Barriers and Optimal Investment[1]

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Abstract

This paper analyzes the impact of different types of barriers on the decision to invest using a simple framework based on stochastic discount factors. Our intuitive approach proposes an alternative to the real options methodology that does not rely on the "smooth-pasting condition." An application to MacDonald and Siegel's canonical investment problem (1986) shows that the standard investment threshold over-estimates the optimal threshold when the lower barrier is absorbing and under-estimates it when the lower barrier is reflecting.

Key words: investment; uncertainty; irreversibility; barriers; real options.

JEL classification: D92, D81, E22.

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[1] The helpful comments of participants at the 2003 Econometrics Society Summer Meetings at Northwestern University, at the 7th Annual International Conference on Real Options, and at the 2003 Meetings of the European Economic Association are gratefully acknowledged. I am, of course, responsible for all remaining errors.

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I. Introduction

Barriers are often assumed away in the stochastic investment literature, yet intuitively they should matter. This paper fills this gap for simple investment problems by making two contributions. First, it extends the canonical investment model of MacDonald and Siegel (1986) when the cost of the investment is fixed. It shows that their investment threshold overestimates the optimal investment threshold with a lower absorbing barrier and underestimates this threshold with a lower reflecting barrier. Our numerical results show that the nature of the lower barrier is important for investment decisions at higher levels of uncertainty. Second, this paper generalizes stochastic discount factors to the case where the autonomous stochastic variable of interest is constrained by a barrier. This provides an intuitive alternative to the conventional real options methodology that can be readily extended to more complex investment problems.

When to pay a constant (sunk) amount I for a payoff X that follows an autonomous diffusion process is probably the most basic investment problem. As such, it has already received a lot of attention (e.g., see McDonald and Siegel 1986, Dixit and Pindyck 1994, or Dixit, Pindyck, and Sødal 1999, and the references herein). Surprisingly, however, with the exception of Brock, Rothschild, and Stiglitz (1982), who analyze a basic problem of stochastic capital theory, the potential impact of a lower barrier on the decision to invest has not been analyzed in this canonical framework.[1] Instead, a lower unattainable barrier (unreachable in finite time) is usually assumed in order to derive a closed-form solution. An example is 0 with the geometric Brownian motion (GBM) for the perpetual call option.

Intuitively, however, we expect to invest more conservatively in the presence of a lower absorbing barrier, which, if reached, makes investing permanently unattractive, than if a lower reflecting barrier allows the investment payoff to rebound and grow larger with volatility.[2] An absorbing barrier could result, for example, from demand shifts following innovations by competitors (in electronics, pharmaceuticals…), from gradual changes in tastes, from bankruptcy if the investment opportunity is a call option to purchase another firm, or from the disappearance of a natural resource (overfishing may permanently depress a fish stock, for example). Conversely, a reflecting barrier may arise from government imposed price floors (as for some agricultural commodities), or when a resource has residual value from alternate uses. For example, the owner of a vacant urban plot of land can erect a commercial building if the economy is booming, or build a surface parking lot if the real estate market is depressed.

In addition, the real options approach (Dixit and Pindyck 1994), which is now the standard methodology for solving simple stochastic investment problems, may deter non specialists because of its reliance on a technical condition (often called "smooth-pasting") for which the underlying theory is hardly accessible to most economists (Sødal 1998). By contrast, the stochastic discount factors approach presented herein relies essentially on concepts from deterministic optimization problems, so it should be appealing to economists with little background in finance.

This paper is organized as follows. Section 2 introduces the stochastic discount factor approach and shows that it is equivalent to the real options methodology. Section 3 analyzes the decision to invest in the presence of a reflecting or an absorbing barrier in a canonical framework and presents results from a numerical illustration. Section 4 concludes. An appendix outlines most of the proofs presented herein.

II. Barriers and Optimal Stopping Rules

II.1 *Assumptions and Definitions*

Consider again the simple framework where a firm can invest a fixed amount I for an investment whose expected net present value follows the time homogeneous diffusion process

$$dX = \alpha(X)dt + \sigma(X)dw, \tag{1}$$

where dw is an increment of a standard Wiener process (Karlin and Taylor 1981) and X is defined on an interval Γ of the form $[L,R]$, $(L,R]$, $[L,R)$, or (L,R) with $-\infty \leq L < R \leq +\infty$.[3] Once undertaken, this investment is irreversible. A discussed in MacDonald and Siegel (1986), this framework reflects some of the key characteristics of the problem of a monopolist or of the problem of a firm in a competitive industry enjoying temporary rents (provided X exhibits a decreasing trend). For simplicity, the value of this investment opportunity is discounted using a constant discount rate.[4]

