I begin with an example that explains the foundations of this section.

Example **2** Consider the public goods example of Section 2. Suppose that transfers are defined as $t_i(\hat{\theta}) = \rho_i \hat{\theta}_i \hat{\theta}_j + E_{\theta_j}[t_i(\hat{\theta}_i, \theta_j)] - \rho_i \hat{\theta}_i E_{\theta_j}[\theta_j]$ for $i = 1, 2$ and $j \neq i$, where t_i is given by the expected externality mechanism. If $\rho_1 = 2\frac{1}{2}$ and $\rho_2 = -1/2$, the game induced by the mechanism is supermodular and truthtelling is the unique Bayesian equilibrium (See Example 3). For $\rho_1 = 3\frac{1}{5}$ and $\rho_2 = 1/2$, the supermodular game induced by the mechanism has now a smallest and a largest equilibrium. In the smallest equilibrium, agent 1 announces 0 for any type below $c_1 \approx 0.47$ and $\theta_1 - c_1$ for types above, and agent 2 announces 0 for any type below $c_2 \approx 0.55$ and $\theta_2 - c_2$ for types above. In the largest equilibrium, agent 1 announces $\theta_1 + c_1$ for any type below $1 - c_1$ and 1 for types above, and agent 2 announces $\theta_2 + c_2$ for any type below $1 - c_2$ and 1 for types above. Moreover, increasing ρ_1 to 4 and ρ_2 to 1 produces extremal equilibria with $c_1 = c_2 = 1$ and $c_1 = c_2 = 0$; the smallest equilibrium is the smallest profile of the entire space where each agent always announces her smallest type, and the largest equilibrium is the largest profile of the entire space where each agent always announces her largest type. Increasing ρ_1 and ρ_2 has had three negative consequences: *i*) By increasing these parameters above (resp.) 5/2 and -1/2, we have generated two new equilibria. By increasing them more, *ii*) we have enlarged the size of the interval prediction to be the whole space, so the Milgrom-Roberts theorem is of little help now *iii*) the truthful equilibrium has become locally unstable.

Before presenting the formal definitions and the results, I discuss some new concepts. Think of the degree of complementarity between the variables of a function as given by its cross-partial derivatives. Large cross-partials mean that the degree of complementarity is high, and vice-versa. In Example 2, the transfers produce more complementarities as ρ_i increases. *Optimal supermodular implementation* involves designing a mechanism whose induced supermodular game has the weakest complementarities among supermodular mechanisms. The rationale behind optimal supermodular implementation is clear from Example 2. First, it is the best compromise between learning, stability and multiplicity of equilibria. Adding complementarities improves learning and stability, but too much complementarity may yield untruthful equilibria. Second, optimal supermodular implementation provides the tightest interval prediction around the truthful equilibrium (Proposition 2). This is hinted at by Example 2, because the extremal equilibria move apart as the degree of complementarity increases.

Next I define those concepts formally and I prove the claim that relates the size of interval prediction to the degree of complementarity. As mentioned above, the cross-partial derivatives offer a way of measuring complementarities in twice-differentiable environments. It is natural to say that a transfer function $\tilde{t}$ generates larger complementarities than t, denoted $\tilde{t} \succeq_{\text{ID}} t$, if $\partial^2 \tilde{t}_i(\hat{\theta})/\partial\hat{\theta}_i\partial\hat{\theta}_j \geq \partial^2 t_i(\hat{\theta})/\partial\hat{\theta}_i\partial\hat{\theta}_j$ for all $\hat{\theta}$, j and i. The next definition formalizes this idea and extends it to non-differentiable transfers.

Definition 4 *Define the ordering relation $\succeq_{ID}$ on the space of transfer functions such that $\tilde{t} \succeq_{ID} t$ if, for all $i \in N$ and for all $\theta''_i > \theta'_i$ and $\theta''_{-i} >_{-i} \theta'_{-i}$, $\tilde{t}_i(\theta''_i, \theta''_{-i}) - \tilde{t}_i(\theta''_i, \theta'_{-i}) - \tilde{t}_i(\theta'_i, \theta''_{-i}) + \tilde{t}_i(\theta'_i, \theta'_{-i}) \geq t_i(\theta''_i, \theta''_{-i}) - t_i(\theta''_i, \theta'_{-i}) - t_i(\theta'_i, \theta''_{-i}) + t_i(\theta'_i, \theta'_{-i})$.*

For twice-differentiable transfers, this definition is equivalent to the condition that the cross-partial derivatives of each $\tilde{t}_i$ are larger than those of t_i.

While $\succeq_{\text{ID}}$ is transitive and reflexive on the space of transfer functions, it is not antisymmetric. Consider the set of $\succeq_{\text{ID}}$-equivalence classes of transfers, denoted $\mathcal{T}$.[20]

The next proposition shows that if a transfer function generates more complementarities than another transfer function, then it induces a game whose interval prediction is larger than the interval prediction of the game induced by the other transfer. This result is also interesting for the theory of supermodular games, as it relates the degree of complementarity to the size of the interval prediction.[21]

For any $t \in \mathcal{T}$ and supermodular implementable $f = (x, t)$, let $\overline{\theta}^t(.)$ and $\underline{\theta}^t(.)$ denote the extremal equilibria of the induced game.

Proposition 2 *Let decision rule $x(.)$ and the valuation functions be such that $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$. For any supermodular implementable scf (x, t'') and (x, t') with $t'', t' \in \mathcal{T}$, if $t'' \succeq_{ID} t'$, then $[\underline{\theta}^{t'}(.), \overline{\theta}^{t'}(.)] \subset [\underline{\theta}^{t''}(.), \overline{\theta}^{t''}(.)]$.*

This proposition provides the foundation for the next definition. If a scf is supermodular implementable and its transfers generate the weakest complementarities, then it is optimally supermodular implementable. This gives the tightest interval prediction around the truthful equilibrium.

Definition 5 *A scf $f = (x, t^*)$ is optimally supermodular implementable if it is supermodular implementable and $t \succeq_{ID} t^*$ for all transfers $t \in \mathcal{T}$ such that (x, t) is supermodular implementable.*

The next result determines which decision rules are optimally supermodular implementable. The result uses the following property of decision rules. Say that a decision rule $x : \Theta \mapsto (x_i(\theta))$ is *dimensionally reducible* if, for each $i \in N$, there are twice-continuously differentiable functions $h_i : \mathbb{R}^2 \to X_i$ and $r_i : \Theta_{-i} \to \mathbb{R}$ such that $r_i(.)$ is increasing and $x_i(\theta) = h_i(\theta_i, r_i(\theta_{-i}))$ for all $\theta \in \Theta$. The condition is trivially true when there are two individuals. If there are more than two, a player's decision rule can depend on her own type directly, but it must depend on her opponents' types indirectly through a real-valued aggregate. Taking types in $[0, 1]$, it excludes, for example, x for which $x_1(\theta) = \theta_1^{\theta_2\theta_3} + \theta_1 + \theta_2 + \theta_3$.[22]

[20] Any quasi-order is transformed into a partially ordered set using equivalence classes.

[21] See Milgrom and Roberts [42] (p.189-190) for a related result.

[22] To see why, $h_1(\theta_1, r_1) = \theta_1^{z(r_1)} + \theta_1 + r_1$ for some $z : \mathbb{R} \to \mathbb{R}$ and $r_1(\theta_{-1}) = \theta_2 + \theta_3$. But there is no z such that $z(\theta_2 + \theta_3) = \theta_2\theta_3$ for all θ_{-1}, because $z(0 + 1) \neq z(.5 + .5)$.

Theorem **3** *Let the valuation functions be twice-continuously differentiable and $f = (x, t)$ be a scf whose decision rule is dimensionally reducible. If f is implementable, then there are transfers t^* such that (x, t^*) is optimally supermodular implementable.*

The theorem says that, in twice-continuously differentiable environments, all implementable scf whose decision rule satisfies the dimensionality condition are optimally supermodular implementable.

5.3.2 Unique Implementation and Full Efficiency

After providing conditions for the smallest interval prediction, it is natural to study situations where truthtelling is the unique equilibrium of the induced supermodular game. All learning dynamics then converge to the equilibrium. This is the concept of *unique supermodular implementation.* As a by-product, it implies coalition-proof Nash implementation by Milgrom and Roberts [43].

This section also supports what appears to be a conflict between full efficiency and learning. Example 1 already delivered the message: Sometimes the designer must sacrifice either learning or efficiency. Either she modifies the expected externality mechanism and secures learning at the price of a balanced budget (full efficiency), or she loses the strong learning properties by balancing budget via the expected externality mechanism.

Definition **6** *A scf $f = (x, t)$ is uniquely supermodular implementable if it is supermodular implementable and the truthful equilibrium is the unique Bayesian equilibrium.*

The main result (Theorem 4) gives sufficient conditions for a scf to be uniquely supermodular implementable. If truthtelling is an equilibrium and if the mechanism induces utility functions whose complementarities between announcements are smaller than the complementarities between own announcement and type, then the truthful equilibrium is unique. This result is followed by Proposition 3 which focuses on optimal transfers. These transfers indeed produce the smallest interval prediction, so a natural question to ask is when they actually lead to unique implementation.

The results use the concepts of γ-increasing differences and bounded complements. The first one strengthens the condition of Proposition 1 by requiring that the marginal expected value be "sufficiently" increasing in a player's announcement. The second condition is defined as follows. The valuation functions and the decision rule produce *bounded complements* if $(-V, x)$ has bounded substitutes. Likewise, say that $u_i \circ f$ has bounded complements if the same definition is satisfied when transfers are included.

Theorem **4** *Let the valuation functions be continuously differentiable, and let $f = (x, t)$ be a scf with a differentiable decision rule whose composition with the valuations has γ-increasing differences and is ω-Lipschitz. Suppose $u_i \circ f$ has complements bounded by κ_i and transfers are β-Lipschitz. If f is supermodular implementable and $\kappa_i < \gamma_i/(n-1)$, then it is uniquely supermodular implementable.*

Proposition **3** *Let the valuation functions be twice-continuously differentiable, and let $f = (x, t)$ be a scf with a dimensionally reducible decision rule whose composition* *with the valuations has γ-increasing differences and is ω-Lipschitz. Letting*

$$\kappa_i = \max_{j \neq i} \max_{(\hat{\theta}, \theta_i) \in \Theta \times \Theta_i} \left(\frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} - \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right),$$

if $\kappa_i < \gamma_i/(n-1)$ for all i, then (x, t^*) is uniquely supermodular implementable.

The proofs appear in Appendix B. On the one hand, for large γ_i, the complementarities between own announcement and type are so strong that players tend to announce high types regardless of their opponents' deceptions. This favors uniqueness. On the other hand, for large κ_i, the complementarities between players' announcements are so strong that it is source of multiplicity. The theorem provides a cutoff between those forces so that, for any profile greater (smaller) than the truthful equilibrium, there is a player for whom it is not optimal to increase (decrease) her truthful strategy to the strategy of that profile. From Theorem 4, the proposition is quite intuitive. To make the induced game supermodular for each vector of types, we may have to add complementarities which are unnecessarily large for some types, but just sufficient for some other types. The condition ensures these differences across types are not too large.

The following examples show that there are cases where it is straightforward to apply the previous results. Besides, Example 4 describes a situation where the original mechanism induces a non-supermodular game with multiple equilibria and where the truthful equilibrium is unstable. Yet the decision rule is uniquely supermodular implementable by Theorem 4. Interestingly, this also illustrates how weak implementation can be turned into strong implementation.

