Inequality and Network Structure*

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Abstract

We explore the manner in which the structure of a social network constrains the level of inequality that can be sustained among its members, based on the following considerations: (i) any distribution of value must be stable with respect to coalitional deviations, and (ii) joint deviations require communication and coordination, so the network structure itself determines the coalitions that may form. We show that if players can jointly deviate only if they form a clique in the network, then the degree of inequality that can be sustained depends on the cardinality of the maximum independent set. For bipartite graphs, this criterion allows us to obtain a complete ordering of networks with respect to the levels of inequality they can sustain. For general graphs, we obtain a partial ordering with the following property: extremal inequality cannot be larger in a network with a smaller maximum independent set. These results extend naturally to the case in which a group of players can deviate jointly if they are all within some fixed distance (possibly greater than one) of each other.

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1 Introduction

In 494 BCE, the *plebs* of the Roman Republic, seeking relief from judicial harassment, indebtedness and poverty, left Rome *en masse* and threatened to settle permanently outside its walls, as a result extracting major concessions from the Roman patricians (Livy, 1960). Plantation owners in Hawaii a century ago expressly hired workers who spoke different native languages to ensure that communication among them would be limited, thus discouraging labor action (Takaki, 1983). U.S. employer efforts in the 1930s to build firm loyalty by sponsoring social activities led to stronger bonds between workers that they could use to mobilize their collective power and form effective unions (Estlund, 2003). And geographically dispersed outsourcing by firms over the past decade has been profitable not only because of efficiency gains, but also because it limits the opportunities for suppliers to communicate and coordinate their actions.

A recurrent theme in these examples is the central role of coalitional deviations in determining the distribution of income. This motivates us to explore formally the manner in which the structure of a social network constrains the level of inequality that can be sustained among its members. Specifically, we develop a model of inequality on networks based on the following considerations: (i) any distribution of value must be stable with respect to coalitional deviations, and (ii) the set of feasible coalitions is itself constrained by the requirement that only groups of players that are mutually connected can jointly deviate. That is, we allow for deviations only by groups of individuals who form a *clique* in the network. A payoff distribution is said to be stable if there is no clique that can profitably deviate. The main research question is then the following: What is the relationship between the structure of the network and the maximum level of stable inequality?

To compare payoff distributions in terms of their level of inequality, we adopt the standard criterion of Lorenz dominance and define a value distribution to be *extremal* if it is stable with respect to clique deviations and does not Lorenz dominate any other stable distribution. Note that since Lorenz dominance provides only a partial ordering of value distributions, the extremal distribution for any given network may not be unique, and extremal distributions for different networks may be incomparable.

Our main contribution is to establish a connection between extremal inequality on a network and the cardinality of its maximum independent set.[1] This connection is especially

[1] An independent set in a network is a set of vertices such that no pair of vertices in the set are connected to each other. An independent set is maximum if there is no independent set with greater cardinality. Even though a network may have multiple maximum independent sets, they must (by definition) all have the same cardinality.

strong in the special case of bipartite networks, which have unique extremal distributions and can be completely ordered; we show that bipartite networks with larger maximum independent sets can sustain greater levels of extremal inequality. For general graphs with arbitrary clique sizes, a weaker result holds: for any two networks, extremal inequality cannot be greater in the one with the smaller maximum independent set.[2] Since the size of the maximum independent set is a proxy for the sparseness of a network, this means that inequality will be limited in dense networks.

Our framework can be easily extended to include the case in which players can jointly deviate if they are all within distance k of each other (the case of clique deviations corresponds to $k = 1$). We explore the manner in which extremal inequality changes as k is varied. Although inequality (weakly) declines as k increases, it can do so at different rates in different networks. As a result, the ranking of networks by the extent of extremal inequality is not invariant in k.

A number of recent papers have explored the determinants of inequality in equilibrium networks (see Section 2 for details). In these papers, an agent's central position confers the ability to gain larger shares of the surplus, the intuition being that essential intermediaries can extract rents through their control of flows between players that are not otherwise connected. These "middleman" models are implicitly based on the idea that competition reduces inequality, and monopoly increases it. The centrality measures thus explain distributional advantage by analyzing how well connected the rich are. While these intuitions are undoubtedly correct in many settings, our model stresses another dimension that influences inequality: the inability of the poor to form viable coalitions. Intuitively, if the network is dense, inequality will be hard to sustain as disadvantaged players can communicate and coordinate on joint actions. Conversely, if the network is sparse, peripheral players can more readily be exploited. Hence, in our approach, inequality is sustained not by the well-connectedness of the rich, but by the isolation of the poor.

2 Related literature

The idea that network structure influences the allocation of value was initially proposed in a seminal paper by Myerson (1977), who assumed that a coalition of individuals could generate value if and only if they were all connected to each other along some path that

[2] Interestingly, Bramoullé and Kranton (2007) find that independent sets play a central role in an entirely different context: the private provision of local public goods in a noncooperative setting; also see Corbo, Calvó-Armengol, and Parkes (2007).

did not involve anyone outside the coalition. Such paths could be of arbitrary length, which entails the implicit assumption that communication through intermediaries is as effective as direct communication in the process of coalition formation. In contrast, we assume that deviating coalitions require direct communication (or at least sufficiently short paths) between members. Myerson's work motivated a significant literature on communication games (see Slikker and van den Nouweland, 2001, for a survey) and more generally on games on combinatorial structures (Bilbao, 2000). For instance, focusing on hierarchies, Demange (2004) restricts coalitional deviations to teams and shows that there exist stable solutions for a wide range of games.

Our approach differs in two important respects from this line of work. First, while the aim of much of the literature cited above is to give a characterization of different solution concepts, to investigate their relation with each other, and to provide conditions for the existence of solutions in general classes of games, our focus is on the maximum degree of inequality that can be sustained in a restricted set of games where existence of stable distributions is guaranteed. Second, the coalitional deviations we allow do not generally constitute a combinatorial structure. For example, two feasible coalitions may overlap in our setting, without there being a feasible coalition (other than the complete network) that contains both, in contrast with the settings considered by Myerson (1977) and Demange (2004), for example. As we discuss in more detail in Section 6, this makes it difficult to completely characterize the degree of inequality that can be sustained in general networks.[3]

A number of writers have explored determinants of the degree of inequality in equilibrium networks. Goyal and Vega-Redondo (2007) propose an allocation rule whereby connections produce a surplus that is shared with essential intermediaries in the network (see also Hojman and Szeidl, 2008). This model captures the intuition of Burt (2005, p. 4) that "people who do better are somehow better connected," the underlying idea being that centrally located individuals may hold up players that are not directly connected, or for other reasons secure a large share of the goods or services that flow through the network. A number of centrality measures have been proposed, including the number of neighbors of a player, his closeness (mean shortest path to other players) and his betweenness (the fraction of shortest paths between all pairs of players in a network that include the player); see, for instance, Jackson

[3] Dutta and Ray (1989) have explored the interaction between stability and equality when there are no constraints on the set of coalitions that can form. They propose a solution concept that, among a set of allocations that satisfy core-like participation constraints, selects the one that is most egalitarian in terms of Lorenz dominance. Since they do not restrict the set of coalitions that can form, it is not possible to explore the manner in which inequality varies with network structure in their framework.

(2008). Inequality in these indices of centrality are thought to induce corresponding levels of inequality in the allocation of value in the network, a supposition for which there is some empirical evidence (see, for instance, Brass, 1984; Podolny and Baron, 1997).

As noted above, our work differs from these papers in focusing on the isolation of the poor rather than the well-connectedness of the rich as a determinant of inequality. Another important difference is that they employ an exogenously given profile of payoff functions that determines for each network the allocation of value between players. The focus is accordingly on the level of inequality that arises in equilibria of the network formation model. In contrast, we take networks to be exogenously given, and investigate the extent of inequality that is supportable in light of the posited rules on coalitional deviation.[4]

Finally, Bloch, Genicot, and Ray (2008) study the stability of insurance networks for different levels of communication. They define a network to be q-stable if no player wants to renege on his commitments to make transfers if all players within distance q of the victim sever their ties with him. As in the current paper, information transmission across network links (over limited distances) plays a crucial role in this work, and the sparseness of the network is an important determinant of the viability of various allocations.[5] However, while Bloch et al. study the stability of insurance norms in different networks, we focus on sustainable levels of inequality. Furthermore, the notion of sparseness differs: while sparseness in our setting is determined by the size of the maximum independent set, in the context of Bloch et al. sparseness is captured by the minimal length of cycles among triples of agents. Notably, both the concept of sparseness as employed by Bloch et al. (2008) and the relevant notion for our context are not easily captured by standard network properties such as density or clustering or common centrality measures (see Section 4), suggesting that for certain economic problems, other measures than those currently studied in the literature may be more appropriate.