To simplify our analysis, we suppose that:

- Assumption 1: X is regular on Γ, i.e., there is a non-zero probability that X can reach any point of Γ in finite time starting from any other point in Γ. This is useful to express the investor's decision in terms of values of X instead of time. Moreover, the infinitesimal trend of X, $\alpha(.)$, and the infinitesimal standard deviation, $\sigma(.)$, are continuous on Γ; in addition, $\sigma(.)>0$ on the interior of Γ.
- Assumption 2: X admits a finite barrier at $\ell \in (L,R)$. We focus mostly on two cases: either ℓ is reflecting, so X simply rebounds upon reaching ℓ, or it is absorbing, so the investment opportunity disappears as soon as X hits $\ell < I$.

Let us now recall a couple of useful definitions about barriers. Although we focus on lower barriers, these concepts can easily be extended to upper barriers.

Definition 1. A lower barrier $\ell \in [L, R)$ is said to be attracting if there is a non-zero probability that X reaches ℓ before any interior point x. Let $p_{\ell;x}(y)$ denote the probability that X reaches ℓ before x starting from y. Conversely, if ℓ is non-attracting, then X is certain to reach any interior point x before ℓ, and thus $p_{\ell;x|y} = 1$. □

It is important to note that the attracting property of a barrier holds for all interior points as a result of the requirement that the function $\sigma(.)$ be strictly positive on (L,R) and the definition of $p_{\ell;x}(y)$. Indeed, Karlin and Taylor (1981) show that

$$p_{\ell;x}(y) = \frac{S(y,x)}{S(\ell,x)}, \tag{2}$$

where, for $L<x_1< x_2<R$, the scale function $S(x_1, x_2)$ is defined by

$$S(x_1,x_2) = \int_{x_1}^{x_2} \exp\left(\int_{\xi_0}^{\xi} \frac{-2\alpha(\zeta)}{\sigma^2(\zeta)} d\zeta \right) d\xi. \tag{3}$$

In Equation (3), ξ_0 is an arbitrary constant that has no bearing on the value of $p_{\ell;x|y}$; changing ξ_0 is akin to multiplying the numerator and the denominator of $p_{\ell;x|y}$ by the same constant. From (2) and (3), we see that ℓ is attracting if and only if $\lim_{\xi \to \ell+} S(\xi, z)$ is finite for $z \in (\ell, R)$.

Definition 2. Let $x \in (\ell, R)$ and $y \in (\ell, x)$. A lower barrier ℓ is said to be attainable if and only if the expected time it takes X to reach either ℓ or x starting from y is finite. If ℓ is not attainable, it is unattainable. □

Unattainable barriers may or may not be attracting, but all attainable barriers are attracting (Karlin and Taylor 1981, Chapter 15). Barriers may then be classified as follows:

- Attainable and attracting, which include reflecting and absorbing barriers;
- Unattainable but attracting, such as $+\infty$ for a Brownian motion with a positive trend; and
- Unattainable and unattracting.

Let us now examine specific absorbing, reflecting, and unattainable barriers in the context of our simple investment problem.

II.2 Objective Function

Let us first assume that a lower barrier ℓ is absorbing, so as soon as X hits ℓ, X stays stuck at $\ell < I$, and the investment opportunity disappears. This elementary problem embodies two key differences compared to its deterministic counterpart. First, because of uncertainty, we don't know how long it may take for X to reach the investment threshold x_A^*. Second, and most importantly here, X never reaches x^* if it hits ℓ first. With this in mind, if $y=X(0)$, to find the investment threshold we need to solve

$$\underset{x}{Max}\, D^{A}_{x|y;\ell}[x - I], \tag{4}$$

where $D^{A}_{x|y;\ell} \equiv E\left(e^{-\rho T^{A}_{x|y;\ell}}\right)$ is the expected discount factor for $T^{A}_{\ell,x|y}$, the elapsed time between time 0 (now) and the first time X hits x conditional on not hitting ℓ first.