Example **3** Consider the public goods example of Section 2. Recall that agents' valuation functions are $V_1(x, \theta_1) = \theta_1 x - x^2$ and $V_2(x, \theta_2) = \theta_2 x + x^2/2$. The decision rule is $x(\theta) = \theta_1 + \theta_2$. Since $\partial x_i(\theta)/\partial\theta_i = 1$ and $\partial^2 V_i(x, \theta_i)/\partial x \partial \theta_i = 1$ for $i = 1, 2$, it implies $\gamma_i = 1$, $i = 1, 2$. Moreover, $\partial^2 V_i(x(\hat{\theta}), \theta_i)/\partial\hat{\theta}_1\hat{\theta}_2 = -2$ if $i = 1$ and 1 if $i = 2$. By Proposition 3, $\kappa_i = 0$ for $i = 1, 2$ and (x, t^*) is uniquely supermodular implementable.

Example **4** Reconsider the public goods example of Section 2. Instead of the expected externality transfers t, suppose that the designer uses $\tilde{t}_i(\hat{\theta}) = t_i(\hat{\theta}) - 3\hat{\theta}_i\hat{\theta}_j + \frac{3}{2}\hat{\theta}_i$ for $i = 1, 2$. There, the game induced by the mechanism is not supermodular with respect to the natural order on $\mathbb{R}$. This game has many equilibria, two of them being the following. In the first one, agent 1 always announces 0 for any type and agent 2 always announces 1 for any type. In the second one, agents switch roles. Moreover, truthtelling is an equilibrium, but it is highly unstable. Any perturbation results in a departure from truth-revealing. However, this situation falls into Theorem 4.

The rest of this subsection deals with the multiple equilibrium problem under the budget balance condition. The next proposition shows that there are scf that are uniquely supermodular implementable with balanced transfers. It gives sufficient conditions in order that the transfers identified in Theorem 2 yield truthtelling as a unique equilibrium. The proof is in Appendix B.

PROPOSITION 4 *Let $n \geq 3$. Let decision rule $x(.)$ and the valuation functions be continuously differentiable such that their composition has γ-increasing differences and produces complements and substitutes bounded (resp.) by τ_i and T_i. If the decision rule is allocation-efficient and $\tau_i - T_i < \gamma_i/(n-1)$, then (x, t^{BB}) is uniquely supermodular implementable.*

The next examples illustrate some interesting implications of Proposition 4. There are situations, like Example 5, where the proposition can be used easily. Also, it is well-known that dominant-strategy implementation may be incompatible with balancing budget (Green and Laffont [22] and Laffont and Maskin [31]). In Example 6, the proposition allows to balance budget in cases where dominant strategies cannot; while the solution concept is weaker, truthtelling is the unique Bayesian equilibrium of a supermodular game. The last example illustrates the potential conflict in supermodular implementation between budget balancing and the multiple equilibrium problem. One may argue that a second-best approach could be appropriate: Choosing the "best" scf among those that have nice learning characteristics.

EXAMPLE 5 Consider the same setting as the public goods example of Section 2 with an additional player, player 3, whose type is independently distributed from the other player's types in $\Theta_3 = [0, 1]$. Player 3's valuation function is $V_3(x, \theta_3) = \theta_3 x$. Let $X = [0, 3]$ and $x(\theta) = \theta_1 + \theta_2 + \theta_3$. Then x is allocation-efficient and $\gamma_i = 1$ for $i = 1, 2, 3$. Since $\tau_i = T_i$ for $i = 1, 2, 3$ and $T_1 = -2$, $T_2 = 1$, $T_3 = 0$, Proposition 4 says that for any $\{\rho_i\}$ such that $2 < \rho_1 < 2\frac{1}{2}$, $-1 < \rho_2 < -\frac{1}{2}$, $0 < \rho_3 < \frac{1}{2}$, (x, t^{BB}) is uniquely supermodular implementable with a balanced budget.

EXAMPLE 6 In the public goods example of Section 2, let $\Theta_1 = \Theta_2 = [2, 3]$. Add a third player, player 3, whose type is independently distributed from the other player's types in $\Theta_3 = [2, 3]$. Player 3's valuation function is $V_3(x, \theta_3) = \theta_3 x - \ln x$. Letting $X = [5, 10]$, the allocation-efficient decision rule is $x(\theta) = \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \sqrt{(\theta_1 + \theta_2 + \theta_3)^2 - 4})$. By Theorem 3.1 in [31], the decision rule is dominant strategy implementable only if transfers are of the Groves form. The necessary condition for those transfers to balance budget (Theorem 4.1 in [31]) is violated. Nevertheless, since $\tau_1 - T_1 \approx 0.022$ and $\gamma_1 > 1$, $\tau_2 - T_2 \approx 0.03$ and $\gamma_2 > 1$, $\tau_3 - T_3 \approx 0.011$ and $\gamma_3 > 1$, Proposition 4 implies that x is uniquely supermodular implementable with a balanced budget.

EXAMPLE 7 Add player 3 in the public goods setting of Section 2. Player 3's type is uniformly distributed in $[0, 1]$ and independently from the other player's types. Her valuation function is $V_3(x, \theta_3) = \theta_3 x - x^3$. The decision rule $x(\theta) = \frac{1}{6}(\sqrt{1 + 12(\theta_1 + \theta_2 + \theta_3)} - 1)$ is allocation-efficient and dimensionally reducible. The designer has the choice between the budget balanced transfers t^{BB} and the optimal transfers t^*. On the one hand, if she prefers full efficiency, then she must choose $\rho_1 \geq 8$, $\rho_2 \geq 5$ and $\rho_3 \geq 6$ in order for the balanced transfers to induce a supermodular game. From Proposition 2, she will choose the binding values. The supermodular game induced by (x, t^{BB}) admits untruthful extremal equilibria and its interval prediction is the whole space. On the other hand, if she prefers strong learning properties, then she will use optimal supermodular implementation. It turns out that optimal transfers induce a supermodular game where truthtelling is the unique Bayesian equilibrium, but they are not balanced.

Before turning to applications of supermodular implementation, some remarks are in order.

Remarks.

1. There are two usual ways of obtaining uniqueness in a Bayesian game. One is to impose conditions on the utility functions and the other is to impose conditions on the information structure. Theorem 4 belongs to the first class. In Example 2, there is a point beyond which increasing ρ_i results in multiple equilibria. This shows that, in this context, any result of the first class will resemble Theorem 4 and involve a cutoff expression in terms of complementarities (κ) and some other parameter (γ).[23] The usefulness of Theorem 4 also comes from its recommendation of an explicit upper bound on the degree of complementarity (e.g ρ_i) generated by the transfers.

2. Optimal supermodular implementation is based on the idea of imposing the weakest "admissible" amount of complementarity. But weak complementarities might imply a low speed of convergence of learning dynamics towards truthtelling. This is not necessarily true. Sometimes, when strictly-dominant strategy implementation is possible, the optimal transfers coincide with the dominant strategy transfers (E.g in the leading example). Then the optimal transfers guarantee the fastest convergence; however, they sometimes deliver a slower convergence than it is possible. Although convergence is possible in one period in Example 7, it takes longer under the optimal transfers.[24] In addition, there exist scf which are uniquely supermodular implementable without being dominant strategy implementable (Example 8).[25]

3. Neither unique nor optimal supermodular implementation implies the other. The truthful equilibrium may be unique, although the transfers are not optimal, and the transfers could be optimal but the truthful equilibrium not unique.

[23] Note that increasing differences in own type and announcement is a concavity condition in the game with transfers. Thus the property of γ-increasing differences is related to the notion of γ-concavity (Rockafellar [48]), which is a form of strong concavity. Other papers (See e.g Bisin et al. [6]) exploit similar trade-offs between Lipschitz constants to obtain uniqueness results, which conveys the idea that utility functions which are "more concave than supermodular" favor uniqueness. Theorem 4 is also inspired by recent theories of uniqueness in Bayesian games (Mason and Valentinyi [37]).

[24] If the composition of any player's valuation function with the decision rule has strictly increasing differences in type and own announcement, Mookherjee and Reichelstein [45] implies that there exist transfers resulting in strictly-dominant strategy implementation (See Proposition 2 in Mookherjee and Reichelstein [45] and the discussion that follows).

[25] When Mookherjee and Reichelstein [45] applies, the choice of transfers is narrow, whereas the present results may provide a whole range of transfers compatible with unique supermodular implementation. In the public goods example, there are infinitely many ρ_1 and ρ_2 resulting in unique supermodular implementation; but it must be that $\rho_1 = 2$ and $\rho_2 = -1$ to achieve (strictly) dominant strategy implementation.

6 Applications

6.1 Principal Multi-Agent Problem

This subsection applies the theory to contracting between some agents and a principal. Consider the traditional principal-agent problem with hidden information. A principal contracts with n agents. Agent i's type space is $[\underline{\theta}_i, \overline{\theta}_i]$. Types are independently distributed according to a common prior $\phi = \times\phi_i$ which admits a bounded density with full support. Let $X_i \subset \mathbb{R}$ be compact. Each agent i exerts some observable effort $x_i \in X_i$, and she bears a cost or disutility $c_i(x_i, \theta_i)$ from producing effort x_i when she is of type θ_i. From the vectors of efforts $x = (x_1, \dots, x_n)$ and types $\theta = (\theta_1, \dots, \theta_n)$, the principal receives utility $w(x, \theta)$. The principal faces the problem of designing an optimal contract subject to incentive constraints and reservation utility constraints for the agents. A contract is a function that maps each possible agents' type into effort and transfer levels. The principal's problem can be stated as

$$(x^*, \tilde{t}) \in \operatorname*{argmax}_{f=(x,t)} E_\theta \left[w(x(\theta), \theta) - \sum_{i=1}^{n} t_i(\theta) \right] \tag{8}$$

subject to

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i}) - c_i(x_i(\theta_i, \theta_{-i}), \theta_i)] \ \geq \ E_{\theta_{-i}}[t_i(\theta'_i, \theta_{-i}) - c_i(x_i(\theta'_i, \theta_{-i}), \theta_i)], \ \forall\, \theta'_i, \theta_i \tag{9}$$

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i}) - c_i(x_i(\theta_i, \theta_{-i}), \theta_i)] \ \geq \ \overline{u}_i, \ \forall\, \theta_i \tag{10}$$

Condition (9) requires truthtelling to be an equilibrium. Condition (10) is an interim participation constraint, as agents may opt out of the mechanism if it does not meet their reservation utility.

Suppose that the underlying functions w, c_i and ϕ are smooth and guarantee the existence of a solution such that x^* is dimensionally reducible. Applying Theorem 3, there are transfers t^* such that (x^*, t^*) is optimally supermodular implementable and solves (8) subject to (9) and (10). In words, if the principal is in a position to engage in a smooth revenue-maximizing and incentive-compatible contract which allows voluntary participation, then she can also turn it into a supermodular contract where agents lie "as little as possible" in equilibrium.

At this level of generality, it is difficult to appreciate the strength of optimal supermodular implementation, so I present a simple example in the spirit of the team production model of McAfee and McMillan [34].