[4]Note that even in the original framework of Myerson (1977), the standard intuition need not apply. Kalai, Postlewaite, and Roberts (1978) show that the central player in a star network can be worse off at a core allocation than he would be at any core allocation in a complete network.

[5]Also see Vega-Redondo (2006) on the effect of network density and cohesiveness on the effectiveness of information transmission in repeated Prisoner's Dilemma games.

3 Distributions on networks

3.1 Networks

Players are located on a network. A *network* is a pair (N, g), where $N = \{1, \ldots, n\}$ is a set of *vertices* and g is an $n \times n$ matrix, with $g_{ij} = 1$ denoting that there is a *link* or edge between two vertices i and j, and $g_{ij} = 0$ meaning that there is no link between i and j. A link between i and j is denoted by $\{i, j\}$. We focus on undirected networks, so $g_{ij} = g_{ji}$. Moreover, we set $g_{ii} = 0$ for all i. In the following, we fix the vertex set N and denote a network by the matrix g. If $g_{ij} = 1$, that is, if there is a link between i and j, we say that i and j are *neighbors* or, alternatively, that they are *adjacent* in g. A *clique* is a set of pairwise adjacent vertices. The number of neighbors of a vertex is termed its *degree*. The degree vector of a network is a vector $d = (d_1, \ldots, d_n)$, where d_i is the degree of vertex i. The *degree distribution* $\bar{d}$ is a permutation of the elements of d that places them in a (weakly) increasing order: $\bar{d}_1 \leq \bar{d}_2 \leq \cdots \leq \bar{d}_n$.

A *path* between two vertices i and j in a network g is a list of vertices $i_1, i_2, \ldots, i_K$ such that $i_1 = i$ and $i_K = j$, and $g_{i_t i_{t+1}} = 1$. If $i = j$, the path $i_1, i_2, \ldots, i_K$ is called a *cycle*. If there is a path between any two vertices in the network, we say that the network is *connected*. The *length* of a path $i_1, i_2, \ldots, i_K$ is $K - 1$. The *distance* $d_{ij}(g)$ between two vertices i and j in network g is defined as follows. If $i = j$, then $d_{ij}(g) = 0$. If $i \neq j$, then $d_{ij}(g)$ is equal to the length of the shortest path between i and j in g, if such a path exists, and ∞ otherwise.

An *independent set* in a network is a set of vertices that are pairwise nonadjacent. A set of vertices forms a *maximum independent set* in g if it is an independent set and there is no independent set in g with a strictly higher cardinality. Note that while a network may have multiple (maximum) independent sets, the cardinality of a maximum independent set is unique.

We derive several results for bipartite networks. A network is *bipartite* if its vertex set is the union of two disjoint (possibly empty) independent sets; see Figure 3.1. It can be shown that a network is bipartite if and only if it does not have a cycle of odd length. The class of bipartite networks contains the set of trees, which are networks without a cycle (see Figure 3.1(c)).

3.2 Stable allocations

Consider a set of players located on a network. The players jointly generate a surplus, which is a function of the network size. The surplus is divided among the players in the

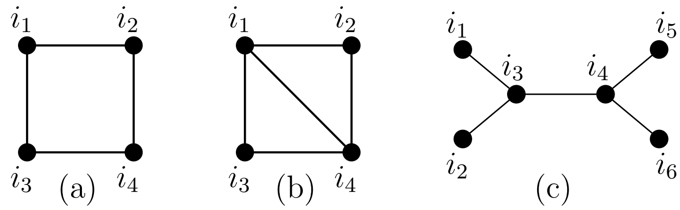


Figure 3.1: (a) A bipartite network, with independent sets $\{i_1, i_4\}$ and $\{i_2, i_3\}$. (b) A network that is not bipartite: there do not exist two independent sets that partition the set of vertices. (c) A bipartite network that is a tree.

network, in such a way that no coalition of players that form a clique in the network can profitably deviate.

More precisely, consider a set of players $N = \{1, \dots, n\}$, and a network g with vertex set N, so that each player is associated with a vertex. As in Myerson (1977), players generate value if they are connected by some path. The value generated by a set of players S is given by $f(|S|)$ such that $f(0) = 0$. Without loss of generality, we assume that the network is connected, that is, there is a path between each pair of players so that the value of g is $f(n)$. Hence, we make two important assumptions on the value function f. First, we assume that the value generated by a group of players does not depend on the way they are connected. This models settings such as those discussed in Section 1, where groups of individuals jointly produce some surplus, independent of any (pre-existing) social connections among them. Second, we assume that players are identical in terms of their productivity. Both assumptions allow us to focus on the inequality induced solely by the social *structure*: for a given number of players, each network produces the same value, so that any difference in outcomes must be linked to the network structure.

The generated surplus is divided among the players. The distribution of the surplus is determined by the deviating coalitions that can form. We assume that only cliques can jointly deviate. Formally, an *allocation* is any vector $x = (x_1, \dots, x_n) \in \mathbb{R}^N$. An allocation x is *feasible* if $x_i \geq 0$ for all $i \in N$, and

$$\sum_{i \in N} x_i \leq f(n). \tag{3.1}$$

We say that an allocation x is *stable* on g if no clique can gain by deviating: for each clique C in g,

$$\sum_{i \in C} x_i \geq f(|C|). \tag{3.2}$$

That is, for an allocation to be stable, the members of each clique have to get at least as much collectively under the allocation as they would if they were to deviate collectively and form their own network. In Section 6 we allow for players to coordinate deviations over larger distances.

It is immediate that the set of feasible and stable allocations, being a set of vectors satisfying a set of weak inequalities (3.1) and (3.2), is closed and convex. The definition of the set of feasible and stable allocations is reminiscent of the definition of the core in transferable-utility games (TU-games). The difference is that while inequality (3.2) needs to hold for *all* coalitions for x to be in the core, we only require the inequality to hold for subsets of players that are sufficiently close in the network. Hence, the set of feasible and stable allocations is a superset of the core of an appropriately defined TU-game where the value function is extended to all coalitions. If f is a (weakly) convex function, the egalitarian allocation defined by $x_i = f(n)/n$ is always stable, so that the set of feasible and stable allocations is nonempty. Even if the function f is strictly convex, the TU-game with the value function extended to all coalitions will typically not be a convex game in the sense of Shapley (1971); see Van den Nouweland and Borm (1991).

3.3 Inequality

We want to compare allocations in terms of the inequality they generate. Corresponding to any allocation x is a *distribution* $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$. The distribution $\bar{x}$ is simply a permutation of the elements of x that places them in (weakly) increasing order: $\bar{x}_1 \leq \bar{x}_2 \leq ... \leq \bar{x}_n$. We say that a distribution $\bar{x}$ is feasible and stable on g if there exists a corresponding allocation that is feasible and stable on g. While the egalitarian distribution is always stable, there may be multiple stable distributions in general, some of which may be characterized by high levels of inequality.

To compare distributions in terms of the level of inequality, we use the criterion of Lorenz dominance. Consider two distributions $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$, $\bar{y} = (\bar{y}_1, ..., \bar{y}_n) \in \mathbb{R}^n_+$ such that

$$\sum_{i \in N} \bar{x}_i = \sum_{i \in N} \bar{y}_i = f(n).$$

Then, we say that $\bar{x}$ *Lorenz dominates* $\bar{y}$ if, for each $m = 1, ..., n$,

$$\sum_{i=1}^{m} \bar{x}_i \geq \sum_{i=1}^{m} \bar{y}_i,$$

with strict inequality for some m. If $\bar{x}$ Lorenz dominates $\bar{y}$, we say that $\bar{x}$ is a *more equal* distribution than $\bar{y}$. If $\bar{x}$ does not Lorenz dominate $\bar{y}$ and $\bar{y}$ does not Lorenz dominate $\bar{x}$, we say that $\bar{x}$ and $\bar{y}$ are *incomparable*.