From Karlin and Taylor (1981), we know that for $y \in (\ell, x)$, $W(y) \equiv D^{A}_{x|y;\ell}$ verifies the linear, second-order, ordinary differential equation

$$\frac{\sigma^2(y)}{2}\frac{d^2W(y)}{dy^2} + \alpha(y)\frac{dW(y)}{dy} - \rho W(y) = 0. \tag{5}$$

By definition, $D^{A}_{x|\ell;\ell} = 0$ ($T^{A}_{\ell,x|y} = +\infty$ because starting from ℓ, X never reaches x) and $D^{A}_{x|x;\ell} = 1$, so the two boundary conditions needed to fully define $W(y)$ are simply

$$W(\ell) = 0 \text{ and } W(x^{*}_{A}) = 1. \tag{6}$$

Then, if $W_1(y)$ and $W_2(y)$ are two independent solutions of (5) defined over $[\ell, x^{*}_{A}]$,

$$D^{A}_{x^{*}_{A}|y;\ell} = \frac{W_2(\ell)W_1(y) - W_1(\ell)W_2(y)}{W_2(\ell)W_1(x^{*}_{A}) - W_1(\ell)W_2(x^{*}_{A})}. \tag{7}$$

Let us now suppose instead that $\ell \in (L, R)$ is reflecting. Because X is regular, it will always reach the investment threshold denoted here by x^{*}_{R}, but we don't know how long it will take. This problem can be written:

$$\underset{x}{Max}\, D^{R}_{x|y;\ell}(x - I), \tag{8}$$

where $D^{R}_{x|y;\ell} \equiv E\left(e^{-\rho T^{R}_{x|y;\ell}}\right)$ is the expected value of the discount factor for $T^{R}_{x|y;\ell}$, the random duration between time 0 when $y=X(0)$ and the moment where X first hits x. $W(y) \equiv D^R_{x|y;\ell}$ also verifies (5) and a boundary condition for finding $W(y)$ is simply

$$W(x_R^*) = 1. \tag{9}$$

We also need to write a boundary condition at ℓ to express that it is reflecting. We have:

Lemma 1: If ℓ is reflecting then

$$\frac{dW}{dy}|_{y=\ell} = 0. \tag{10}$$

Proof. Let us therefore suppose that, at time 0, $X=\ell$. In the neighborhood of ℓ, X behaves as a Brownian motion with infinitesimal mean $\alpha(\ell)$ and variance $\sigma^2(\ell)$. Now consider a discrete approximation of the Brownian motion, as in Dixit (1993). Since X cannot take a value lower than ℓ, after a small time increment Δt, X moves up by a small deterministic increment $\Delta\ell>0$ (i.e., $X(\Delta t)=\ell+\Delta\ell$), where $\Delta\ell \approx \sqrt{\Delta t} \gg \Delta t$. Then, for $x\in(\ell,R)$,

$$W(\ell) = E_0\{\exp(-\rho \int_0^{T_{x|\ell}} d\tau)\} = E_0\{\exp(-\rho \int_0^{\Delta t} d\tau)\exp(-\rho \int_{\Delta t}^{T_{x|\ell}} d\tau)\}$$

$$= [1\text{-}\rho\Delta t+o(\Delta t)]E_0\{\exp(-\rho \int_0^{T_{x|\ell+\Delta\ell}} d\tau)\} = [1\text{-}\rho\Delta t+o(\Delta t)]W(\ell+\Delta\ell)$$

$$= [1\text{-}\rho\Delta t+o(\Delta t)][W(\ell)+\Delta\ell \frac{dW}{dy}|_{y=\ell} + o(\Delta\ell)].$$

The transition from the first to the second line above relies on the law of total probability and the Markov property (Karlin and Taylor 1981). The transition from the second to the third line is a

Taylor extension of $W(\ell + \Delta\ell)$. Simplifying, dividing by $\Delta\ell$, and taking $\Delta\ell$ to 0, gives (10). □

If $W_1(y)$ and $W_2(y)$ again denote two independent solutions of (5), when we combine the two boundary conditions (9) and (10), we get

$$D^R_{x^*_R|y;\ell} = \frac{W_2'(\ell)W_1(y) - W_1'(\ell)W_2(y)}{W_2'(\ell)W_1(x^*_R) - W_1'(\ell)W_2(x^*_R)}. \tag{11}$$

In economics, a lower barrier is typically assumed to be unattainable to simplify derivations. This seems to imply, however, that lower barriers have a negligible impact on the investment decision. If ℓ is unattainable, let us denote relevant discount for our simple investment problem by $D_{x|y}$. Mathematically, one of the two independent solutions of (5) and its first derivative typically goes to infinity when x approaches ℓ (think of zero for the geometric Brownian motion); suppose it is the case for $W_2(y)$. Then it is easy to see from (7) and (11) that viewing ℓ as the limit of either an absorbing or a reflecting barrier leads to the same discount factor, and therefore to the same investment threshold.