Example **8** There are two agents, 1 and 2, whose types are independently uniformly distributed in $[0, 3]$. Players exert some effort to produce an observable contribution x_i. The amount of effort e_i necessary for x_i is $e_1(x, \theta_1) = (3 - \theta_1)(x_1 - x_2) + x_1$ and $e_2(x, \theta_2) = (3 - \theta_2)(x_2 + x_1)$. Larger contributions require larger effort and higher ability levels decrease marginal effort. But agent 2 generates positive externalities on her counterpart, whereas 1 has negative externalities. Given $x = (x_1, \dots, x_n)$, the principal only knows the density $f(y|x)$ of output y given x. Suppose f has support $[0, \overline{y}]$, $\overline{y} > 0$. The principal has utility function $u(y, x, \theta)$ and perceives costs as $c_p(x, \theta)$. Costs are given by the sum of the production cost and the cost of inducing the agents to reveal their private information. Therefore, the above constrained maximization (8) comes down to

$$x^*(\theta) \in \underset{(x_1,\ldots,x_n)}{\operatorname{argmax}} E_{y|x}[u(y,x,\theta)] - c_p(x,\theta). \tag{11}$$

For simplicity, let $u(y,x,\theta) = y + \theta_2(\theta_1 x_1 + x_2)$ and let the production costs be such that $c_p(x,\theta) = x_1^2/2 + \theta_1 E(\theta_2)x_1 + x_2^2/2 + \theta_1 x_2$. The decision rule obtained from (11) is $(x_1^*(\theta), x_2^*(\theta)) = (\theta_2\theta_1 - \theta_1 E(\theta_2), \theta_2 - \theta_1)$. Let i's valuation function be $V_i(x,\theta_i) = -c(e_i(x,\theta_i))$ where $c(e_i) = e_i$.[26] Decision rule $x^*(.)$ satisfies the conditions of Proposition 7 of Appendix B, so there exist transfers t such that (x^*, t) is implementable. Constructing optimal transfers from (16) and (17) gives $t_1^*(\hat{\theta}) = -\hat{\theta}_1^2/2 - 3\hat{\theta}_1 + 4\hat{\theta}_2\hat{\theta}_1$ and $t_2^*(\hat{\theta}) = -5\hat{\theta}_2^2/4 + 3\hat{\theta}_2 + 3\hat{\theta}_2\hat{\theta}_1$. Truthtelling is the unique Bayesian equilibrium of the supermodular game induced by the mechanism with optimal transfers.

6.2 Approximate Supermodular Implementation

In this section, I generalize some of the previous results within the context of approximate (or virtual) implementation.[27] Supermodular implementation is useful in many applications where the outcome space is at least as "rich" as the type spaces. When type sets are finite, Theorem 1 always applies. When the type sets and the outcome space are continua, it applies to general models such as principal multi-agent and public goods models. The recourse to approximate implementation is justified when the outcome space is finite while type sets are continuous; I describe two binary-choice models that violate bounded substitutes. The results for approximate implementation apply to public goods, auctions and bilateral trading (Myerson and Satterthwaite [46].)

Consider the following auction model. Let buyer i's type space be $\Theta_i \equiv [\underline{\theta}_i, \overline{\theta}_i]$. Buyer i's utility function takes the linear form $u_i(x_i, \theta_i) = \theta_i x_i + t_i$. Consider the allocation-efficient decision rule which attributes the good to the agent with the highest type. For $i \in N$ and all θ,

$$x_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \max\{\theta_j : j \in N\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \sum_{j\in N} x_j^*(\theta) = 1 \tag{12}$$

Take $N = \{1,2\}$. For any $\theta_2'' > \theta_1'' > \theta_2' > \theta_1'$, $x_1(\theta_1'', \theta_2'') - x_1(\theta_1', \theta_2'') - x_1(\theta_1'', \theta_2') + x_1(\theta_1', \theta_2') = -1$. Hence, for substitutes to be bounded, there must exist T such that $-\theta_1 \geq T(\theta_1'' - \theta_1')(\theta_2'' - \theta_2')$ for all $\theta_1 \in \Theta_1$. But this is clearly impossible as we can maintain the order $\theta_2'' > \theta_1'' > \theta_2' > \theta_1'$ while $\theta_1' \uparrow \theta_2'$ and $\theta_1'' \downarrow \theta_2'$. So Proposition 1 does not apply.

Consider now a situation in which n agents must decide whether to undertake a public project with cost c. The decision rule $x(.)$ takes values in $\{0,1\}$. Let i's type

[26] The one-dimensional condensation property of Mookherjee and Reichelstein [45] is violated. There exists no $h_1 : X \to \mathbb{R}$ such that $c(e_1(x,\theta_1)) = D_1(h_1(x), \theta_1)$ for some $D_1 : \mathbb{R} \times \Theta_1 \to \mathbb{R}$. Moreover, note that $V_1(x^*(\hat{\theta}), \theta_1)$ does not have increasing differences in $(\hat{\theta}_1, \theta_1)$, so $x^*(.)$ is not dominant-strategy implementable by Definition 5 in Mookherjee and Reichelstein [45].

[27] See Abreu and Matsushima [1] and Duggan [17].

space be $[\underline{\theta}_i, \overline{\theta}_i]$. Agents' utility function takes the same linear form. The allocation-efficient decision rule is $x^*(\theta) = 1$ if $\sum_{i \in N} \theta_i \geq c$ and 0 otherwise. A similar reasoning establishes that substitutes are unbounded.

Clearly, the problem is caused by the lack of smoothness in those decision rules. Approximate implementation solves this difficulty. According to the next definition, a scf is approximately supermodular implementable if, in any ϵ-neighborhood of that scf, there exists a supermodular implementable scf.

Definition 7 *A decision rule $x(.)$ is approximately (optimally) supermodular implementable, if there exists a sequence of (optimally) supermodular implementable scf $\{(x_n, t_n)\}$ such that, for $1 \leq p < \infty$, $\lim_{n \to \infty} (\int_\Theta |x_{n,i} - x_i|^p)^{\frac{1}{p}} = 0$ for all i.*

The next two results say that, for twice-continuously differentiable valuations that satisfy increasing differences, monotone $L_p(\Theta)$-decision rules are approximately supermodular implementable. The main idea is that smooth scf satisfy the bounded substitutes assumption and they are dense in L_p-spaces. Moreover, if the decision rule also satisfies the dimensionality condition, then it is approachable by optimally supermodular implementable scf.

Proposition 5 *Let the valuation functions be twice-continuously differentiable such that $\partial V_i(x_i, \theta_i)/\partial \theta_i$ is increasing in x_i. If the decision rule $x_i(.) \in L_p(\Theta)$ is increasing in $\hat{\theta}_i$, then $x(.)$ is approximately supermodular implementable.*

Proposition 6 *Let the valuation functions be twice-continuously differentiable such that $\partial V_i(x_i, \theta_i)/\partial \theta_i$ is increasing in x_i. If $x(.)$ is a decision rule for which there exist a bounded function h_i and a continuous function r_i such that $x_i(\theta) = h_i(\theta_i, r_i(\theta_{-i}))$, h_i is increasing in θ_i and r_i is strictly increasing for all $i \in N$,[28] then $x(.)$ is approximately optimally supermodular implementable.*

In the above auction and public goods settings, it follows as a corollary of Proposition 6 that the efficient decision rules are optimally and approximately supermodular implementable.[29] The bargaining mechanism of Myerson and Satterthwaite ([46], p.274) also satisfies the assumptions of Proposition 6 and as such, the decision rule is approachable by a sequence of optimally supermodular implementable decision rules. The expected gains from trade along the sequence converge to the maximal expected gains.

Propositions 5 and 6 work under the assumption that if we cannot exactly implement a scf and maintain learning properties, then we may be willing to accomplish these goals for arbitrarily close scf. This suggests that there may be a dilemma between close implementability and stability or learning. This supports Cabrales [7] where a similar trade-off is formalized for Abreu and Matsushima [1] and [2].

Remark. Even though Proposition 6 applies to all bounded and monotone decision rules, it involves conditions that imply dominant strategy implementability by Mookherjee and Reichelstein [45]. The choice between (optimal) supermodular approximate

[28]The function r_i is strictly increasing if $r_i(\theta''_{-i}) > r_i(\theta'_i)$ whenever $\theta''_{-i} \gg \theta'_{-i}$.

[29]Consider the auction setting. Clearly, V is C^2 and $\partial V_i(x_i, \theta_i)/\partial \theta_i = x_i$ is increasing in x_i. For all i, let $h_i(\theta_i, r_i) = 1$ if $\theta_i > r_i$ and 0 otherwise. The function h_i is bounded and increasing in θ_i. Now choose $r_i(\theta_{-i}) = \max\{\theta_j : j \neq i\}$ which is continuous and strictly increasing.

implementation and dominant strategy implementation is ambiguous, because the existence of dominant strategies does not prevent adaptive dynamics from converging to an "unwanted" equilibrium,[30] to a non-equilibrium profile or simply from cycling in the induced game (See Saijo et al. [49]).

7 A Revelation Principle for Supermodular Bayesian Implementation

Supermodular implementation is widely applicable in quasilinear environments even though the paper limits attention to direct mechanisms. For general preferences, however, direct mechanisms may be restrictive. The Revelation Principle says that direct mechanisms cause no loss of generality under traditional weak implementation. But how restrictive are direct mechanisms in supermodular Bayesian implementation for general preferences?

It is particularly relevant to analyze this question, because the challenge in any supermodular design problem is to specify an ordered message space and an outcome function so that agents adopt monotone best-responding behaviors. The set of all possible message spaces and orders on those spaces is so large that it might seem intractably-complex. A Supermodular Revelation Principle gives conditions so that, if a scf is supermodular implementable, then there exists a direct-revelation mechanism that supermodularly implements this scf truthfully. It is a technical insight which reduces the space of mechanisms to consider to the space of direct-revelation mechanisms. The question is complex because it is combinatorial in essence; it pertains to the existence of orders on type spaces that make the (induced) direct-revelation game supermodular.

Example 9 in Appendix A shows that, unfortunately, there exist supermodular implementable scf that are not truthfully supermodular implementable. Consequently, the revelation principle fails to hold in general for supermodular implementation. Nevertheless, it exists in a weaker form, as captured by the next theorem. Although it is not as general as the traditional revelation principle, it measures the restriction imposed by direct mechanisms and gives conditions that may warrant their use.

As mentioned in Section 4, there are issues related to non-Euclidean message spaces that justify the next and more general definition of supermodular implementability.

DEFINITION **8** *The mechanism Γ supermodularly implements the scf $f(.)$ if there exists a Bayesian equilibrium $m^*(.)$ such that $g(m^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, and if the (ex-ante) induced game $\mathcal{G}$ is supermodular.*

THEOREM **5** *(*THE SUPERMODULAR REVELATION PRINCIPLE FOR FINITE TYPES[31]*) Let type space Θ_i be a finite set for $i \in N$. If there exists a mechanism $(\{(M_i, \succeq_i)\}, g)$ that supermodularly implements the scf f such that there is a Bayesian equilibrium $m^*(.)$ for which $g \circ m^* = f$ and $m_i^*(\Theta_i)$ is a lattice, then f is supermodular implementable.*

[30] An unwanted equilibrium can be an equilibrium in dominant strategies whose outcome is different from the social choice function or it can be a "non-dominant" strategy equilibrium.

[31] In Mathevet [38], I generalize the definition of supermodular implementability to incorporate orders that are not pointwise orders. This allows to prove a supermodular revelation principle for continuous types.

Corollary 3 *Let type spaces be finite sets. If there exists a mechanism* $(\{(M_i, \succeq_i)\}, g)$ *with totally ordered message spaces that supermodularly implements the scf* f *such that there is a Bayesian equilibrium* $m^*(.)$ *for which* $g \circ m^* = f$*, then* f *is supermodular implementable.*

According to the supermodular revelation principle, limiting attention to direct mechanisms amounts to restricting one's scope to mechanisms where the equilibrium strategies are lattice-ranged. When the range of the equilibrium strategies is a lattice, it is possible to construct an order on each player's type space that makes it order-isomorphic to the range of her equilibrium strategy. By order-isomorphism, type spaces become lattices under this order and it also preserves supermodularity from the indirect mechanism to its direct version. Therefore, the transmission channel is the range of the equilibrium strategies. Besides, the theorem states conditions that are verifiable a posteriori. It may be useful to know when a complex mechanism can be replaced with a simpler direct mechanism.

Corollary 3 says that if the designer is only interested in mechanisms where the message spaces are totally ordered, then she can look at direct mechanisms without loss of generality.