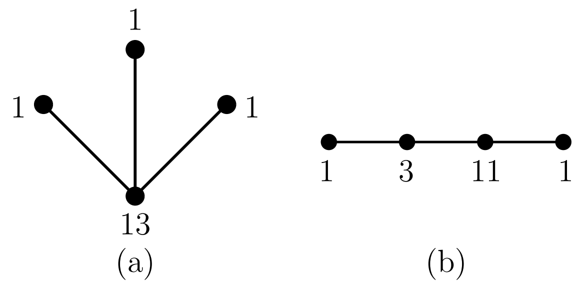


Figure 4.1: (a) The network g of Example 4.1. The numbers represent the unique allocation consistent with the extremal distribution for g. (b) The network g' of Example 4.1. The numbers represent one of the allocations that is consistent with the extremal distribution for g'.

We call a stable distribution $\bar{x}$ on g which is feasible *extremal* if there is no distribution $\bar{y}$ that is stable and feasible such that $\bar{x}$ Lorenz dominates $\bar{y}$. Since the Lorenz dominance criterion only provides a partial order on the set of feasible and stable distributions, there may be multiple extremal distributions for a given network. We say that a network g has a *unique extremal distribution* if the set of extremal distributions on g is a singleton.

4 Examples

The concepts of stability and extremal distributions may be illustrated with a few examples.

Example 4.1 Suppose $f(m) = m^2$ for $m = 0, 1, \ldots, n$, and consider the networks g and g' depicted in Figure 4.1(a) and (b), respectively. The value of both networks is $f(4) = 16$. The conditions for stability require that each individual is assigned at least $f(1) = 1$, and each pair of neighbors is assigned at least $f(2) = 4$. The networks g and g' have a unique extremal distribution, given by $\bar{x} = (1, 1, 1, 13)$ and $\bar{x}' = (1, 1, 3, 11)$, respectively. Hence, $\bar{x}'$ dominates $\bar{x}$. The extremal distribution for g corresponds to a unique allocation, as depicted in Figure 4.1(a): The player with three neighbors receives 13 while the other players each get 1. By contrast, the extremal distribution for g' is consistent with many different allocations to vertices. Any allocation such that two unconnected nodes receive 1 and the other two are assigned 3 and 11 is stable. An example of such an allocation is shown in Figure 4.1(b). $\triangleleft$

In Example 4.1, what properties of network g allow it to support a more unequal distribution than g'? One possibility is the fact that the distribution of the number of neighbors that each player has in g is itself more unequal than that in g'. Could the fact that the degree

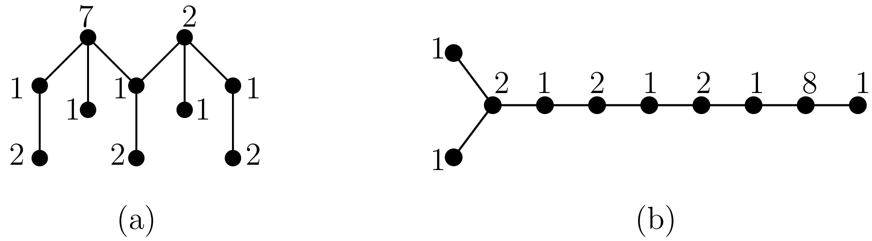


Figure 4.2: (a) The network h of Example 4.2. (b) The network h' of Example 4.2. The numbers represent one of the allocations consistent with the unique extremal distribution in each case.

distribution $\bar{d}'$ for g' Lorenz dominates the degree distribution $\bar{d}$ for g be related to the fact that $\bar{x}'$ Lorenz dominates $\bar{x}$? As the following example shows, the answer is negative.

Example 4.2 Suppose $f(1) = 1$, $f(2) = 3$, and $f(10) = 20$. Consider the networks h and h' in Figure 4.2(a) and (b), respectively. In both cases, the value generated by the network is equal to 20. The stability conditions require that each individual be assigned at least $f(1) = 1$, and each pair of neighbors be assigned at least $f(2) = 3$. Both networks have a unique extremal distribution, given by

$$\bar{x} = (1, 1, 1, 1, 1, 2, 2, 2, 2, 7),$$
$$\bar{x}' = (1, 1, 1, 1, 1, 1, 2, 2, 2, 8).$$

Hence, $\bar{x}$ Lorenz dominates $\bar{x}'$. ◁

In Example 4.2, h' can sustain greater inequality than h. This is the opposite of what one would predict based on inequality in the degree distributions of h and h', which are given by:

$$\bar{d} = (1, 1, 1, 1, 1, 2, 2, 3, 3, 3)$$
$$\bar{d}' = (1, 1, 1, 2, 2, 2, 2, 2, 2, 3),$$

respectively. Clearly $\bar{d}'$ Lorenz dominates $\bar{d}$, even though $\bar{x}$ Lorenz dominates $\bar{x}'$. The level of extremal inequality sustainable in a network therefore does not depend in a straightforward manner on inequality of the degree distribution.

Like a player's degree, his betweenness is often taken as a measure of a player's prominence and as a determinant of a player's payoffs. The betweenness of a player i in a network is the number of shortest paths between v and w player i belongs to over the total number of all

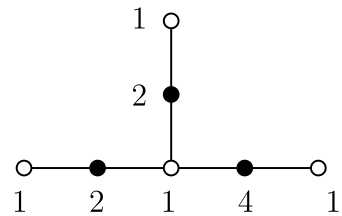


Figure 4.3: The network of Example 4.3. The player with the greatest degree, betweenness and closeness gets the lowest payoff in any extremal allocation.

shortest paths between v and w, averaged over all v and w (see, for example, Jackson, 2008). A player's betweenness may be interpreted as a measure of how essential he is in information transmission between other players. However, inequality in betweenness fares no better in explaining extremal inequality, as the next example demonstrates.

Example 4.3 Suppose $f(1) = 1$, $f(2) = 3$, and $f(7) = 12$. Consider the network in Figure 4.3. The value generated by the network is 12. The stability conditions require that each individual is assigned at least $f(1) = 1$, and that each pair of neighbors is assigned at least $f(2) = 3$. The network has a unique extremal distribution, given by

$$\bar{x} = (1, 1, 1, 1, 2, 2, 4).$$

This distribution is consistent with different allocations to the players, but in any such allocation, each player represented by an open circle ($\circ$) is assigned $f(1) = 1$. This includes the player with the highest degree. This player also has the highest betweenness (0.43), more than double than that of his neighbors, both of whom receive higher payoffs. $\triangleleft$

Taken together Examples 4.2 and 4.3 reveal that a focus on inequality in the degree or betweenness in attempting to understand the extent of inequality in social networks is misleading in two respects. First, networks with more equal degree or betweenness distributions may be capable of sustaining greater inequality than those with more unequal distributions. And second, by either measure, well-connected players can do substantially worse than less well-connected players in a given network. Inspection of Figure 4.3 also shows that another important centrality measure, closeness, also fails to predict high payoffs.[6] In the following two sections, we show that rather than the degree or betweenness distribution, it is the cardinality of the largest independent sets in a network that is most informative about the extent to which inequality can be sustained. Section 5 provides a complete characterization for the special case of bipartite networks, while Section 6 gives a somewhat weaker result for general networks.

[6] The closeness of a player in the network is the average length of the shortest paths to other players (Jackson, 2008).

5 Bipartite networks

In this section, we first show that any bipartite network has a unique extremal distribution when some conditions on the value functions are satisfied. We then investigate how the unique extremal distribution changes for bipartite networks when the network structure is varied. The class of bipartite networks is an important one in the network literature in economics, as it it contains the class of trees. This class plays an important role in the network formation literature as in many cases, equilibrium networks are minimally connected.

In addition to the normalization $f(0) = 0$, we make the following assumptions on f:[7]

A1: $f(2) \geq 2f(1)$;

A2: $2f(n) \geq nf(2)$.

It can be readily checked that these conditions are satisfied if f is increasing and (weakly) convex. If A1 is not satisfied, no allocation exists that is both feasible and stable for nonempty networks. If A2 is not satisfied, an allocation that is stable and feasible may exist, but our results below will not hold for all bipartite networks.