II.3 First Order Necessary Condition

As time elapses, y, the current value of X changes randomly. It would thus seem that the first-order necessary condition depends on a random variable, as discussed in Dixit, Pindyck and Sødal (1999).[5] The key to this problem is to note that the first order condition is verified at the optimum x^*, so this condition needs to be written at $x=y=x^*$. This leads to:

Lemma 2. Whether ℓ is absorbing, reflecting, or unattainable, the optimum investment threshold x^* verifies the first order necessary condition

$$\frac{\partial D_{x|y;\ell}}{\partial x}\Big|_{x=y=x^*} (x^* - I) + 1 = 0. \qquad (12)$$

where we omit the subscript "A" or "R" for simplicity. □

Proof. See Proposition 1 below. □

From Lemma 2, the sum of two marginal changes at x^* equals zero: first, waiting a bit longer impacts the present value of the project through the expected discount factor; and second, it affects the net payoff from realizing the investment opportunity (its marginal value is 1 here).

II.4 Link with the Real Options Approach

Let us now examine how the stochastic discount factor approach described above relates to the standard real options approach. As above, $W_1(y)$ and $W_2(y)$ denote two independent solutions of Equation (5) defined over (ℓ, R).

Proposition 1. With either a reflecting or an absorbing barrier at ℓ, the standard real options approach and our approach are equivalent. □

Proof. Consider first the absorbing case. From Dixit and Pindyck (1994), the value of the option to invest I to get x, denoted by $\varphi(x)$, verifies the Bellman equation (5), so let us write it

$\varphi(x) = A_1 W_1(x) + A_2 W_2(x)$, where A_1 and A_2 are two unknown constants to be determined simultaneously with the investment threshold; denoted here by x_a^* to distinguish it from x_A^*, which solves (12). Since ℓ is absorbing, the option to invest at ℓ is 0 so that

$$A_1 W_1(\ell) + A_2 W_2(\ell) = 0. \tag{13}$$

When the option is exercised, at x_a^*, it is exchanged for the net value of the investment (the "continuity condition"). $\varphi(x_a^*) = x_a^* - I$ implies

$$A_1 W_1(x_a^*) + A_2 W_2(x_a^*) = x_a^* - I. \tag{14}$$

Since there are three unknowns (A_1, A_2, and x_a), another condition (the "smooth-pasting condition") is needed (Dixit and Pindyck 1994). Here, it equals

$$A_1 W_1'(x_a^*) + A_2 W_2'(x_a^*) = 1. \tag{15}$$

Combining (13) and (15) gives A_1 and A_2; inserting these expressions in Equation (14) gives

$$-\frac{W_2(\ell)W_1'(x_a^*) - W_1(\ell)W_2'(x_a^*)}{W_2(\ell)W_1(x_a^*) - W_1(\ell)W_2(x_a^*)}(x_a^* - I) + 1 = 0. \tag{16}$$

This is also Equation (12) so $x_a^* = x_A^*$.

We proceed similarly for the reflecting case. The value of the option to invest, denoted here by $\psi(x)$, again verifies the Bellman Equation (5), so $\psi(x) = B_1 W_1(x) + B_2 W_2(x)$, where B_1 and B_2 are two unknown constants, and we denote by x_r^* the investment threshold. To find the boundary condition at ℓ, the logic followed to derive Equation (10) leads to $\psi'(\ell) = 0$ (see also Dixit 1993), so that

$$B_1 W_1^{'}(\ell) + B_2 W_2^{'}(\ell) = 0. \tag{17}$$

The continuity and smooth-pasting conditions (Equations (14) and (15)) are similar, so the equation followed by the investment threshold is

$$-\frac{W_2^{'}(\ell)W_1^{'}(x_r^*) - W_1^{'}(\ell)W_2^{'}(x_r^*)}{W_2^{'}(\ell)W_1(x_r^*) - W_1^{'}(\ell)W_2(x_r^*)}(x_r^* - I) + 1 = 0, \tag{18}$$

which again is equivalent to (12), so $x_r^* = x_R^*$. □

In this simple framework, the value of the option to invest is simply the net present value of the investment at $y \in (l, x^*)$. As expected, the investor seeks the investment threshold that maximizes the value of the option to invest at exercise.

More importantly, the stochastic discount factor approach provides an intuitive alternative to the real options approach that does not rely on the smooth-pasting condition. This approach can readily be extended using functional forms defined in Karlin and Taylor (1981; Chapter 15) to more complex problems involving multiple payoffs as well as flows of costs or benefits.