The theorem only gives sufficient conditions for revelation; but in those cases where a supermodular direct mechanism exists while the lattice condition is violated, the existence of an order has little or nothing to do with a revelation principle. In the spirit of Echenique [19], there may be conditions on the scf and the utility functions such that an order exists for which the game is supermodular. Since this existence would not follow from implementability, it is not a revelation approach.

8 Interpreting Learning in Bayesian Games

The learning literature has a straightforward application to games of incomplete information which is the approach taken in the paper. In the context of Bayesian implementation, the learning results of supermodular games find a natural interpretation in the ex-ante Bayesian game. Loosely, learning at the ex-ante stage may be interpreted as pre-playing the mechanism. At this stage, agents do not know their own type and they can be viewed as practicing the induced game repeatedly. Each agent submits a deception at each round until the designer collects the agreed-upon profile of deceptions, and types are revealed. Until then, no outcome is actually implemented. Learning at the ex-ante stage may also mean that agents are actually playing the mechanism repeatedly with independently and identically distributed types across periods. As a round begins, the agents do not know their own type yet, hence they submit a deception. By the end of the round, they learn their type and behave according to their deception. An outcome is then implemented at the end of each round. Here the designer is only interested in implementing the desired outcome in the long-run.

Although the learning results only apply directly to the ex-ante Bayesian game, they can be interpreted in the interim formulation. The interim Bayesian game inherits the complementarities, because most results work by showing that the ex-post game is supermodular for every type profile. However, the problem at this stage comes from the interpretation of learning and the technical difficulties related to the Milgrom-Roberts learning theorem. To illustrate the first difficulty, suppose that there are two agents. At the interim stage, each agent knows her own type and so she makes a single announcement at each period that the mechanism is repeated. But to compute her expected utility, an agent uses the prior distribution and the opponent's deception telling her what is played for each type. Since the opponent no longer announces a deception, an agent is unable to compute her expected utility. One way of interpreting learning then is to consider that there is a continuum of agents, and that prior belief ϕ actually represents the distribution of types in this population. An agent now faces a continuum of announcements (one for each opponent) as the mechanism is repeated, hence she can compute her expected utility. The interpretation of the process, however, becomes evolutionary in nature. We are now interested in that the observed proportions of types converge to the true proportions in the population. On the technical side, Van Zandt [57] shows that there are issues in applying some results of supermodular games to interim Bayesian games. But his results can be used to show that the Milgrom-Roberts learning theorem applies to the interim Bayesian game.[32]

9 Conclusion

This paper introduces a theory of implementation where the mechanisms implement scf in supermodular game forms. Supermodular implementation differs from the previous literature by its explicit purpose and methodology. The paper does not put an end to the question of learning and stability in incentive design and implementation, but it explicitly attacks it and provides answers to this important yet neglected question. Given that mechanisms are designed to achieve some equilibrium outcome, it is rather important to design mechanisms that enable boundedly rational agents to learn to play some equilibrium outcome. The methodology consists in inducing supermodular games rather than starting explicitly with a solution concept. Of course, supermodularity implies properties of iterative dominance, but it has stronger theoretical and experimental implications (See e.g Camerer [10]). The mechanisms derive their properties from the game that they induce and not directly from the solution concept.

Beyond the results, the paper brings out basic questions about learning and the design problem. We may wonder whether there is a price to pay for learning or stability in terms of efficiency. The trade-off appears quite clearly in this framework; sometimes the designer must sacrifice learning for full efficiency or vice-versa. In the public goods example, the designer can modify the expected externality mechanism and secure learning at the price of a balanced budget, or she can use the expected externality mechanism to balance budget but she loses the strong learning properties. This may be related to the specifics of supermodular implementation, but it is an interesting issue. We may also wonder whether there is a price to pay for learning or stability in terms of closeness of the decision rule implemented. This has obvious implications in

[32] The results in [39] can only be directly applied to the ex-ante version $\mathcal{G}$. However, Lemma 2, Proposition 3, Lemma 5 and Proposition 5 in Van Zandt [57] are particularly useful to apply the learning theorem to the interim formulation. This requires, however, that the utility functions (with transfers) be continuous in the announcement profile.

terms of efficiency. Cabrales [7] also suggests a dilemma between learning and close implementability for the Abreu-Matsushima mechanism, and it is verified in the supermodular implementation framework. Although this dilemma may be related to the specifics of these frameworks, it is a question with potentially important consequences.

The paper raises issues that have not been discussed. The multiple equilibrium problem in supermodular implementation suggests an alternative solution, namely strong implementation. Strong implementation requires all equilibria of the mechanisms to yield desired outcomes. Instead of relying on weak implementation, supermodular implementation could be based on strong implementation which would justify indirect mechanisms. Even under strong implementation, learning dynamics may cycle within the interval prediction and players may learn to play a non-equilibrium profile. Although strong supermodular implementation cannot substitute for unique supermodular implementation, it is an avenue to explore.

Like many Bayesian mechanisms, the present mechanisms are parametric in the sense that they rely on agents' prior beliefs. Thus the designer uses information other than that received from the agents (Hurwicz [26]). It may be interesting to design nonparametric supermodular mechanisms. This is yet another justification for indirect mechanisms, as nonparametric direct Bayesian mechanisms impose dominant-strategy incentive-compatibility (Ledyard [32]).

Finally, it is important to pursue testing supermodular games. Since supermodular Bayesian implementation provides a general framework, it is a good candidate for experimental tests. From a practical viewpoint, discretizing type spaces may simplify the players' task of announcing deceptions at each round. But there are also simple environments with continuous types where announcing a deception is equivalent to choosing a real number, such as the leading public goods and the team-production examples.[33].

Appendix A

This example shows that the revelation principle fails to hold in general for supermodular Bayesian implementation.

Example **9** Consider two agents, 1 and 2, with type spaces $\Theta_1 = \{\theta_1^1, \theta_1^2\}$ and $\Theta_2 = \{\theta_2^1, \theta_2^2, \theta_2^3\}$. Prior beliefs assign equal probabilities to all $\theta \in \Theta$. Let $X = \{x_1, \ldots, x_{12}\}$ be the outcome space. Agent 1's preferences are given by utility function $u_1(x_n, \theta_1)$ such that:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
$u_1(x_n, \theta_1^1)$	-10	0	16	-13	-2	33	-21	-2	18	-19	0	36
$u_1(x_n, \theta_1^2)$	-10	0	16	-21	-2	18	-13	-2	33	-19	0	36

Let u_2 be a constant function. Let the scf f be defined as follows

$f(.,.)$	θ_2^1	θ_2^2	θ_2^3
θ_1^1	x_4	x_5	x_6
θ_1^2	x_7	x_8	x_9

[33] In the public goods example of Section 2, announcing an optimal deception comes down to choosing an intercept in a compact set (See (1)). In Example 8, optimal deceptions are characterized by a positive slope.

Consider the following indirect mechanism $\Gamma = ((M_1, \succeq_1), (M_2, \succeq_2), g)$. Agent 1's message space is $M_1 = \{\underline{m}_1, m_1^1, m_1^2, \overline{m}_1\}$; $\succeq_1$ is such that m_1^1 and m_1^2 are unordered, $\overline{m}_1$ is the greatest element and $\underline{m}_1$ is the smallest element. Agent 2's message space is $M_2 = \{\underline{m}_2, m_2^1, \overline{m}_2\}$; $\succeq_2$ is such that $\overline{m}_2 \succeq_2 m_2^1 \succeq_2 \underline{m}_2$. The outcome function g is given by

$g(.,.)$	$\underline{m}_2$	m_2^1	$\overline{m}_2$
$\underline{m}_1$	x_1	x_2	x_3
m_1^1	$f(\theta_1^1, \theta_2^1)$	$f(\theta_1^1, \theta_2^2)$	$f(\theta_1^1, \theta_2^3)$
m_1^2	$f(\theta_1^2, \theta_2^1)$	$f(\theta_1^2, \theta_2^2)$	$f(\theta_1^2, \theta_2^3)$
$\overline{m}_1$	x_{10}	x_{11}	x_{12}

I show that mechanism Γ supermodularly implements f in Bayesian equilibrium. Given u_2 is constant, any strategy $m_2 : \Theta_2 \to M_2$ is a best-response to any strategy of 1. So, consider strategy $m_2^*(.)$ such that $m_2^*(\theta_2^1) = \underline{m}_2$, $m_2^*(\theta_2^2) = m_2^1$ and $m_2^*(\theta_2^3) = \overline{m}_2$. Since for all m_1 we have

$$\begin{array}{rcl} \sum_{m_2} u_1(g(m_1^1, m_2), \theta_1^1) & > & \sum_{m_2} u_1(g(m^1, m_2), \theta_1^1) \\ \sum_{m_2} u_1(g(m_1^2, m_2), \theta_1^2) & > & \sum_{m_2} u_1(g(m_1, m_2), \theta_1^2) \end{array}$$

1's best-response $m_1^*(.)$ to $m_2^*(.)$ is such that $m_1^*(\theta_1^1) = m_1^1$ and $m_1^*(\theta_1^2) = m_1^2$. So $(m_1^*(.), m_2^*(.))$ is a Bayesian equilibrium and $g \circ m^* = f$. Moreover, for each θ_1, $u_1(g(m_1, m_2), \theta_1)$ is supermodular in m_1 and has increasing differences in (m_1, m_2). This implies that u_1^g is supermodular in $m_1(.)$ and has increasing differences in $(m_1(.), m_2(.))$, because $\Sigma_1(\Theta_1)$ is endowed with the pointwise order. Therefore, Γ supermodularly implements f in Bayesian equilibrium, because 2's utility is constant.

Does this imply that there exists a mechanism $(\{(\Theta_i, \geq_i)\}, f)$ which truthfully implements f in supermodular game form? By means of contradiction, suppose there is such a mechanism. Then $(\Theta_1, \geq_1)$ must be totally ordered, for otherwise $\Sigma_1(\Theta_1)$ cannot be a lattice. Assume $\theta_1^2 >_1 \theta_1^1$. Let $\theta_i^k(.) = \theta_i^k$ regardless of i's true type. Let $\theta_1^T(.)$ be the truthful strategy for 1 and let $\theta_1^L(.)$ be constant lying. Note $\theta_1^1(.) <_1 \theta_1^T(.), \theta_1^L(.)$. Moreover, θ_2^1 and θ_2^2 must be ordered, because $\Sigma_2(\Theta_2)$ is a lattice. Thus $\theta_2^1(.)$ and $\theta_2^2(.))$ are ordered.

Since the direct mechanism must induce a supermodular game, $u_1^f(\hat{\theta}_1(.), \hat{\theta}_2(.))$ must satisfy the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$.[34] Given

$$\begin{array}{rcl} -2 = u_1^f(\theta_1^T(.), \theta_2^2(.)) & \geq & u_1^f(\theta_1^1(.), \theta_2^2(.)) = -2 \\ -13 = u_1^f(\theta_1^T(.), \theta_2^1(.)) & > & u_1^f(\theta_1^1(.), \theta_2^1(.)) = -17 \end{array}$$

u_1^f satisfies the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$ only if $\theta_2^1 >_2 \theta_2^2$. But

$$-2 = u_1^f(\theta_1^L(.), \theta_2^2(.)) \geq u_1^f(\theta_1^1(.), \theta_2^2(.)) = -2$$

does not imply $-21 = u_1^f(\theta_1^L(.), \theta_2^1(.)) \geq u_1^f(\theta_1^1(.), \theta_2^1(.)) = -17$. The single-crossing property is violated. Now assume $\theta_1^1 >_1 \theta_1^2$. Note $\theta_1^1(.) >_1 \theta_1^T(.), \theta_1^L(.)$. Given

$$\begin{array}{rcl} -2 = u_1^f(\theta_1^1(.), \theta_2^2(.)) & \geq & u_1^f(\theta_1^L(.), \theta_2^2(.)) = -2 \\ -17 = u_1^f(\theta_1^1(.), \theta_2^1(.)) & > & u_1^f(\theta_1^L(.), \theta_2^1(.)) = -21 \end{array}$$

[34]The single-crossing property, defined in Section 3, is implied by increasing differences.

u_1^f satisfies the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$ only if $\theta_2^1 >_2 \theta_2^2$. But

$$-2 = u_1^f(\theta_1^1(.), \theta_2^2(.)) \geq u_1^f(\theta_1^T(.), \theta_2^2(.)) = -2$$

does not imply $-17 = u_1^f(\theta_1^1(.), \theta_2^1(.)) \geq u_1^f(\theta_1^T(.), \theta_2^1(.)) = -13$. The single-crossing property is violated. The scf f is not truthfully supermodular implementable, although it is supermodular implementable.