We first consider uniqueness of the extremal distribution. Let A be a maximum independent set in g. Let $\ell \in N \setminus A$ be an arbitrary player not belonging to A. Define the allocation x^* by

$$x_i^* = \begin{cases} f(1) & \text{if } i \in A, \\ f(2) - f(1) & \text{if } i \in N \setminus (A \cup \{\ell\}), \\ f(n) - |A|f(1) - (n - |A| - 1)(f(2) - f(1)) & \text{if } i = \ell. \end{cases} \tag{5.1}$$

The corresponding distribution is denoted by $\bar{x}^*$.

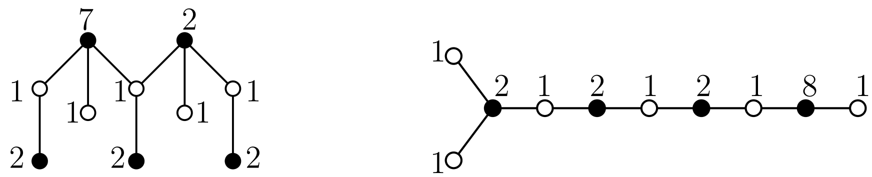
Theorem 5.1 *Suppose assumptions A1 and A2 are satisfied. If g is a bipartite network, then $\bar{x}^*$ is its unique extremal distribution.*

The proof can be found in the appendix; the idea behind it is the following. The allocation x^* assigns $f(1)$ to each player in a maximum independent set, $f(2) - f(1)$ to all players not in the maximum independent set except ℓ, and the remainder to ℓ. Under any other stable and feasible allocation, the total value allocated to the t players with the smallest

[7] In case n is odd, the condition A1 combined with

A2': $2f(n) \geq nf(2) - (f(2) - 2f(1))$,

(which is slightly weaker than A2 if A1 holds) are sufficient for our results to hold.



(a) (b)

Figure 5.1: (a) The network h of Example 4.2. (b) The network h' of Example 4.2. The numbers represent one of the allocations consistent with the extremal distribution in each case. In both (a) and (b), vertices belonging to a maximum independent set are marked by white circles ($\circ$)

assignment must always be at least as large as this sum under x^* for any t, so that any stable and feasible distribution that is not equal to $\bar{x}^*$ must Lorenz dominate it. We show this by dividing the set of players into pairs of neighbors (which together need to get at least $f(2)$ if the allocation is to be stable) and "unmatched" players (who need to get at least $f(1)$ under any stable allocation). Using this, we show that the proposed allocation x^* satisfies all the constraints implied by stability in such a way as to minimize the cumulative sum of the t smallest assignments, making it the unique extremal distribution.

As a corollary of Theorem 5.1, we find that bipartite networks can be ranked in terms of extremal inequality by the cardinality of their maximum independent sets. Hence, even though the Lorenz dominance relation is not a complete order, we obtain a complete order on the set of bipartite networks.

Corollary 5.2 *Suppose assumptions A1 and A2 are satisfied. Consider any two bipartite networks g, g' with vertex set N. Let A and A' denote any maximum independent sets, and $\bar{x}$ and $\bar{x}'$ the unique extremal distributions in g and g' respectively. Then, $\bar{x} = \bar{x}'$ if and only if $|A| = |A'|$. If $|A| \neq |A'|$, then $\bar{x}$ Lorenz dominates $\bar{x}'$ if and only if $|A| < |A'|$.*

We can now return to Example 4.2 to apply this result. Although h has for instance a more unequal degree distribution then h',[8] the cardinality of its maximum independent set is 5, as compared to 6 for h'. A direct application of Theorem 5.1 implies that h' can sustain a more unequal payoff distribution; see Figure 5.1.

When we impose limits on the size of the deviating coalitions, it is possible to extend these results to general networks. This seems to be a natural restriction, since a joint deviation

[8]It can easily be checked that the same holds for betweenness, while h' has a more unequal distribution of closeness than h.

requires coordination, which may be difficult to obtain in large groups. When the size of a deviating coalition is at most 2, Theorem 5.1 and Corollary 5.2 extend to general networks.[9] However, the examples in the next section show that when larger coalitions are allowed, these results no longer hold. Nevertheless, we will see that the size of the maximum independent set is still a key determinant of extremal inequality in general networks when coalitions of arbitrary size are allowed.

6 General networks and broader coalitions

6.1 General networks

What can one say about more general networks? First of all, it is easy to see that the complete network, in which all players are directly connected, has a unique extremal distribution if the function f is convex. Since each subset of players forms a clique in the complete network, the corresponding distribution is the most egalitarian among the extremal distributions for all networks with the same player set. For other networks, there are two issues to consider: the uniqueness of the extremal distribution for a given network, and the ordering of networks with respect to their extremal distributions. Unfortunately, the proof of Theorem 5.1 cannot be easily extended to more general networks (when we do not impose any restrictions on the size of deviating coalitions). The reason is that in the case of bipartite networks, only deviations by individual players or by neighbors are allowed, so that the unique extremal distribution can easily be characterized. By contrast, general networks can contain cliques with three or more players, so that deviations by larger groups are allowed. In that case, candidate extremal distributions are harder to characterize.

Even if a uniqueness result could be obtained, it would not be as straightforward to rank networks in terms of the inequality of their extremal distribution, as the following example shows. This example demonstrates that two networks that are in the same equivalence class with respect to the cardinality of their maximum independent sets may nevertheless be unambiguously ranked with respect to their extremal distributions.

Example 6.1 Suppose $f(m) = m^2$ for $m = 0, 1, \ldots, n$, and consider the networks q and q' in Figure 6.1(a) and (b), respectively. Both networks have unique extremal distributions,

[9] We are grateful to Brian Rogers for suggesting this result.

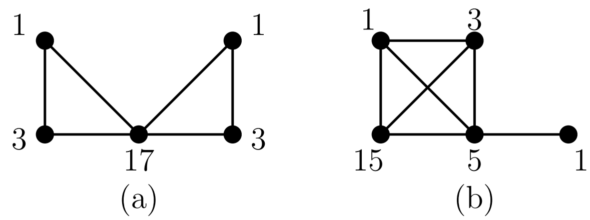


Figure 6.1: (a) The network q of Example 6.1. The numbers represent one of the allocations consistent with the unique extremal distribution for q. (b) The network q' of Example 6.1. The numbers represent represent one of the allocations consistent with the unique extremal distribution for q'.

given by

$$\bar{x} = (1, 1, 3, 3, 17),$$
$$\bar{x}' = (1, 1, 3, 5, 15),$$

so $\bar{x}'$ Lorenz dominates $\bar{x}$. $\triangleleft$

The previous example shows that two networks with the *same* cardinality of their maximum independent sets can be unambiguously ranked with respect to their extremal distributions. In contrast, the following example shows that two networks that *differ* in the cardinality of their maximum independent set cannot necessarily be ranked with respect to their extremal distributions.

Example 6.2 Suppose $f(m) = m^2$ for $m = 0, 1, \ldots, n$, and consider the networks r and r' in Figure 6.2(a) and (b), respectively. Both networks have unique extremal distributions, given by

$$\bar{x} = (1, 1, 1, 1, 1, 3, 3, 3, 3, 83),$$
$$\bar{x}' = (1, 1, 1, 1, 1, 1, 3, 5, 7, 79).$$

These two distributions are incomparable based on the Lorenz criterion. $\triangleleft$

The examples above demonstrate that the results obtained in Section 5 do not generalize to arbitrary networks. However, we can obtain a somewhat weaker result when we impose stronger assumptions on the value function f:

A3: For all $m, \ell = 0, \ldots, n-1$ such that $m > \ell$,

$$f(m+1) - f(m) \geq f(\ell + 1) - f(\ell).$$

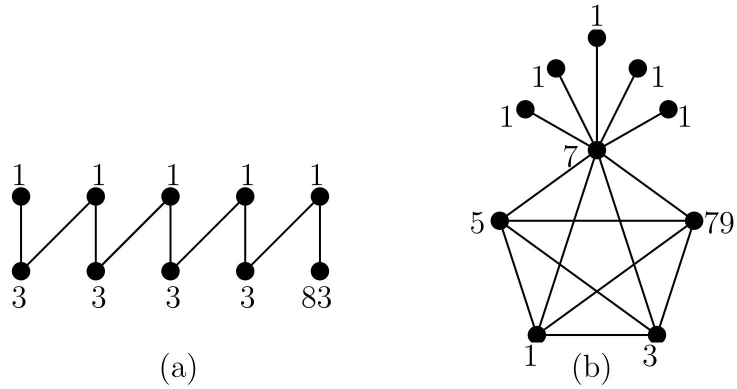


Figure 6.2: (a) The network r of Example 6.2. The numbers represent one of the allocations consistent with the unique extremal distribution for r. (b) The network r' of Example 6.2. The numbers represent represent one of the allocations consistent with the unique extremal distribution for r'.