III. Application to a Simple Investment Problem

Let us now consider the case where *X* follows the geometric Brownian motion (GBM)

$$dX = \mu X dt + \sigma X dz, \tag{19}$$

where μ and σ>0 are respectively the infinitesimal trend and volatility parameters. It is well known that the GBM is regular and that ℓ=0 is unattainable.[6] The case ℓ=0 is discussed in details in Dixit and Pindyck (Chapter 5).

For convenience, we define the dimensionless parameters

$$\kappa \equiv 1-\frac{2\alpha}{\sigma^2},\ \lambda = \frac{\rho}{\alpha}.$$

κ provides an index of variability for X: for $\alpha>0$, the more negative κ is, the less volatile X is; conversely, a value of κ between 0 and 1 indicates high volatility for X. λ on the other hand, scales the discount factor with the expected rate of growth of X; as show below, λ must be less than one in order for our investment problem to have a finite solution even with an absorbing lower barrier.

Two independent solutions of (5) here are

$$W_1(\xi) = \xi^{\theta_1} \text{ and } W_2(\xi) = \xi^{\theta_2}, \tag{20}$$

where θ_1 and θ_2 verify

$$\frac{\sigma^2}{2}\theta^2 + \left(\alpha - \frac{\sigma^2}{2}\right)\theta - \rho = 0, \tag{21}$$

so that

$$\theta_1 = \frac{\kappa}{2} + \sqrt{\frac{\kappa^2}{4} + \lambda(1-\kappa)} > 0,\ \theta_2 = \frac{\kappa}{2} - \sqrt{\frac{\kappa^2}{4} + \lambda(1-\kappa)} < 0. \tag{22}$$

A little bit of algebra shows that

$$(\theta_1 > 1) \Leftrightarrow (\rho > \alpha). \tag{23}$$

Now suppose that there is a barrier on X at $\ell>0$ with $\ell<I$, otherwise the investor is guaranteed to make money, and denote I/ℓ by J; hence, $J>1$.

Let us first suppose that $\ell>0$ is absorbing. Inserting (20) into (7) leads to

$$D^A_{x|y;\ell} = \frac{\ell^{\theta_2} y^{\theta_1} - \ell^{\theta_1} y^{\theta_2}}{\ell^{\theta_2} x^{\theta_1} - \ell^{\theta_1} x^{\theta_2}}. \tag{24}$$

The resulting first order necessary condition, based on (12), can be written $F_A(x/\ell) = 0$, where

$$F_A(z) \equiv (1-\theta_1) z^{\theta_1 - \theta_2 + 1} + \theta_1 J z^{\theta_1 - \theta_2} + (\theta_2 - 1) z - \theta_2 J. \tag{25}$$

Likewise, if $\ell>0$ is reflecting, inserting (20) into (11) gives

$$D^R_{x|y;\ell} = \frac{\theta_2 \ell^{\theta_2 - 1} y^{\theta_1} - \theta_1 \ell^{\theta_1 - 1} y^{\theta_2}}{\theta_2 \ell^{\theta_2 - 1} x^{\theta_1} - \theta_1 \ell^{\theta_1 - 1} x^{\theta_2}}, \tag{26}$$

and (12) becomes $F_R(x/\ell) = 0$, with

$$F_R(z) \equiv (1-\theta_1) z^{\theta_1 - \theta_2 + 1} + \theta_1 J z^{\theta_1 - \theta_2} + \theta_1 \frac{\theta_2 - 1}{\theta_2} z - \theta_1 J. \tag{27}$$

where again $J \equiv I/\ell$. $F_A(.)$ and $F_R(.)$ are dimensionless. We have:

Proposition 2. $F_A(x/\ell) = 0$ (ℓ absorbing) and $F_R(x/\ell) = 0$ (ℓ reflecting) admit unique solutions, denoted respectively by x^*_A and x^*_R if and only if $\lambda>1$ (i.e., $\rho>\alpha$). If $\lambda\leq 1$, it is optimal to wait forever since the discounted expected value of the investment keeps on growing. □