This example suggests that the conditions of Theorem 5 are somewhat minimally sufficient. Agent 1's equilibrium strategy is indeed not lattice-ranged and the scf is not truthfully supermodular implementable. Whereas this example might indicate that the pointwise-order structure causes revelation to fail, this is not the case. Allowing more general order structures does not weaken the conditions for a revelation principle (See Mathevet [38]).

Appendix B

The following lemma shows that if the complete information payoffs are supermodular and have increasing differences, then the ex-ante payoffs are supermodular and have increasing differences.

Lemma 1 *Assume $(M_i, \geq_i)$ is a lattice for $i \in N$. Suppose that, for each $\theta_i \in \Theta_i$, $u_i(g(m_i, m_{-i}), \theta_i)$ is supermodular in m_i for each m_{-i} and has increasing differences in (m_i, m_{-i}). Then u_i^g is supermodular in $m_i(.) \in \Sigma_i(M_i)$ for each $m_{-i}(.)$ and has increasing differences in $(m_i(.), m_{-i}(.)) \in \Sigma_i(M_i) \times \left\{\prod_{j \neq i} \Sigma_j(M_j)\right\}$.*

The proof is omitted because it is simple.

The proof of the next Proposition is also omitted, because the result is standard and its proof is similar to that of Proposition 23.D.2 in Mas-Colell et al. [36].

Proposition 7 *Consider valuation functions V and a decision rule $x(.)$ such that $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$.*

(i)If the scf $f = (x, t)$ is truthfully Bayesian implementable, then for all $\hat{\theta}_i$

$$E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] = -E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i)] + \int_{\underline{\theta}_i}^{\hat{\theta}_i} \frac{\partial E_{\theta_{-i}}[V_i(x_i(s, \theta_{-i}), s)]}{\partial \theta_i} ds + \epsilon(\underline{\theta}_i) \quad (13)$$

(ii) Let the decision rule $x(.)$ be such that $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]/\partial \theta_i$ is increasing in $\hat{\theta}_i$ for each θ_i and $i \in N$. If transfers t satisfy (13), then the scf $f = (x, t)$ is implementable.

Proof of Proposition 2: Let (x, t'') and (x, t') be any supermodular implementable scf such that $t'', t' \in \mathcal{T}$ and $t'' \succeq_{\text{ID}} t'$. For any supermodular implementable scf, the induced game has a smallest and a greatest equilibrium along with a truthful equilibrium in between. Let $\theta_i^T(.)$ denote player i's truthful strategy, that is, $\theta_i^T(\theta_i) = \theta_i$ for all θ_i. Let $\mathcal{G}_\ell$ be the game $\mathcal{G}$ where the strategy spaces are restricted from $\Sigma_i(\Theta_i)$ to $[\inf \Sigma_i(\Theta_i), \theta_i^T(.)]$, and let $\mathcal{G}_u$ be the game $\mathcal{G}$ where the strategy spaces are restricted from $\Sigma_i(\Theta_i)$ to $[\theta_i^T(.), \sup \Sigma_i(\Theta_i)]$. Since closed intervals are sublattices and $\mathcal{G}$ is supermodular, those modified games $\mathcal{G}_\ell$ and $\mathcal{G}_u$ are supermodular games. Moreover, $\mathcal{G}_\ell$ must have the same least equilibrium as game $\mathcal{G}$ and the truthful equilibrium is its largest equilibrium. Likewise, $\mathcal{G}_u$ has the same greatest equilibrium as game $\mathcal{G}$ and the truthful equilibrium is its smallest equilibrium. Let $u_i^f(\hat{\theta}(.), t) = E_\theta[V_i(x_i(\hat{\theta}(\theta)), \theta_i)] + E_\theta[t_i(\hat{\theta}(\theta))]$. I show that (i) In $\mathcal{G}_\ell$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has decreasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$ and (ii) In $\mathcal{G}_u$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has increasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. In those modified games, this shows how the untruthful extremal equilibrium varies in response to changes in transfers with respect to $\succeq_{\text{ID}}$. Before proving (i) and (ii), note that Proposition 7 implies that all transfers t_i such that (x, t) is implementable have the same expected value $E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ up to a constant. Taking any implementable scf $(x, \tilde{t})$, those transfers can thus be written $t_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[\tilde{t}_i(\hat{\theta}_i, \theta_{-i})]$ for some function $\delta_i : \Theta \to \mathbb{R}$. First consider $\mathcal{G}_\ell$ and let δ'' and δ' be the δ functions corresponding to t'' and t'. Choose any $\theta_i''(.) > \theta_i'(.)$ and notice that for any deception $\hat{\theta}_{-i}(.)$, $\hat{\theta}_j(\theta_j) \leq \theta_j$ for all θ_j and $j \neq i$. Moreover, note $t'' \succeq_{\text{ID}} t'$ implies $\delta'' \succeq_{\text{ID}} \delta'$. Hence for all $i \in N$,

$$\begin{aligned} &E_\theta[\delta_i''(\theta_i''(\theta_i), \theta_{-i}) - \delta_i''(\theta_i''(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] - E_\theta[\delta_i''(\theta_i'(\theta_i), \theta_{-i}) - \delta_i''(\theta_i'(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] - \\ &- E_\theta[\delta_i'(\theta_i''(\theta_i), \theta_{-i}) - \delta_i'(\theta_i''(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] + E_\theta[\delta_i'(\theta_i'(\theta_i), \theta_{-i}) - \delta_i'(\theta_i'(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] \geq 0 \end{aligned} \tag{14}$$

But (14) is equivalent to

$$u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.), t'') + u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.), t') - u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.), t') - u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.), t'') \leq 0 \tag{15}$$

for each $\hat{\theta}_{-i}(.)$, which implies that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has decreasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. It follows from Theorem 6 in Milgrom-Roberts [39] that the smallest equilibrium in $\mathcal{G}_\ell$ is decreasing in t. The same argument applies to $\mathcal{G}_u$. There, all deceptions $\hat{\theta}_{-i}(.)$ are such that $\hat{\theta}_j(\theta_j) \geq \theta_j$ for all θ_j and $j \neq i$. As a result, the sign in (14) is reversed, which implies $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has increasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. The greatest equilibrium in $\mathcal{G}_u$ is thus increasing in t. Q.E.D

Proof of Theorem 3: Suppose $f = (x, t)$ is implementable and x is dimensionally reducible. Letting

$$\delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) = -\int_{\underline{\theta}_i}^{\hat{\theta}_i} \int_{r_i(\underline{\theta}_{-i})}^{r_i(\hat{\theta}_{-i})} \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(h_i(s_i, r_i), \theta_i)}{\partial r_i \partial s_i} \, dr_i \, ds_i \tag{16}$$

for all $\hat{\theta} \in \Theta$, I show that

$$t_i^*(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] \tag{17}$$

is well-defined and that (x, t^*) is optimally supermodular implementable. By Proposition 1, $E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ is well-defined and given by (13). Since V_i and h_i are C^2 on an open set containing compact set Θ_i, $\min_{\theta_i \in \Theta_i} \partial^2 V_i(h_i(s_i, r_i), \theta_i)/\partial r_i \partial s_i$ exists, it is continuous in (r_i, s_i) by the Maximum Theorem and it is bounded. Hence $\delta_i : \Theta \to \mathbb{R}$

is continuous, which implies that it is Borel-measurable. Since δ_i is also bounded, $E_{\theta_{-i}}[\delta_i(.,\theta_{-i})]$ is well-defined and so is $t_i^* : \Theta \to \mathbb{R}$. The next step is to verify the continuity requirements. As a continuous function on a compact set, δ_i is uniformly continuous in $\hat{\theta}$, and so $E_\theta[t^*(\hat{\theta}(\theta))]$ is continuous in $\hat{\theta}_{-i}(.)$. Since V is C^2, (13) is usc in $\hat{\theta}_i$ and so is $E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})]$ by Proposition 1, which implies $E_\theta[t_i^*(\hat{\theta}(\theta))]$ is usc in $\hat{\theta}_i(.)$. Put together, u_i^f satisfies the continuity requirements. Finally I prove that (x,t^*) is optimally supermodular implementable. Note $E_{\theta_{-i}}[t_i^*(\hat{\theta}_i,\theta_{-i})] = E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})]$ and thus (x,t^*) is implementable. By construction, t_i^* is twice-differentiable[35] and

$$\frac{\partial^2 t_i^*(\hat{\theta}_i,\hat{\theta}_{-i})}{\partial\hat{\theta}_i\partial\hat{\theta}_j} = \frac{\partial^2 \delta_i(\hat{\theta}_i,\hat{\theta}_{-i})}{\partial\hat{\theta}_i\partial\hat{\theta}_j} = \frac{\partial}{\partial\hat{\theta}_j}\int_{r_i(\underline{\theta}_{-i})}^{r_i(\hat{\theta}_{-i})} -\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(h_i(\hat{\theta}_i,r_i),\theta_i)}{\partial r_i\partial s_i}\,dr_i$$
$$= -\left(\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(h_i(\hat{\theta}_i,r_i(\hat{\theta}_{-i})),\theta_i)}{\partial r_i\partial s_i}\right)\frac{\partial r_i(\hat{\theta}_{-i})}{\partial\hat{\theta}_j}. \quad (18)$$

Because

$$-\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(x_i(\hat{\theta}_i,\hat{\theta}_{-i}),\theta_i)}{\partial\hat{\theta}_i\partial\hat{\theta}_j} = -\min_{\theta_i\in\Theta_i}\left(\frac{\partial^2 V_i(h_i(\hat{\theta}_i,r_i(\hat{\theta}_{-i})),\theta_i)}{\partial r_i\partial s_i}\frac{\partial r_i(\hat{\theta}_{-i})}{\partial\hat{\theta}_j}\right) \quad (19)$$

and $r_i(.)$ is an increasing function, (18) and (19) are equal. Therefore, $\partial^2[V_i(x_i(\hat{\theta}),\theta_i) + t_i^*(\hat{\theta})]/\partial\hat{\theta}_i\partial\hat{\theta}_j$ is equal to

$$\left(\frac{\partial^2 V_i(h_i(\hat{\theta}_i,r_i(\hat{\theta}_{-i})),\theta_i)}{\partial r_i\partial s_i} - \min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(h_i(\hat{\theta}_i,r_i(\hat{\theta}_{-i})),\theta_i)}{\partial r_i\partial s_i}\right)\frac{\partial r_i(\hat{\theta}_{-i})}{\partial\hat{\theta}_j} \geq 0 \quad (20)$$

for all $\hat{\theta}$, θ_i and j,i, and so (x,t^*) is supermodular implementable. Moreover, for all transfers $t \in \mathcal{T}$ such that (x,t) is implementable, it must be that

$$\frac{\partial^2 t_i(\hat{\theta})}{\partial\hat{\theta}_i\partial\hat{\theta}_j} \geq -\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(x_i(\hat{\theta}_i,\hat{\theta}_{-i}),\theta_i)}{\partial\hat{\theta}_i\partial\hat{\theta}_j} = \frac{\partial^2 t_i^*(\hat{\theta})}{\partial\hat{\theta}_i\partial\hat{\theta}_j}$$

for all $\hat{\theta}$ and j,i. This implies that (x,t^*) is optimally supermodular implementable.