Assumption A3 states that f is (weakly) convex. Clearly, A3 implies A1 and A2. It is easy to verify that $f(m) \geq mf(1)$ for all m when $A3$ is satisfied. The following result provides a partial characterization of extremal inequality in general networks.

Theorem 6.3 *Suppose f satisfies A3, and consider two networks g and g' with vertex set N. Let A and A' denote any maximum independent set in g and g', respectively. If $|A| < |A'|$, then there exists an extremal distribution $\bar{x}'$ for g' such that no extremal distribution $\bar{x}$ in g is Lorenz dominated by $\bar{x}'$.*

The proof can be found in the appendix. Note that Theorem 6.3 does not require the extremal distributions to be unique. It also leaves open the possibility that the extremal distributions for two different networks are incomparable, or that extremal inequality does not change when the cardinality of the maximum independent set changes. The reason we cannot rule this out is twofold. First, there may be multiple extremal distributions for a given network. Second, even if all extremal distributions are unique, the restricted core may change in a nontrivial and unexpected way (cf. Kalai et al., 1978). What Theorem 6.3 does rule out is that extremal inequality in a network with a smaller maximum independent set is greater than in a network with a larger one. We now turn to a natural application of Theorem 6.3, allowing players to coordinate a deviation over larger distances.

6.2 Broader coalitions

So far, we have only allowed for deviations of cliques. This presumes that players can coordinate on a deviation only if they are all directly connected, that is, if the distance

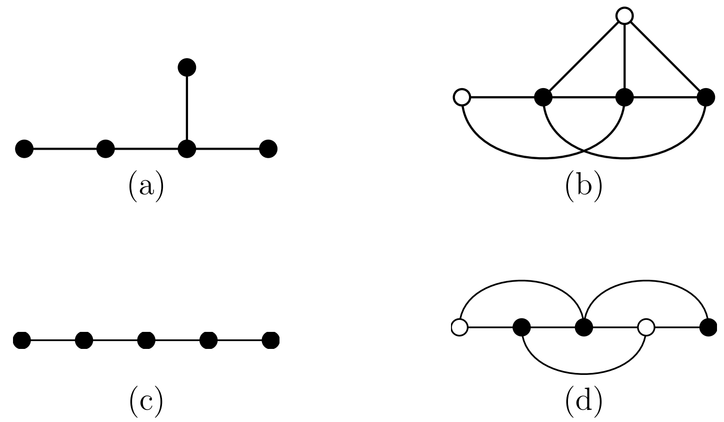


Figure 6.3: (a) The network s of Example 6.4. (b) The 2-power s^2 of s. (c) The network $\tilde{s}$ of Example 6.4. (d) The 2-power $\tilde{s}^2$ of $\tilde{s}$. In both (b) and (d), the vertices in a maximum independent set are marked with an open circle ($\circ$).

between each pair of players in the coalition is equal to one. What happens if we allow for deviations by coalitions of players that are all within distance k of each other in the network?

Given a network, define a *k-coalition* to be a set of players that are all within distance k of each other. As before, the value that a k-coalition C can obtain on its own is $f(|C|)$. We say that an allocation x is *k-stable* on g if, for each k-coalition C in g,

$$\sum_{i \in C} x_i \geq f(|C|).$$

Hence, no k-coalition can profitably deviate from a k-stable allocation. Stability, as defined in Section 3, corresponds to k-stability for $k = 1$. A k-stable distribution $\bar{x}$ on g which is feasible is called *k-extremal* if there is no distribution $\bar{y}$ that is k-stable and feasible such that $\bar{x}$ Lorenz dominates $\bar{y}$.

To analyze this case, it is useful to define the k-power of a network. A k-power g^k of a connected network g is the network with the same vertex set as g, with $g^k_{ij} = 1$ if and only if the distance between i and j in g is at most k (see, for example, Gross and Yellen, 2003). A set of players is a k-coalition in a connected network g if and only if it is a 1-coalition in the k-power g^k of g. Hence, an allocation is k-stable in g if and only if it is stable in g^k.

How does the degree of inequality that can be sustained in a network depend on the network structure in this more general setting? Not surprisingly, a direct extension of Corollary 5.2 does not hold, as the following example illustrates: ranking networks in terms of the cardinality of their maximum independent sets does not provide a ranking in terms of the inequality that they can sustain.

Example 6.4 Suppose $f(m) = m^2$ for $m = 0, 1, \ldots, n$, and let $k = 2$. Consider the networks s and $\tilde{s}$ in Figure 6.3(a) and (c), respectively. The 2-extremal distributions of s and $\tilde{s}$ are the extremal distributions of s^2 and $\tilde{s}^2$, respectively. It can be easily verified that for both networks there is a unique extremal distribution. While for both networks, the cardinality of the maximum independent set of the 2-powers is equal to 2 (see Figure 6.3(b) and (d), respectively), the unique 2-extremal distribution $(1, 1, 3, 5, 15)$ for s Lorenz dominates the unique 2-extremal distribution $(1, 1, 3, 3, 17)$ for $\tilde{s}$. ◁

While the previous example shows that the size of the maximum independent set does not fully characterize extremal inequality, the following result shows that the degree of inequality that can be sustained in a network weakly decreases when we increase k:

Corollary 6.5 *For any network g and k, k' such that $k' > k$, if $\bar{x}', \bar{x}$ are extremal distributions in g for k and k', respectively, then $\bar{x}' = \bar{x}$, $\bar{x}'$ Lorenz dominates $\bar{x}$, or $\bar{x}$ and $\bar{x}'$ cannot be compared with respect to Lorenz dominance.*

This result directly follows from Theorem 6.3, since increasing k weakly decreases the size of the maximum independent set. Intuitively, a group of players that forms a k-coalition in a network g is a k'-coalition in g for $k' > k$, so that increasing k limits the degree of inequality that can be sustained.

Interestingly, while the degree of inequality that can be sustained in a network weakly decreases for any network if k increases (Corollary 6.5), this decrease occurs at very different rates for different networks, as the following example shows.

Example 6.6 Consider the star network g_{star} and the line network g_{line} depicted in Figure 6.4(a) and (c), respectively, and suppose $f(m) = m^2$ for $m = 0, 1, \ldots n$. When $k = 1$, Theorem 5.1 shows that there is a unique extremal distribution; by Corollary 5.2, the unique extremal distribution $\bar{x}^1_{line}$ for the line Lorenz dominates the unique extremal distribution $\bar{x}^1_{star}$ for the star.