Proof. Let us first suppose that ℓ is absorbing. We note that

$$\begin{cases} F_A(J) = J(J^{\theta_1 - \theta_2} - 1) > 0, \\ F'_A(J) = (1-\theta_2)(J^{\theta_1 - \theta_2} - 1) > 0, \end{cases} \tag{28}$$

since $J \equiv I/\ell > 1$, $\theta_1 > 0$, and $\theta_2 < 0$. Differentiating $F_A(z)$ twice gives

$$\begin{cases} F_A^{'}(z) = (1-\theta_1)(\theta_1-\theta_2+1)z^{\theta_1-\theta_2} + \theta_1(\theta_1-\theta_2)Jz^{\theta_1-\theta_2-1} + \theta_1\dfrac{\theta_2-1}{\theta_2}, \\ F_A^{''}(z) = (\theta_1-\theta_2)z^{\theta_1-\theta_2-2}\left\{(1-\theta_1)(\theta_1-\theta_2+1)z + \theta_1(\theta_1-\theta_2-1)J\right\}, \end{cases} \tag{29}$$

so $F_A^{''}(z)$ has the sign of the linear function $f(z) = (1-\theta_1)(\theta_1-\theta_2+1)z + \theta_1(\theta_1-\theta_2-1)J$ and $f(J) = (\theta_1-\theta_2)(1-\theta_1-\theta_2)J^{\theta_1-\theta_2-1}$ has the sign of $1-\theta_1-\theta_2 = \dfrac{2\alpha}{\sigma^2}$, i.e. the sign of α. Knowing the sign of $F_A^{''}(z)$ allows us to make inferences about $F_A^{'}(z)$ and $F_A(z)$. The reflecting case is handled similarly after noting that $F_R^{''}(z) = F_A^{''}(z)$. Table 1 summarizes the variations of $F_A(z)$, $F_R(z)$ and their first two derivatives on $(J,+\infty)$. Details are provided in the appendix. □

Let us now compare x_A^* and x_R^*. We have:

Proposition 3. Suppose that $\rho>\alpha$ so the investment problem admits a unique solution. Assuming an unattainable barrier (i.e., assuming $\ell=0$) overestimates the optimal investment threshold if $\ell>0$ is absorbing, and it underestimates the optimal investment threshold if $\ell>0$ is reflecting:

$$x_A^* < \frac{\theta_1}{\theta_1-1}I < x_R^*, \tag{30}$$

where $\dfrac{\theta_1}{\theta_1-1}I$ is the investment threshold if $\ell=0$ (McDonald and Siegel 1986). □

Proof. See the appendix. □

This result implies that the wedge between the critical value x* and I is influenced by the presence and the nature of a lower barrier. This wedge is smaller when ℓ is absorbing and larger when ℓ is reflecting. Since we only have implicit expressions for x_A^* and x_R^*, we need to compare them numerically. Before illustrating our results on a numerical example, let us see how x_A^* and x_R^* vary with uncertainty (σ), the cost of the project (I), and the lower barrier (ℓ) when ρ is fixed.[7]

Proposition 4. x_R^* increases with uncertainty (σ), and both x_A^* and x_R^* increases with the cost of the project (I). However, x_A^* decreases as ℓ increases while x_R^* increases with ℓ. □

Proof. See the appendix. □

Proposition 4 is compatible with the findings of Brock, Rothschild, and Stiglitz (1982). The investment threshold increases with the cost of the project (I) as the investor needs to wait longer to secure higher gains. Moreover, the investment threshold increases with uncertainty when the lower barrier is reflecting because more uncertainty increases expected net gains. Likewise, a higher lower reflecting barrier truncates the low values of X from below, thus increasing the expected net present value of the project; it is therefore optimal to invest later. However, a higher lower absorbing barrier increases the likelihood that the project will loose its value so the investor needs to act more swiftly. Unfortunately, it is not possible to conclude analytically how x_A^* increases with σ, so we conduct a numerical investigation.

Suppose here that I=\$1, ρ=5% per year, and α=2% per year, so λ>1. We vary σ between 0 and 1.0 (the unit of σ is $(\text{year})^{-0.5}$), and ℓ between \$0.0 and \$0.9 to see how x_A^*, $x^* = \frac{\theta_1}{\theta_1 - 1} I$, and x_R^* vary with these parameters. Results are presented on Figures 1A to 2B.

From Figure 1A, we see that for relatively low values of the volatility (more generally, for negative values of κ when α>0), there is little difference between x_A^*, x^*, and x_R^*. But for higher values of σ (for $\kappa \in (0,1)$ if α>0), the difference between these optimal investment thresholds can be substantial: when σ=0.4 for example, $x_A^* = \$4.46$, $x^* = \$5.00$, and $x_R^* = \$6.09$. These differences matter as they are captured in the present value of the investment opportunity (Figure 1B). For X(0)=\$1.5, when κ=0, the present value of the investment opportunity for the absorbing case is only 2.6% below that of the unattainable case (ℓ=0), and 5.4% above present value of the investment opportunity for the reflecting case. However, when σ=0.4, these differences jump to –12.9% and +51.2% respectively.