Q.E.D

Proof of Theorem 4: By way of contradiction, suppose that the truthful equilibrium is not the unique Bayesian equilibrium. Since the scf is supermodular implementable, there exist a greatest and a smallest equilibrium in the game induced by the mechanism. So, one of these extremal equilibria must be strictly greater/smaller than the truthful one. Suppose that the greatest equilibrium, denoted $(\overline{\theta}_i(.))_{i\in N} \in \prod\Sigma_i(\Theta_i)$, is strictly greater than the truthful equilibrium. That is, for all $i \in N$, $\overline{\theta}_i(\theta_i) \geq \theta_i$ for a.e θ_i, and there exists $N^* \neq \emptyset$ such that, for all $i \in N^*$, $\overline{\theta}_i(\theta_i) > \theta_i$ for all θ_i in some subset of types with positive measure.

I evaluate the first-order condition of agent i's maximization program at the greatest equilibrium; then, I bound it from above by an expression which cannot be positive for

[35]See previous footnote.

all players (hence the contradiction). Consider player i's interim utility for type θ_i against $\overline{\theta}_{-i}(.)$:

$$E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))]. \tag{21}$$

Since $V_i \circ x_i$ and t_i are (resp.) ω_i- and β_i-Lipschitz and both differentiable in $\hat{\theta}_i$ for all $\hat{\theta}_{-i}$, we can apply the Bounded Convergence Theorem to show that for any deception $\hat{\theta}_{-i}(.)$ the first-derivative of (21) with respect to $\hat{\theta}_i$ is

$$E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)}{\partial \hat{\theta}_i}\right] + E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))}{\partial \hat{\theta}_i}\right]. \tag{22}$$

Since $u_i \circ f$ has complements bounded by κ_i, we have

$$E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)}{\partial \hat{\theta}_i} + \frac{\partial t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))}{\partial \hat{\theta}_i} - \frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i} - \frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right] \tag{23}$$

$$\leq \int_{\Theta_{-i}} \kappa_i \sum_{j \neq i} (\overline{\theta}_j(\theta_j) - \theta_j) \phi_{-i}(\theta_{-i}) d\theta_{-i} = \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] \tag{24}$$

By (23) and (24),

$$(22) \leq \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i}\right] + E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right]. \tag{25}$$

By part (i) of Proposition 7,

$$E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right] = -E_{\theta_{-i}}\left[\left.\frac{\partial V_i(x_i(\theta'_i, \theta_{-i}), \hat{\theta}_i)}{\partial \theta'_i}\right|_{\theta'_i = \hat{\theta}_i}\right].$$

Therefore, (25) implies

$$(22) \leq \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i} - \frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i)}{\partial \theta'_i}\right]. \tag{26}$$

If, as claimed, it is optimal for each player i to play $\overline{\theta}_i(\theta_i)$ for a.e type θ_i, then the RHS of (26) evaluated at $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ must be positive for a.e θ_i and all $i \in N$. To see why, let $\Theta_i^* \subset \Theta_i$ be the set of types θ_i for which the RHS of (26) is strictly negative when evaluated at $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$. Note that Θ_i^* is measurable by definition, because the RHS of (26) is a measurable function in θ_i when $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$. By way of contradiction, suppose there is a player $i \in N$ for whom Θ_i^* has strictly positive measure. Since the RHS of (26) is greater than (22), if $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ then (22) is strictly negative for all $\theta_i \in \Theta_i^*$. But for types $\theta_i \in \Theta_i^*$, $[\underline{\theta}_i, \overline{\theta}_i(\theta_i)]$ is available to player i. Thus there exists $\varepsilon > 0$ for which the deception $\theta_i^* : \Theta_i \to \Theta_i$ defined as $\theta_i^*(\theta_i) = \overline{\theta}_i(\theta_i) - \varepsilon \mathbf{1}_{\Theta_i^*}$ for all θ_i gives i a strictly greater utility than $\overline{\theta}_i(.)$. Notice $\theta_i^*(.) \in \Sigma_i(\Theta_i)$ because $\overline{\theta}_i(.) \in \Sigma_i(\Theta_i)$, so $\theta_i^*(.)$ improves on $\overline{\theta}_i(.)$ which is a contradiction. As a result, Θ_i^* has null measure.

Since it is optimal for each player i to play $\overline{\theta}_i(\theta_i)$ for a.e type θ_i, the RHS of (26) at $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ is positive for a.e θ_i and all $i \in N$. However, this leads to the following contradiction. If the RHS of (26) is positive for a.e θ_i, then

$$\begin{aligned} 0 \leq\ & \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_i}\left[\frac{\partial E_{\theta_{-i}}[V_i(x_i(\overline{\theta}_i(\theta_i), \theta_{-i}), \theta_i)]}{\partial \hat{\theta}_i} - \right. \\ & \left. - \frac{\partial E_{\theta_{-i}}[V_i(x_i(\overline{\theta}_i(\theta_i), \theta_{-i}), \overline{\theta}_i(\theta_i))]}{\partial \hat{\theta}_i}\right] \\ \leq\ & \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + \gamma_i E_{\theta_i}[\theta_i - \overline{\theta}_i(\theta_i)] \text{ for all } i \in N, \end{aligned} \tag{27}$$

where the last inequality follows from γ_i-increasing differences. Since $\kappa_i/\gamma_i < 1/(n-1)$ by hypothesis and ϕ_j has full support for all j, (27) implies

$$\sum_{j \neq i} \frac{1}{n-1} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] \geq E_{\theta_i}[\overline{\theta}_i(\theta_i) - \theta_i] \text{ for all } i \in N, \text{ and}$$

$$\sum_{j \neq i} \frac{1}{n-1} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] > E_{\theta_i}[\overline{\theta}_i(\theta_i) - \theta_i] \text{ for all } i \in \{i : \{j \neq i\} \cap N^* \neq \emptyset\}.$$

Hence

$$\sum_{i \in N} \sum_{j \neq i} \frac{1}{n-1} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] > \sum_{i \in N} E_{\theta_i}[\overline{\theta}_i(\theta_i) - \theta_i]$$

which is a contradiction because both sides are equal by definition. It is not optimal for all $i \in N$ to play $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ for a.e θ_i. Thus, there is no equilibrium that is greater than the truthful equilibrium. The same argument applies to show that there is no equilibrium that is smaller than the truthful equilibrium. Truth-revealing is the unique equilibrium. Q.E.D

Proof of Proposition 3: Since the valuations and the decision rule produce γ-increasing differences, $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]/\partial \theta_i$ is strictly increasing in $\hat{\theta}_i$. Let transfers be the optimal transfers defined by (16) and (17), where t_i is given by (13). By assumption, $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$, so Proposition 7 and Theorem 3 imply (x, t^*) is supermodular implementable. It follows from (20) that $u_i \circ f$ has bounded complements, because V and x are C^2. The bound κ_i on complements is computed as follows,

$$\kappa_i = \max_{j \neq i} \max_{(\hat{\theta}, \theta_i) \in \Theta \times \Theta_i} \left(\frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} - \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right).$$

Since x is dimensionally reducible and V is C^2, the first derivative of t_i^* in $\hat{\theta}_i$ is uniformly bounded above. Hence transfers are β_i-Lipschitz in $\hat{\theta}_i$. Applying Theorem 4 completes the proof. Q.E.D

Proof of Theorem 2: Let

$$H_i(\hat{\theta}_{-i}) = -\left(\frac{1}{n-1}\right)\sum_{j\neq i} E_{\tilde{\theta}_{-j}}\left[\sum_{k\neq j} V_k(x_k(\hat{\theta}_j, \tilde{\theta}_{-j}), \tilde{\theta}_k)\right],$$

and for $\rho_i \in \mathbb{R}$, let

$$\delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \sum_{j\neq i} \rho_i \hat{\theta}_i \hat{\theta}_j.$$

Define

$$t_i^{BB}(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\tilde{\theta}_{-i}}\left[\sum_{j\neq i} V_j(x_j(\hat{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)\right] + H_i(\hat{\theta}_{-i}) -$$
$$- \frac{1}{n-2}\sum_{j\neq i}\sum_{k\neq i,j} \rho_j \hat{\theta}_j \hat{\theta}_k + \frac{1}{n-2}\sum_{j\neq i}\sum_{k\neq i,j} \rho_j \hat{\theta}_j E(\theta_k). \quad (28)$$

First, (x, t^{BB}) is implementable because $x(.)$ is allocation-efficient and

$$E_{\theta_{-i}}[t_i^{BB}(\hat{\theta}_i, \theta_{-i})] = E_{\tilde{\theta}_{-i}}\left[\sum_{j\neq i} V_j(x_j(\hat{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)\right] + E_{\theta_{-i}}[H_i(\theta_{-i})],$$

which is the expectation of the transfers in the expected externality mechanism. Second, note that for all θ,

$$\sum_{i\in N}\left(\delta_i(\theta_i, \theta_{-i}) - \frac{1}{n-2}\sum_{j\neq i}\sum_{k\neq i,j} \rho_j \theta_j \theta_k\right) = \sum_{i\in N} \delta_i(\theta_i, \theta_{-i}) - \frac{1}{n-2}\sum_{i\in N}\sum_{j\neq i}(n-2)\rho_i \theta_i \theta_j = 0$$

and

$$\sum_{i\in N}\left(\frac{1}{n-2}\sum_{j\neq i}\sum_{k\neq i,j} \rho_j \theta_j E(\theta_k) - E_{\theta_{-i}}[\delta_i(\theta_i, \theta_{-i})]\right) =$$
$$= \frac{1}{n-2}\sum_{i\in N}\sum_{j\neq i}(n-2)\rho_i \theta_i E(\theta_j) - \sum_{i\in N} E_{\theta_{-i}}[\delta_i(\theta_i, \theta_{-i})] = 0,$$

hence

$$\sum_{i\in N} t_i^{BB}(\theta) = \sum_{i\in N} E_{\tilde{\theta}_{-i}}\left[\sum_{j\neq i} V_j(x_j(\theta_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)\right] + \sum_{i\in N} H_i(\theta_{-i}) = 0,$$

because transfers are balanced in the expected externality mechanism. Furthermore, t_i^{BB} is clearly continuous in $\hat{\theta}_{-i}$ for each $\hat{\theta}_i$ and usc in $\hat{\theta}_i$ for each $\hat{\theta}_{-i}$. From standard arguments, $E_\theta[t_i^{SM}(\hat{\theta}_i(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))]$ is continuous in $\hat{\theta}_{-i}(.)$ and usc in $\hat{\theta}_i(.)$. Next I show that it is possible to take ρ_i so that the complete information payoffs have increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$. By assumption, there exists a lower bound T_i on the substitutes from $V_i \circ x_i(.)$. Set $\rho_i > -T_i$. Choose any $\theta''_{-i} \geq_{-i} \theta'_{-i}$ and $\theta''_i > \theta'_i$. From (28), note

$$t_i^{BB}(\theta''_i, \theta''_{-i}) - t_i^{BB}(\theta''_i, \theta'_{-i}) - t_i^{BB}(\theta'_i, \theta''_{-i}) + t_i^{BB}(\theta'_i, \theta'_{-i}) =$$
$$= \delta_i(\theta''_i, \theta''_{-i}) - \delta_i(\theta''_i, \theta'_{-i}) - \delta_i(\theta'_i, \theta''_{-i}) + \delta_i(\theta'_i, \theta'_{-i}). \quad (29)$$