However, when $k = 2$, the situation is reversed. In the case of the star, all players can now form deviating coalitions, while for the line, the two players at the end of the star can still not coordinate a joint deviation. This is most easily seen by considering the 1-coalitions in the 2-powers of the line and the star, shown in Figure 6.4(b) and (d), respectively. This has implications for the degree of inequality that can be sustained. Also for $k = 2$, the extremal distributions for the line and star are unique; however, the unique extremal distribution $\bar{x}^2_{star}$ for the star now Lorenz dominates the unique extremal distribution for the line $\bar{x}^2_{star}$. ◁

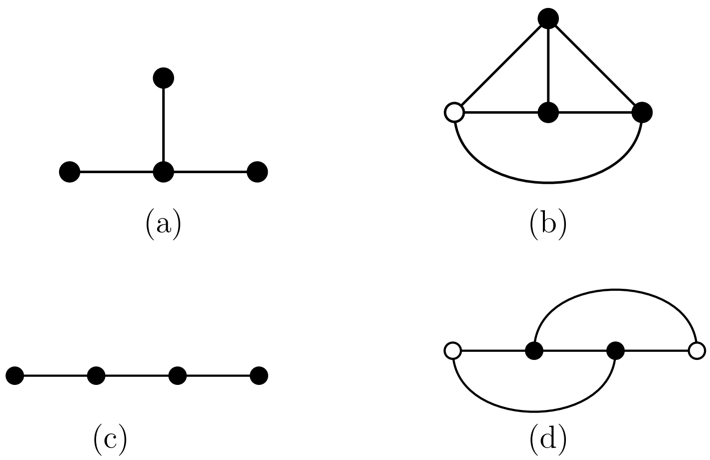


Figure 6.4: (a) The star network g_{star} of Example 6.6. (b) The 2-power g^2_{star} of g_{star}. (c) The network g_{line} of Example 6.6. (d) The 2-power g^2_{line} of $\tilde{s}$. In (b) and (d) the vertices in a maximum independent set are marked with an open circle ($\circ$)

Can we use other network properties to obtain a more complete characterization of extremal inequality for different values of k? Intuitively, one might think that the diameter or the characteristic path length of a network determines the rate at which inequality decreases when k increases. The diameter is the maximum distance between any two players in a connected network, while the characteristic path length is the average distance between any two players. Indeed, in Example 6.6, the line network both has a smaller diameter and a smaller characteristic path length than the star network. The increase in k from $k = 1$ to $k = 2$ has a larger impact on the set of feasible coalitions in the star network than in the line, because the star network directly becomes fully connected. However, this intuition is incorrect, as the following example shows.

Example 6.7 Again, assume $f(m) = m^2$ for $m = 0, 1, \ldots, n$, and take $k = 2$. Consider the networks t, t' and t'' in Figure 6.5(a), (c), and (e), respectively. The diameter of t is 3, the diameter of t' is 4, and the diameter of t'' is 5. The ordering in terms of the characteristic path lengths is the same: t has a characteristic path length of 1.87, t' has a characteristic path length of 2.07, and t'' has a characteristic path length of 2.33. The 2-powers of t, t' and t'' are shown in Figure 6.5(b), (d) and (f), respectively. Then, the unique 2-extremal distribution for t is (1,1,3,5,7,19). For t', the unique 2-extremal distribution is (1,1,1,3,5,25), and for t'' it is (1,1,3,3,5,23). That is, for $k = 2$, the unique k-extremal distributions for t and t'' Lorenz dominate the unique extremal distribution for t': the degree of inequality that can be sustained changes nonmonotonically with the diameter and the characteristic path length. ◁

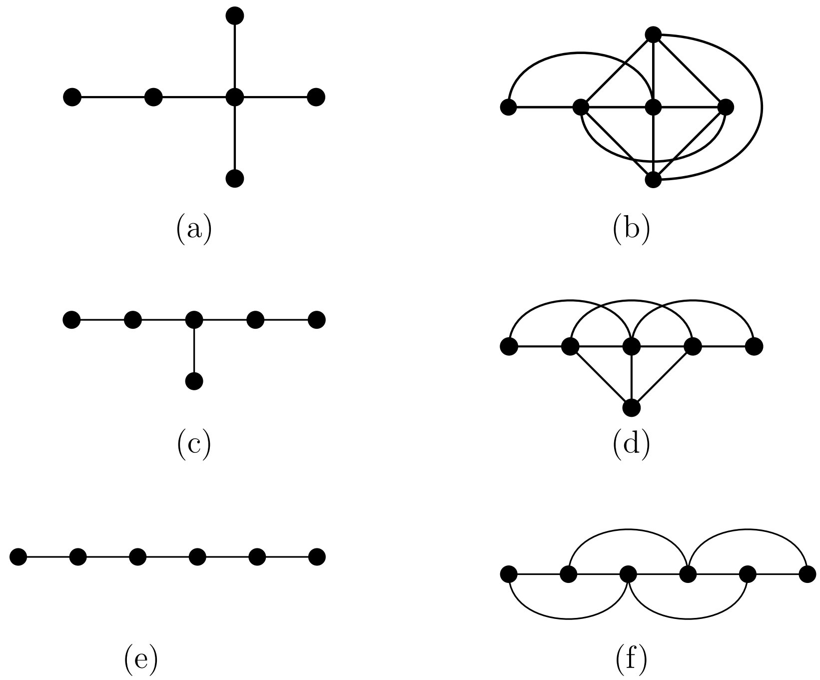


Figure 6.5: (a) The network t of Example 6.7. (b) The 2-power of t. (c) The network t' of Example 6.7. (d) The 2-power of t'. (e) The network t'' of Example 6.7. (f) The 2-power of t''.

6.3 Discussion

For general networks, the degree of inequality that can be sustained depends on the network structure in a nontrivial way. While it may be possible to show that there exists a unique extremal distribution, we conjecture that there is no network property that will give a complete ranking of networks in terms of the inequality they can sustain in general networks and for arbitrary coalitions. Intuitively, it seems that one would need to take into account the full clique structure. While the full clique structure can be summarized by a single number (the cardinality of the maximum independent set) for bipartite networks, this is not true for the general case. Moreover, as discussed in Section 1, feasible coalitions may overlap in our framework, without there being a feasible coalition (other than the complete network) that includes both, unlike in the settings considered by e.g. Myerson (1977) and Demange (2004). Technically, this means that there is no natural assignment of players to cliques in general networks, a crucial step to obtain a complete characterization for bipartite networks in the proof of Theorem 5.1. This implies that there is no natural "order" that we can use for general networks to determine players' reservation value. This seems to suggest that the full clique structure—the number of (maximal) cliques, their sizes and the overlap between different cliques—cannot be summarized by a simple metric, so that it is not possible to provide a more complete characterization of extremal inequality for general networks.

7 Conclusions

In this paper, we have studied how the degree of inequality that can be sustained on a network depends on its structure. The starting point of our analysis is the intuitive idea that players can jointly deviate only if they are sufficiently close to each other in the network. The key network property is the size of the maximum independent set.

Returning to the examples with which we began, the size of the maximum independent set provides a framework for understanding distributional conflicts on these networks. The Roman patricians conceded to a population that was densely connected by common residence and kinship and hence was able to coordinate a mass defection. The strategy of outsourcing leads to supplier networks with a large maximum independent set, minimizing the capacity of suppliers for coordinated deviation and redistributing the surplus to the purchasing firm. Factory employment, common language and company sponsored social events among workers have the opposite effect, reducing the cardinality of the maximum independent set and with it, the firm's feasible claims on the surplus of the network.

Indeed, many historians and archaeologists have noted the contrast between the stability of the high levels of inequality characteristic of relationships between a landed class and dependent farmers in ancient societies and the frequent challenges to unequal distribution of the surplus in industrial production during the modern era (Hobsbawm, 1964; Trigger, 2003). Agrarian inequality is based on the infrequent delivery of crops by otherwise isolated farmers, while inequality between industrial employers and workers is based on the daily delivery of the worker's own labor to a common site (the factory). As a result employees have direct links based on their common place of employment, while share croppers and other agrarian producers do not.

Our results also shed light on discussions on different organizational structures. For instance, while tree-like structures or hierarchies can be efficient in processing information (Bolton and Dewatripont, 1994) or can lead to stable collaboration on a wide range of problems (Demange, 2004), our study points out that these networks can lead to very unequal outcomes, since they tend to be rather sparse, with large maximum independent sets. However, our analysis also shows that if informal ties between workers can develop, possibly spanning different hierarchical levels, then sustainable levels of inequality can decline rapidly. This suggests a possible answer to the puzzle posed by Demange (2004) of how to select between different hierarchical forms if all are equally effective at promoting stable cooperation: when designing an organizational structure, inequality, in addition to stability and efficiency, can be an important consideration.