The importance of the location of the lower barrier, in addition to its nature, is highlighted in Figures 2A and 2B. When ℓ is relatively "far" from the optimal threshold (which depends also on the volatility of X), the lower barrier has relatively little impact on the investment decision ($\ell \leq$\$0.25 on Figure 2A). As ℓ gets closer to I, however, its impact starts to be felt: for ℓ =\$0.50 for example, $x_A^* = \$3.50$, and $x_R^* = \$4.03$ (for ℓ=0 here, $x^* = \$3.72$); these values change to \$3.02 and \$4.03 respectively for ℓ=0.80. Again, these differences matter: when

ℓ=0.80 for example, the present value of the investment opportunity is 22.0% lower for ℓ absorbing and 74.7% higher for ℓ reflecting compared to the case ℓ=0 (Figure 2B).

In addition, a comprehensive numerical investigation around the parameter values selected did not reveal a parameter combination that decreases x_A^* when σ increases. This may not be the case if X followed another process, as mentioned in Brock, Rothschild, and Stiglitz (1982, page 42) who find that, in general, the effect of an increase in the variance of X is ambiguous in the presence of an absorbing barrier.

IV. Conclusions

While barriers are often assumed away in stochastic investment problems, this paper shows that barriers matter when uncertainty is high enough. We provide an intuitive methodology based on stochastic discount factors to derive simple investments rules for autonomous diffusion process in the presence of common types of barriers. Using functionals analyzed in Karlin and Taylor (1981, Chapter 15), this approach can easily be extended to many other investment problems, including for example barriers with more complex payoffs or investments that modify a monetary flow.

An illustration based on a canonical investment problem (a particular case of MacDonald and Siegel 1986) shows that investment rules based on the perpetual call option may overestimate the investment threshold in the presence of a lower absorbing barrier and may underestimate the investment threshold with a lower reflecting barrier.

These results have implications for testing empirically the theory of investment under uncertainty. For example, empirical real options models applied to dataset that include investment opportunities with reflecting and absorbing barriers may yield inconclusive or biased results if the nature and location of different barriers is not accounted for. More generally, barriers may play an important role in the solution of stochastic investment problems when volatility is high enough.

Future work could consider the impact of barriers on investment opportunities with time limits, analyze the interplay between barriers and discount rates, and revisit the pricing formulas of financial options when the underlying is limited by a barrier.

Appendix

Proof of Proposition 2. Let us first suppose α<0, so $f(J)<0$. From (23), $\theta_1>1$, so $\lim_{z\to+\infty} f(z)=-\infty$; since $f(z)$ is linear, these 2 inequalities imply that $f(z)<0$ on (J,+∞); the same holds for $F_A^{''}(z)$, so $F_A^{'}(z)$ is strictly decreasing on (J,+∞). From (28), $F_A^{'}(J)>0$, and since $\lim_{z\to+\infty} F_A^{'}(z)=-\infty$ (as $\theta_1>1$), $F_A(z)$ first increases and then decreases on (J,+∞). From $F_A(J)>0$ (see (28)) and $\lim_{z\to+\infty} F_A(z)=-\infty$ (again, $\theta_1>1$; see (25)), we infer that $F_A(z)$ has a unique zero on (J,+∞).

Let us now suppose 0≤α<ρ. From (23), $\theta_1>1$, $\lim_{z\to+\infty} f(z)=-\infty$, but now $f(J)\geq 0$, $F_A^{''}(z)$ starts positive on (J,+∞), and then becomes strictly negative. As a result, $F_A^{'}(z)$ first increases and then decreases towards -∞ (from $\theta_1>1$); as $F_A^{'}(J)>0$ (see (28)), $F_A(z)$ first increases and then decreases. From $F_A(J)>0$ and $\lim_{z\to+\infty} F_A(z)=-\infty$, we conclude as above.

Finally, suppose α≥ρ. Again $f(J)\geq 0$ but now $\theta_1\leq 1$ so $\lim_{z\to+\infty} f(z)=+\infty$, making both $f(z)$ and $F_A^{''}(z)$ positive on (J,+∞). Hence, $F_A^{'}(z)$ increases on (J,+∞) and it is strictly positive because $F_A^{'}(J)>0$ (see (28)). Therefore, $F_A(z)$ strictly increases on (J,+∞). $F_A(J)>0$ (from (28)) then implies that $F_A(z)$ has no zero on (J,+∞). Low discounting in this case just does not prevent the expected net present value of the project to keep on increasing.