If the following expression is positive, then $u_i(x_i(\hat{\theta}_i,\hat{\theta}_{-i}),\theta_i)$ has increasing differences in $(\hat{\theta}_i,\hat{\theta}_{-i})$ for all θ_i,

$$V_i(x_i(\theta_i'',\theta_{-i}''),\theta_i)+V_i(x_i(\theta_i',\theta_{-i}'),\theta_i)-V_i(x_i(\theta_i'',\theta_{-i}'),\theta_i)-V_i(x_i(\theta_i',\theta_{-i}''),\theta_i) + \sum_{j\neq i}\rho_i\big(\theta_i''\theta_j''+\theta_i'\theta_j'-\theta_i''\theta_j'-\theta_i'\theta_j''\big). \quad (30)$$

The proof then follows similarly to that of Theorem 1. Q.E.D

Proof of Proposition 4: Since $\tau_i - T_i < \gamma_i/(n-1)$, then $\rho_i = -T_i$ implies $\rho_i + \tau_i < \gamma_i/(n-1)$. By Theorem 2, (x, t^{BB}) is supermodular implementable whenever $\rho_i \geq -T_i$. Because $V_i \circ x_i(.)$ has complements bounded by τ_i, the definition of t_i^{BB} implies that $u_i \circ f$ has complements bounded by $\rho_i + \tau_i$. Theorem 4 completes the proof. Q.E.D

Proof of Proposition 5: Let $O \supset \Theta$ be some open set and define the extension of $x(.)$ from Θ to O. For any $\theta \in O$, let $\iota_1(\theta) = \{j \in N : \theta_j \in [\underline{\theta}_j, \overline{\theta}_j]\}$, $\iota_2(\theta) = \{j \in N : \theta_j < \underline{\theta}_j\}$ and $\iota_3(\theta) = \{j \in N : \theta_j > \overline{\theta}_j\}$. The extension of $x(.)$ from Θ to O, denoted x^e, is such that for all $\theta \in O$, $x^e_{(i,k)}(\theta) = x_{(i,k)}((\theta_j)_{\iota_1(\theta)}, (\underline{\theta}_j)_{\iota_2(\theta)}, (\overline{\theta}_j)_{\iota_3(\theta)})$ for all k and $i \in N$. Note that $x^e_{(i,k)} \in L_p(O)$ and it is increasing in $\hat{\theta}_i$ because $x_{(i,k)}$ is increasing in $\hat{\theta}_i$. By Theorem 12.10 in [4], the space of C^2-functions on O is norm dense in $L_p(O)$, hence there exists a sequence $\{x_n\}$ of C^2-functions from O into $\mathbb{R}$ such that $\lim_{n\to\infty}(\int_O |x_{n,(i,k)} - x^e_{(i,k)}|^p)^{1/p} = 0$ for all k and i. This implies $\lim_{n\to\infty}(\int_\Theta |x_{n,(i,k)} - x_{(i,k)}|^p)^{1/p} = 0$ for all k and all i. Moreover, we can take $\{x_n\}$ such that $x_{n,(i,k)}$ is increasing in θ_i on O_i for all k and i.[36] By definition, V and x are C^2 if there exist open sets $U_i \supset X_i$, $i = 1,\ldots,n$, such that $V : U_i \times O_i \to \mathbb{R}$ and $x : \prod_{i\in N} O_i \to U_i$ are C^2. Therefore, since each Θ_i is compact and V and x_n are C^2, then they form a continuous family, $\partial E_{\theta_{-i}}[V_i(x_{n,(i,k)}(\hat{\theta}),\theta_i)]/\partial\theta_i = E_{\theta_{-i}}[\partial V_i(x_{n,(i,k)}(\hat{\theta}),\theta_i)/\partial\theta_i]$ is increasing in $\hat{\theta}_i$ on Θ_i and substitutes are bounded. Proposition 1 and Theorem 1 imply that, for all n, there exist t_n^{SM} such that $f = (x_n, t_n^{SM})$ is supermodular implementable. Q.E.D

Proof of Proposition 6: The proof begins with an approximation of the functions $h_{(i,k)} : \mathbb{R}^2 \to \mathbb{R}$ and $r_i : \mathbb{R}^n \to \mathbb{R}$ by C^2-functions, and it studies the convergence of the resulting composite function. Let μ^n denote the Lebesgue measure on $\mathbb{R}^n$. Because type sets are compact and h_i is bounded, Theorem 12.10 in [4] guarantees the existence of a sequence of C^2-functions that converges to $h_{(i,k)}$ in $L_1(\mu^2)$-norm. Since h_i is bounded, we can take that sequence so that each element is (uniformly) bounded. From this sequence, Theorem 12.6 in [4] implies that we can extract a subsequence $\{h^m_{(i,k)}\}$ of C^2-functions that converges pointwise to $h_{(i,k)}$ for μ^2-almost all (θ_i, r_i). Now consider function $r_i(.)$. By the Stone-Weierstrass theorem, for all $i \in N$ there exists a sequence of C^2-increasing functions $\{r_i^q\}$ that uniformly converges to r_i.[37] The triangle

[36] Since $x_{i,k}$ is increasing in θ_i, it is always possible to take the members of the approximating sequence to be increasing (See Mas-Colell [35]).

[37] Since r_i is increasing, recall that we can take the members of the approximating sequence to be increasing.

inequality gives

$$\int_\Theta |h^m_{(i,k)}(\theta_i, r^q_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n \leq \int_\Theta |h^m_{(i,k)}(\theta_i, r^q_i(\theta_{-i})) - h^m_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n + \int_\Theta |h^m_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n. \quad (31)$$

The next step is to demonstrate that the second integral in the RHS of (31) converges to zero, as a result of the μ^2-a.e convergence of $h^m_{(i,k)}$.[38] Note that

$$\int_\Theta |h^m_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n = \int_{\Theta_i \times r_i(\Theta_{-i})} |h^m_{(i,k)}(\theta_i, t) - h_{(i,k)}(\theta_i, t)| d\mu \times \mu_{r_i} \quad (32)$$

where $\mu_{r_i} = \mu^{n-1} \circ r_i^{-1}$. One way to proceed is to apply the Radon-Nikodym theorem. To this end, I show that μ_{r_i} is absolutely continuous with respect to μ. By way of contradiction, suppose that $\mu(A) = 0$ for some set A and that there are countable unions of intervals, $\cup_k I^k_j \subset \mathbb{R}$, such that $r_i(\theta_{-i}) \in A$ for all $\theta_{-i} \in \prod_j (\cup_k I^k_j)$. Since $r_i(.)$ is continuous and strictly increasing, $r_i(\prod_j (\cup_k I^k_j))$ must contain some interval I, in which case $I \subset A$ and $\mu(A) > 0$. This is a contradiction. Therefore, for any A such that $\mu(A) = 0$, there is no $\{\cup_k I^k_j\}$ such that $r_i^{-1}(A) \subset \prod_j (\cup_k I^k_j)$, which implies $\mu_{r_i}(A) = 0$. As a result, μ_{r_i} is absolutely continuous with respect to μ. Clearly, both μ_{r_i} and μ are (totally) finite on $r_i(\Theta_{-i})$. By the Radon-Nikodym theorem, there exists f on $r_i(\Theta_{-i})$ such that $\mu_{r_i}(A) = \int_A f d\mu$ for every measurable set $A \subset r_i(\Theta_{-i})$. From (32), it gives

$$\int_\Theta |h^m_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n = \int_{\Theta_i \times r_i(\Theta_{-i})} |h^m_{(i,k)}(\theta_i, t) - h_{(i,k)}(\theta_i, t)| f(t) d\mu^2. \quad (33)$$

Since $|h^m_{(i,k)}(\theta_i, t) - h_{(i,k)}(\theta_i, t)| f(t)$ is integrable and dominated a.e by $Hf(t)$ for $H > 0$ sufficiently large, the limit of the RHS of (33) as $m \to \infty$ is given by the (integral of the) limit of the integrand, and this limit is 0. This result allows to construct the following subsequence from $\{h^m_i(\theta_i, r^q_i(\theta_{-i}))\}$:

1. For each m, take $\alpha(m)$ such that $\int_\Theta |h^{\alpha(m)}_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| \, d\mu^n < 1/2m$.

2. Since $h^{\alpha(m)}$ is C^2, $h^{\alpha(m)}_{(i,k)}(\theta_i, r^q_i(\theta_{-i}))$ converges uniformly to $h^{\alpha(m)}_i(\theta_i, r_i(\theta_{-i}))$ as $q \to \infty$; thus choose $\beta(m)$ such that $\int_\Theta |h^{\alpha(m)}_i(\theta_i, r^{\beta(m)}_i(\theta_{-i})) - h_i(\theta_i, r_i(\theta_{-i}))| \, d\mu^n < 1/2m$.

Along the subsequence so constructed, the LHS of (31) is less than $1/m$ for all m and thus it converges to $h_i(., r_i(.))$ in L_1-norm. In other words, there is a sequence of dimensionally-reducible decision rules $\{x^m_i\}$ that converges to x_i in L_1-space. Implementability of each x^m follows from the fact that $\partial V_i(x_i, \theta_i)/\partial \theta_i$ is increasing in x_i

[38]This is indeed not immediate. Suppose $\lim_{m\to\infty} h^m_{(i,k)} = h_{(i,k)}$ except for $\{(\theta_i, r^*_i) : \theta_i \in I\}$ where I is some interval. If $r_i(.)$ is constant and equal to r^*_i, then $\lim_{m\to\infty} h^m_{(i,k)} = h_{(i,k)}$ μ^2-a.e, but $\int_\Theta |h^m_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n$ does not converge to 0.

and $x_i^m(.)$ is increasing in $\hat{\theta}_i$ for each m. Hence $x^m(.)$ is implementable. Theorem 3 completes the proof. Q.E.D

Lemma 2 *Let $(X, \geq)$ be a complete lattice. For $Y \supset X$, let $\phi : X \twoheadrightarrow Y$ be a correspondence whose range is Y and such that for all $x \in X$, $x \in \phi(x)$ and $\phi(x') \cap \phi(x) = \emptyset$ for all $x' \neq x$. Then, there exists an extension $\geq^*$ of $\geq$ such that:*
(i) $(Y, \geq^*)$ is a complete lattice,
(ii) For all distinct $x, x' \in X$, and all $y \in \phi(x)$, $y' \in \phi(x')$, $y \geq^* y'$ iff $x \geq x'$,
(iii) For all $x \in X$, $\phi(x)$ is a complete chain.

Proof: Define $\geq^*$ on X as $\geq$. Then, for all distinct $x, x' \in X$, and all $y \in \phi(x)$, $y' \in \phi(x')$, let $\geq^*$ be such that $y \geq^* y'$ iff $x \geq x'$. So *(ii)* is satisfied. Finally, complete the definition of $\geq^*$ by using the Well Ordering Principle of set theory. This result implies that, for all $x \in X$, there exists $\succeq$ on $\phi(x)$ such that $(\phi(x), \succeq)$ is a chain, and such that any $B \subset \phi(x)$ has a least upper bound and a greatest lower bound in $\phi(x)$.[39] Define $\geq^*$ to be equal to $\succeq$ on $\phi(x)$ for each $x \in X$. Therefore, for all $x \in X$, $\phi(x)$ is a complete chain and *(iii)* is satisfied. I show next that $(Y, \geq^*)$ is a complete lattice with the order $\geq^*$ just defined on all of Y.