There are numerous avenues for further research; we focus here on two. First, we have provided a characterization of the maximal degree of inequality that can be sustained in a network. While it may be natural to assume that these extremal distributions may indeed materialize in some contexts, because a powerful player can make a take-it-or-leave-it offer to other players or because other stable outcomes may only be arrived at through a complex process (cf. Demange, 2004), there are of course many other stable distributions. It would therefore be of interest to characterize the full set of stable outcomes. This would allow one to calculate the average or expected level of inequality for different networks. Second, we have taken the social network as given. Indeed, in our motivating examples the network that allows individuals to coordinate possible deviations is typically formed for nonstrategic reasons, independent of the value-generating process. However, a recurrent theme in the network literature is that individuals typically create links to improve their position vis-à-vis others (e.g. Goyal and Vega-Redondo, 2007), which can lead to inefficiencies (Jackson, 2008). Communication among workers through membership in a trade union may entail costly links that are redundant from the standpoint of production but that are undertaken as a redistributive strategy. It would therefore be interesting to study the endogenous formation of networks in the current setting, assuming that individuals need to coordinate on the severance of links, and that it may be costly or impossible to form links to players further away in the network. The relationship between inequality and network structure is an economically interesting one, and we leave these and other questions for further research.

Appendix

Proof of Theorem 5.1

We first derive some preliminary results. Lemma 7.1 shows that the set of vertices of any network can be partitioned into a maximum independent set and a set of vertices that are connected to at least one vertex in the maximum independent set.

Lemma 7.1 *Consider a network g with at least two vertices, and let A be a maximum independent set in g. Define*

$$B := \{j \in N \mid \exists i \in A \text{ such that } g_{ij} = 1\}$$

to be the set of vertices that have at least one neighbor in A. Then the sets A and B form a partition of the vertex set N.

Proof. First we show that $A \cap B = \emptyset$. Suppose that there is a vertex $i \in A \cap B$. As $i \in A$ and since A is an independent set, there is no $j \in A$ such that $g_{ij} = 1$. However, we also have $i \in B$. By the definition of B, there exists $m \in A$ such that $g_{im} = 1$, a contradiction.

We now establish that $N = A \cup B$. Suppose there exists $i \in N$ that does not belong to $A \cup B$. Then, by the definition of B, there exists no $j \in A$ such that $g_{ij} = 1$. But then $A \cup \{i\}$ is an independent set, contradicting that A is a maximum independent set. □

Lemma 7.2 is a technical result on bipartite networks, which allows us to derive Corollary 7.4. Corollary 7.4 states that for bipartite networks, there exists an injective mapping from vertices not belonging to a maximum independent set to the vertices in the maximum independent set, in such a way that the vertices that are matched in this way are neighbors in the network.

Before we can derive these results, we need some more definitions. The *endpoints* of an edge $\{i, j\}$ are the vertices i and j. A vertex is *incident* to an edge if it is one of the endpoints of that edge. A vertex without any neighbors is called an *isolated vertex*. An *edge cover* of a network with no isolated vertices is a set of edges L such that every vertex of the network is incident to some edge of L. A *minimum edge cover* of a network without isolated vertices is an edge cover of the network such that there is no edge cover with strictly smaller cardinality, see Figure 7.1. Note that while a network can have multiple (minimum) edge covers, the cardinality of a minimum edge cover is well defined. A *subgraph* of a network (N, g) is a network (N', g') such that

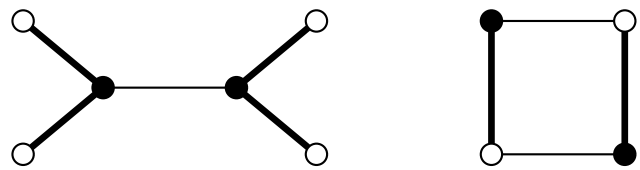


Figure 7.1: Two bipartite networks; in each network, a minimum edge cover is indicated with bold lines, and vertices belonging to one of the maximum independent sets are marked by white circles (∘). Note that while in the network in (a) the minimum edge cover and the maximum independent set are unique, there two maximum independent sets and two minimum edge covers for the network in (b).

(i) the vertex set of $(N'g')$ is a subset of that of (N, g), that is, $N' \subseteq N$;

(ii) the edge set of (N', g') is a subset of (N, g), that is, $g'_{ij} = 1$ implies $g_{ij} = 1$ for all vertices i and j.

An *induced subgraph* is a subgraph obtained by deleting a set of vertices. A *component* of a network (N, g) is a maximal connected subgraph, that is, a subgraph $(N'g')$ that is connected and is not contained in another connected subgraph of (N, g). Given a graph (N, g), the subgraph induced by the set non-isolated vertices is referred to as the *core subgraph* of (N, g).[10] Finally, a *star* is a tree consisting of one vertex adjacent to all other vertices. We refer to this vertex as the *center* of the star.

Lemma 7.2 *Let (M, h) be a bipartite network, and let (M', h') be an induced subgraph of (M, h). For any maximum independent set A of the core subgraph (N, g) of (M', h'), there exists a minimum edge cover $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$ of (N, g) such that*

$$\{i_1, \ldots, i_m\} = A, \qquad \{j_1, \ldots, j_m\} = N \setminus A,$$

and there exists no j_ℓ, j_k, $j_\ell \neq j_k$ such that $i_\ell = i_k$.

Proof. First note that every induced subgraph of a bipartite network is again a bipartite network (that is, the class of bipartite networks is hereditary). Therefore, we can prove the statement in the lemma by proving that for any bipartite network (N, g) and any maximum independent set A of the core subgraph of (N, g), there exists a minimum edge cover $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$ of the core subgraph with the desired properties (cf. West, 2001, Remark 5.3.20). Without loss of generality, we can restrict attention to a bipartite network

[10] Of course, if a network does not have isolated vertices, the core subgraph is just the network itself.

(N, g) without isolated vertices. As before, we fix the vertex set N and denote the network (N, g) by g.

Let A be a maximum independent set in g. We will construct a minimum edge cover $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$ with the desired properties. First note that for any minimum edge cover L' of g, for any vertex i belonging to A, there exists an edge e in L' such that i is an endpoint of e, as otherwise L' would not cover all vertices. Moreover, as A is an independent set, there is no edge in L' with two vertices from A as its endpoints. Hence, without loss of generality, we can take $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$, with

$$\{i_1, \ldots, i_m\} \supseteq A.$$

By the Kőnig-Rado edge covering theorem (e.g. Schrijver, 2003, p. 317), the cardinality of a maximum independent set is equal to the cardinality of a minimum edge cover, so that

$$\{i_1, \ldots, i_m\} = A.$$

Since $\{i_1, \ldots, i_m\} = A$, for the vertices of $N \setminus A$ to be covered by L, we need

$$\{j_1, \ldots, j_m\} \supseteq N \setminus A.$$

As A is an independent set, we have

$$\{j_1, \ldots, j_m\} = N \setminus A.$$

Finally, suppose that there exist distinct j_ℓ, j_k such that $i_\ell = i_k =: i$. First note that for any minimum edge cover Λ the following holds. If both endpoints of an edge e belong to edges in Λ other than e, then $e \notin \Lambda$, because otherwise $\Lambda \setminus \{e\}$ would also be an edge cover of the network, contradicting that Λ is a minimum edge cover. Hence, each component formed by edges of L has at most one vertex with more than one neighbor and is a star. By assumption, j_ℓ and j_k belong to the same component in L; the center of this component is i. Since each vertex in A is associated with at least one edge in L, this means that $|L| > |A|$, which cannot hold by the Kőnig-Rado edge covering theorem. □

Remark 7.3 In Lemma 7.2, we show that for each maximum independent set in the core subgraph of an induced subgraph of a bipartite network, there exists a minimum edge cover such that each vertex i in the core subgraph not belonging to the maximum independent set is matched to a vertex j in the maximum independent set to which it is connected in the network, and there is no other vertex i' in the core subgraph that is matched to j. Note that vertices not belonging to the maximum independent set will typically be connected to multiple vertices in the maximum independent set, see e.g. the network in Figure 7.1(a). ◁

Corollary 7.4 *Let (M, h) be a bipartite network, and let (M', h') be an induced subgraph of (M, h). For any maximum independent set A of (M', h'), there exists an injective mapping π from $M' \setminus A$ to A such that $h'_{i\pi(i)} = 1$ for all $i \in M' \setminus A$.*

Proof. Denote the set of isolated vertices in (M', h') by B. By Lemma 7.2, there exists a minimum edge cover $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$ for the core subgraph (N, g) of (M', h') such that

$$\{i_1, \ldots, i_m\} = A \setminus B, \qquad \{j_1, \ldots, j_m\} = M' \setminus (A \cup B),$$

and there exists no j_m, j_k, $j_m \neq j_k$ such that $i_m = i_k$. Moreover, $B \subseteq A$. Hence, the mapping $\pi : \{j_1, \ldots, j_m\} \to \{i_1, \ldots, i_m\} \cup B$ defined by

$$\pi(j_t) = i_t$$

for $t = 1, \ldots, m$ satisfies the desired properties. □

Finally, Lemma 7.5 establishes that the allocation x^* (Equation 5.1) is feasible and stable for a bipartite network.