First, I prove that it is a partially ordered set. For all $x \in X$, $x \in \phi(x)$ and thus $x \geq^* x$ because $(\phi(x), \geq^*)$ is a chain. This proves reflexivity. Now take $y_1, y_2, y_3 \in Y$ such that $y_1 \geq^* y_2$ and $y_2 \geq^* y_3$. If $y_1 \in \phi(x_1)$, $y_2 \in \phi(x_2)$ and $y_3 \in \phi(x_3)$ where x_1, x_2, x_3 are distinct, then $y_1 \geq^* y_2$ implies $x_1 > x_2$ and $y_2 \geq^* y_3$ implies $x_2 > x_3$. By transitivity of $\geq$, we have $x_1 > x_3$, which implies $y_1 \geq^* y_3$. Suppose that $y_1, y_2 \in \phi(x_1)$ and $y_3 \in \phi(x_3)$ for distinct $x_1, x_3 \in X$. Since $y_2 \geq^* y_3$, we have $x_1 > x_3$ which implies $y_1 \geq^* y_3$. If $y_1, y_2, y_3 \in \phi(x_1)$, then $y_1 \geq^* y_3$ because $(\phi(x_1), \geq^*)$ is a chain, which shows transitivity. Now, if $y_1 \geq^* y_2$ and $y_2 \geq^* y_1$ for some $y_1 \in \phi(x_1)$ and $y_2 \in \phi(x_2)$, then $x_1 = x_2$. Therefore, $y_1, y_2 \in \phi(x_1)$ and so $y_1 = y_2$ because $(\phi(x_1), \geq^*)$ is a chain. This establishes antisymmetry.

Secondly, I prove that $\sup_Y S$ and $\inf_Y S$ exist, so $(Y, \geq^*)$ is a complete lattice. Let $\mathcal{X} \subset X$ be the set of x's whose image intersects S: $x \in \mathcal{X}$ iff $S \cap \phi(x) \neq \emptyset$. If $|\mathcal{X}| = 1$, then $S \subset \phi(x)$ where x is the unique element of $\mathcal{X}$. By definition of $\geq^*$, S has an infimum and a supremum in $\phi(x) \subset Y$. Now assume $|\mathcal{X}| \geq 2$ and let $S(x) = S \cap \phi(x)$ for all $x \in \mathcal{X}$. Note $\{S(x)\}_{x \in \mathcal{X}}$ forms a partition of S. Define $\overline{s}(x) = \sup_Y S(x)$ and $\underline{s}(x) = \inf_Y S(x)$, which exist and belong to $\phi(x)$ by definition of $\geq^*$. Note that if $\sup_Y S$ and $\inf_Y S$ exist, then $\sup_Y S = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$ and $\inf_Y S = \inf_Y(\cup_{\mathcal{X}} \underline{s}(x))$ by associativity. Since $(X, \geq)$ is a complete lattice, $\sup_X \mathcal{X}$ exists; call it $\overline{x}$. If $\overline{x} \in \mathcal{X}$, then $\overline{s}(\overline{x}) = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$ and so $\sup_Y S$ exists. So suppose $\overline{x} \notin \mathcal{X}$. Define $s^* = \inf_Y \phi(\overline{x})$ and note $s^* \in \phi(\overline{x})$. I show $s^* = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$. Since $\overline{x} \notin \mathcal{X}$, $\overline{x} > x$ for all $x \in \mathcal{X}$. This implies $s^* \geq^* \overline{s}(x)$ for all $x \in \mathcal{X}$. Hence s^* is an upper bound for $\cup_{\mathcal{X}} \overline{s}(x)$. Take any upper bound $\overline{y} \neq s^*$ for $\cup_{\mathcal{X}} \overline{s}(x)$. Then $\overline{y} \notin \cup_{\mathcal{X}} \overline{s}(x)$, for if there were $x' \in \mathcal{X}$ such that $\overline{y} = \overline{s}(x')$ then $x' \geq x$ for all $x \in \mathcal{X}$ would imply that $\overline{x} \equiv \sup_X \mathcal{X} = x'$ is in $\mathcal{X}$, a contradiction. Therefore, $\overline{y} \in \phi(\tilde{x})$ for some $\tilde{x} \in X \backslash \mathcal{X}$ and since $\overline{y} \geq^* \overline{s}(x)$ for all

[39] Take $\omega \in \phi(x)$. By the Well Ordering Principle, there is an order that well orders $\phi(x) \backslash \{\omega\}$. Extend this order to all of $\phi(x)$ by setting ω as the greatest element. Let $\succeq$ be the extension. Since $(\phi(x), \succeq)$ is also well ordered, $\inf_{\phi(x)}(S)$ exists for any $S \subset \phi(x)$. Since the set of upper bounds of S contains ω, it has a least element because $\phi(x)$ is well ordered. Hence $\sup_{\phi(x)}(S)$ exists.

$x \in \mathcal{X}$, $\tilde{x} > x$ for all $x \in \mathcal{X}$. Hence $\tilde{x} \geq \overline{x}$. If $\tilde{x} \neq \overline{x}$, then $\overline{y} >^* s^*$, and if $\tilde{x} = \overline{x}$, then $\overline{y} \in \phi(\overline{x})$ implies $\overline{y} \geq^* s^*$. As a result, $s^* = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$. Finally, $\inf_Y S$ exists by a similar argument. Since $(X, \geq)$ is a complete lattice, $\inf_X \mathcal{X}$ exists; call it $\underline{x}$. If $\underline{x} \in \mathcal{X}$, then $\inf_Y(\cup_{\mathcal{X}} \underline{s}(x)) = \underline{s}(\underline{x})$. Otherwise $\inf_Y(\cup_{\mathcal{X}} \underline{s}(x)) = \sup_Y \phi(\underline{x})$. Q.E.D

Proof of Theorem 5: By the traditional revelation principle, (Θ, f) truthfully implements f in Bayesian equilibrium with any order on Θ_i. It remains to prove that there is an order $\geq_i^*$ on Θ_i such that the game induced by $(\{(\Theta, \geq_i^*)\}, f)$ is supermodular. I prove first that, for any $i \in N$, the order $\succeq_i$ on M_i induces an order $\geq_i^*$ on Θ_i such that $(\Theta_i, \geq_i^*)$ is a (complete) lattice. So, $\Sigma_i(\Theta_i)$ is a (complete) lattice with the pointwise order. Second, I establish that under $\geq_i^*$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$ and has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$.

Denote $M_i^* = m_i^*(\Theta_i)$ for all $i \in N$. Define correspondence $[\,] : M_i^* \twoheadrightarrow \Theta_i$ where $[m_i] = \{\theta_i \in \Theta_i : m_i^*(\theta_i) = m_i\}$ is the equivalence class of $m_i \in M_i^*$. Let $\theta^s : M_i^* \rightarrow \Theta_i$ be a selection from $[\,]$. As a mapping from M_i^* to $\theta^s(M_i^*)$, θ^s is a bijection because $m_i \neq m_i'$ necessarily implies $[m_i] \cap [m_i'] = \emptyset$. Since θ^s is a bijection, we can define $\geq_i$ on a subset of Θ_i such that $\theta^s(m_i'') \geq_i \theta^s(m_i')$ if and only if $m_i'' \succeq_i m_i'$. Because θ^s is an order-isomorphism from $(M_i^*, \succeq_i)$ to $(\theta^s(M_i^*), \geq_i)$, it preserves all existing joins and meets. This implies that $(\theta^s(M_i^*), \geq_i)$ is a (complete) lattice because $(M_i^*, \succeq_i)$ is a (complete) lattice. Define the extension $\geq_i^*$ (or simply $\geq^*$) of $\geq_i$ to all of Θ_i, as follows:

1. For any distinct $m_i, m_i' \in M_i^*$ and for all $\theta_i \in [m_i]$, $\theta_i' \in [m_i']$, $\theta_i \geq^* \theta_i'$ if and only if $\theta^s(m_i) \geq_i \theta^s(m_i')$.

2. For all $m_i \in M_i^*$, $([m_i], \geq^*)$ is a complete chain.

By Lemma 2, $(\Theta_i, \geq^*)$ is a (complete) lattice. Thus, $\Sigma_i(\Theta_i)$ is a (complete) lattice with the pointwise order. Endow those lattices with their order-interval topology and the Borel σ-algebra so that all functions are trivially continuous and measurable.

The next step of the proof will use the fact that $m_i^*(.)$ preserves meets and joins, which I prove now. Take any $T \subset \Theta_i$. Since $(M_i^*, \succeq_i)$ and $(\Theta_i, \geq^*)$ are complete lattices, $\sup_{M_i^*}(m_i^*(T))$ and $\sup_{\Theta_i} T$ exist. Denote $\overline{m}_T = \sup_{M_i^*}(m_i^*(T))$. Since $\sup_{\Theta_i} T$ is an upper bound for T, $\geq^*$ implies $m_i^*(\sup_{\Theta_i} T)$ is an upper bound for $m_i^*(T)$ in M_i^*. Thus, $m_i^*(\sup_{\Theta_i} T) \succeq_i \overline{m}_T$. But $\overline{m}_T$ is an upper bound for $m_i^*(T)$, hence $\sup_{[\overline{m}_T]}([\overline{m}_T])$ is an upper bound for T. So, $\sup_{[\overline{m}_T]}([\overline{m}_T]) \geq^* \sup_{\Theta_i} T$, and therefore, $\overline{m}_T \succeq_i m_i^*(\sup_{\Theta_i} T)$. A similar argument applies to show $\inf_{M_i^*}(m_i^*(T)) = m_i^*(\inf_{\Theta_i} T)$.

Now I show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$ and has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. Take any $i \in N$ and for all $j \neq i$, endow Θ_j with $\geq_j^*$ and $\Sigma_j(\Theta_j)$ with the corresponding pointwise order. Endow $\prod \Sigma_j(\Theta_j)$ with the product order. The first step is to show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$. For any $\theta_i''(.)$ and $\theta_i'(.)$, we know $m_i^*(\theta_i'(.)) \vee m_i^*(\theta_i''(.)) = m_i^*(\theta_i'(.) \vee \theta_i''(.))$ and similarly for $\wedge$. Since the mechanism $(\{(M_i, \succeq_i)\}, g)$ supermodularly implements f, $u_i^g(m_i(.), m_{-i}(.))$ is supermodular in $m_i(.)$ for each $m_{-i}(.)$. For any $\hat{\theta}_{-i}(.)$,

$$u_i^g(m_i^*(\theta_i'(.) \vee \theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.))) + u_i^g(m_i^*(\theta_i'(.) \wedge \theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.)))$$
$$\geq u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\hat{\theta}_{-i}(.))) + u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.))),$$

which implies that for any $\hat{\theta}_{-i}(.)$,

$$u_i^f(\theta_i'(.) \vee \theta_i''(.), \hat{\theta}_{-i}(.)) + u_i^f(\theta_i'(.) \wedge \theta_i''(.), \hat{\theta}_{-i}(.)) \geq u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.)) + u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.)).$$

The second step is to show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. For any $\theta_i''(.) \geq_i^* \theta_i'(.)$ and $\theta_{-i}''(.) \geq_{-i}^* \theta_{-i}'(.)$, we know $m_i^*(\theta_i''(.)) \succeq_i m_i^*(\theta_i'(.))$ and $m_{-i}^*(\theta_{-i}''(.)) \succeq_{-i} m_{-i}^*(\theta_{-i}'(.))$. Since the mechanism $(\{(M_i, \succeq_i)\}, g)$ supermodularly implements f, $u_i^g(m_i(.), m_{-i}(.))$ has increasing differences in $(m_i(.), m_{-i}(.))$. For any θ_i,

$$\begin{aligned} u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\theta_{-i}''(.))) - u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\theta_{-i}''(.))) &\geq \\ \geq u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\theta_{-i}'(.))) - u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\theta_{-i}'(.))), \end{aligned}$$

which implies that for any θ_i,

$$u_i^f(\theta_i''(.), \theta_{-i}''(.)) - u_i^f(\theta_i'(.), \theta_{-i}''(.)) \geq u_i^f(\theta_i''(.), \theta_{-i}'(.)) - u_i^f(\theta_i''(.), \theta_{-i}'(.)),$$

and it completes the proof. Q.E.D

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