Lemma 7.5 *Suppose assumptions A1 and A2 are satisfied. Consider a bipartite network g with at least two vertices. Let A be a maximum independent set in g, and let ℓ be an arbitrary player in $N \setminus A$. Then, the allocation x^* is feasible and stable.*

Proof. The allocation x^* is feasible by definition: Condition (i) is satisfied by definition:

$$\sum_{i \in N} x_i^* = f(n).$$

To show that the allocation x is stable, we need to establish the following:

(i) Each player gets at least $f(1)$, that is, $x_i \geq f(1)$ for each $i \in N$.

(ii) Each pair of neighbors gets at least $f(2)$, that is, for each $i, j \in N$ such that $g_{ij} = 1$, $x_i + x_j \geq f(2)$.

Condition (i) is satisfied, since each $f(1) \leq f(2) - f(1) \leq x_\ell$, where the first inequality follows from A1, and the second from A2. To see that (ii) holds, note that by A1 and Lemma 7.1, each pair of neighbors $i, j \in N \setminus \{\ell\}$ gets at least $f(2) - f(1) + f(1) = f(2)$. It then follows from A2 that each pair of neighbors i, ℓ obtains at least $f(2)$. □

We are now ready to prove Theorem 5.1. Consider a bipartite network (N, g). As before, we fix N and denote the network by g. When $|N| = 1$, it is easy to see that the set of feasible and stable allocations is the singleton $\{x^*\}$, so that trivially $\bar{x}^*$ is the unique extremal distribution.

Hence, consider the case $|N| \geq 2$. Let A be a maximum independent set of N, and for each t, define

$$S_t := \sum_{i=1}^{t} \bar{x}_i^*$$

to be the sum of the t smallest assignments under $\bar{x}^*$, and note that

$$S_t^* = \begin{cases} t\, f(1) & \text{if } t \leq |A|; \\ |A|\, f(1) + (t - |A|)\, (f(2) - f(1)) & \text{if } |A| < t \leq n-1; \\ f(n) & \text{if } t = n. \end{cases} \tag{7.1}$$

By Lemma 7.5, x^* is stable and feasible. It remains to show that for any distribution $\bar{y}$ on g that is stable and feasible, either $\bar{y} = \bar{x}^*$ or $\bar{y}$ Lorenz dominates $\bar{x}^*$. Suppose not. Then there exists t such that

$$S_t^* > S_t,$$

where we have defined $S_t := \sum_{i=1}^{t} \bar{y}_i$ to be the sum of the t smallest assignments under $\bar{y}$. Let C_t be any subset of vertices of cardinality t such that

$$\sum_{i \in C_t} y_i = S_t,$$

and let $A_t \subseteq C_t$ be a maximum independent set in the subgraph induced by C_t. Clearly, $|A_t| \leq |A|$.

By Lemma 7.1, the set C_t can be partitioned into A_t and the set B_t that have at least one neighbor in A_t. By Corollary 7.4, there is an injective mapping π from B_t to A_t such that for each $i \in B_t$, $\{i, \pi(i)\}$ is an edge in the subgraph induced by C_t. Define

$$U_t := \{i \in A_t \mid i = \pi(j) \text{ for some } j \in B_t\}$$

to be the set of players in A_t that are matched with a player in B_t by the mapping π.

In a bipartite network, only singleton coalitions or coalitions consisting of pairs of neighbors can form. Hence, by stability of $\bar{y}$, each individual player needs to be assigned at least $f(1)$ under $\bar{y}$. By A1, it holds that $2f(1) < f(2)$. Hence, under a stable allocation, two neighboring players cannot both be assigned $f(1)$.

Combining these results gives

$$\begin{aligned} S_t &= \sum_{i \in C_t} y_i \\ &= \sum_{i \in B_t} (y_i + y_{\pi(i)} + \sum_{i \in A_t \setminus U_t} y_i \\ &\geq \sum_{i \in B_t} f(2) + \sum_{i \in A_t \setminus U_t} f(1) \\ &= \big(t - |A_t|\big) f(2) + \big(|A_t| - (t - |A_t|)\big) f(1) \\ &= t\,(f(2) - f(1)) + |A_t| \big(2f(1) - f(2)\big) \\ &\geq t\,(f(2) - f(1)) + |A| \big(2f(1) - f(2)\big), \end{aligned} \tag{7.2}$$

where the last inequality follows from $|A_t| \leq |A|$ and $2f(1) - f(2) < 0$ (by A1). We need to consider three cases. First, if $t \leq |A|$, then $S_t^* = t\, f(1)$. Since by stability, $y_i \geq f(1)$ for all i, it follows that $S_t^* \leq S_t$. Second, suppose $|A| < t \leq n - 1$. Then it follows from (7.1) and (7.2) that

$$S_t^* = t\,(f(2) - f(1)) + |A|\,(2f(1) - f(2)) \leq S_t.$$

Finally, if $t = n$, then $S_t^* = S_t = f(n)$. Hence, for all t, $S_t^* \leq S_t$, a contradiction. □

Proof of Theorem 6.3

We first construct an allocation that is feasible and stable in g' and gives $f(1)$ to all players in A'. Define the allocation y' by

$$y'_i = \begin{cases} f(1) & \text{if } i \in A', \\ \frac{f(n) - |A'| f(1)}{n - |A'|} & \text{otherwise.} \end{cases}$$

This allocation satisfies the requirement that $y'_i = f(1)$ for all $i \in A'$. Note that by A3, $y'_i \geq f(1)$ for all i.

It can be readily verified that y' is feasible. We now show that it is stable in g'. Let $C \subseteq N$ be a clique in g', and note that either $C \cap A' = \emptyset$ or $|C \cap A'| = 1$. Suppose $|C| = 1, 2, \ldots, n-1$ (the case $|C| = n$ is satisfied by feasibility). Then,

$$\sum_{i \in C} y'_i \geq f(1) + \frac{(c-1)(f(n) - af(1))}{n - a},$$

where $c := |C|$ and $a := |A'|$. We thus need to show that

$$f(1) + \frac{(c-1)(f(n) - af(1))}{n - a} \geq f(c).$$

Rearranging terms gives

$$(c-1)[f(n)-af(1)]+(n-a)f(1)\geq(n-a)f(c).$$

From the convexity of f (assumption A3), it follows that the left-hand side of this expression is linearly increasing in c (for any n and a). Since f is convex, it suffices to show that there exists $m_1 \leq 2$ and $m_2 \geq n-1$ such that that $(\ell-1)[f(n)-af(1)]+(n-a)f(1) \geq (n-a)f(\ell)$ for $\ell = m_1, m_2$. It can easily be verified that this holds for $m_1 = 1$ and $m_2 = n$. Hence, y' is stable. Denote the corresponding distribution by $\bar{y}'$.

Suppose $\bar{z}'$ is extremal. Then, since $\bar{z}'$ is stable, $\bar{z}'_i \geq f(1)$ for all i. If $\bar{z}'_i > f(1)$ for all i, $\bar{y}'$ is also extremal. Otherwise, $\bar{z}' = f(1)$ for all $i = 1, \ldots, |A'|$. Hence, there exists an extremal distribution $\bar{x}'$ in g' such that $\bar{x}'_i = f(1)$ for $i = 1, \ldots, |A'|$.

For any extremal distribution $\bar{x}$ in g, $\bar{x}_i \geq f(1)$ for $i = 1, \ldots, n$. Since A is a maximum independent set in g, any set $S \subseteq N$ with $|S| > |A|$ must contain at least two adjacent vertices. Hence, we cannot have $\bar{x}_i = f(1)$ for some $i > |A|$, so that either $\bar{x}$ is more equal than $\bar{x}'$, or $\bar{x}$ and $\bar{x}'$ are incomparable. $\square$

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