The same results hold for the reflecting case, using the same logic and $F_R^{''}(z)=F_A^{''}(z)$. □

Proof of Proposition 3. From Table 1, the sign of $F_R\left(\frac{x_A^*}{\ell}\right)$ allows us to compare x_A^* and x_R^* since $F_R(z)>0$ for $z\in(J,\frac{x_R^*}{\ell})$ and $F_R(z)<0$ for $z>\frac{x_R^*}{\ell}$. After using that $F_A\left(\frac{x_A^*}{\ell}\right)=0$ to replace the higher powers of $\frac{x_A^*}{\ell}$ with a linear expression in $\frac{x_A^*}{\ell}$, a little algebra shows that $F_R\left(\frac{x_A^*}{\ell}\right)=\frac{\theta_2-\theta_1}{\theta_2}\left\{(1-\theta_2)\frac{x_A^*}{\ell}+\theta_2 J\right\}$. Here $\frac{\theta_2-\theta_1}{\theta_2}>0$ ($\theta_2-\theta_1<0$ and $\theta_2<0$), and as $\frac{x_A^*}{\ell}>J>1$, $(1-\theta_2)\frac{x_A^*}{\ell}+\theta_2 J>(1-\theta_2)J+\theta_2 J=J>0$ so $F_R\left(\frac{x_A^*}{\ell}\right)>0$, and $x_A^*<x_R^*$.

But from $F_A(x_A^*/\ell)=0$, we have $(1-\theta_2)\frac{x_A^*}{\ell}+\theta_2 J=\left(\frac{x_A^*}{\ell}\right)^{\theta_1-\theta_2}\left\{(1-\theta_1)\frac{x_A^*}{\ell}+\theta_1 J\right\}$ so $(1-\theta_2)\frac{x_A^*}{\ell}+\theta_2 J>0$ implies that $(1-\theta_1)\frac{x_A^*}{\ell}+\theta_1 J>0$ and therefore $x_A^*<\frac{\theta_1}{\theta_1-1}I$, which gives half of (30). Recall indeed that $\rho>\alpha$ implies that $\theta_1>1$.

Finally, use $F_R\left(\frac{x_R^*}{\ell}\right)=0$ to replace the linear terms in $\frac{x_R^*}{\ell}$ with higher powers of $\frac{x_R^*}{\ell}$ in $F_A\left(\frac{x_R^*}{\ell}\right)$ to get $F_A\left(\frac{x_R^*}{\ell}\right)=\frac{\theta_1-\theta_2}{\theta_1}\left(\frac{x_R^*}{\ell}\right)^{\theta_1-\theta_2}\left[(1-\theta_1)\frac{x_R^*}{\ell}+\theta_1 J\right]$. The inequality $x_A^*<x_R^*$

implies that $F_A\left(\frac{x_R^*}{\ell}\right) < 0$, so $(1-\theta_1)\frac{x_R^*}{\ell} + \theta_1 J < 0$ and therefore $\frac{\theta_1}{\theta_1 - 1} I < x_R$. □

Let us now outline the comparative statics analysis for the investment thresholds with different types of barriers. We assume that $\lambda > 1$ in order to have a solution.

Proof of Proposition 4. To get started, let us examine how θ_1 and θ_2 vary with $v=\sigma^2$ and ρ. Differentiating (21) gives $\frac{d\theta}{dv} = \frac{-\theta^2(\theta-1)}{v\theta^2 + 2\rho}$, so that

$$\frac{d\theta_2}{dv} > 0, \quad \frac{d\theta_1}{dv} \begin{cases} < 0, & \text{if } \rho > \alpha, \\ \geq 0, & \text{if } \rho \leq \alpha. \end{cases} \tag{A.1}$$

Likewise, $\frac{d\theta}{d\rho} = \frac{2\theta}{v\theta^2 + 2\rho}$ and

$$\frac{d\theta_1}{d\rho} > 0, \text{ and } \frac{d\theta_2}{d\rho} < 0. \tag{A.2}$$

To analyze how the investment threshold x^* changes with one of the model parameters, which we designate generically by ω, we apply the implicit function theorem using either (25) or (27) to get a relationship that can be written $G(z,\omega) = 0$ (with the appropriate subscript). Then

$$\frac{dx^*}{d\omega} = -\frac{\partial G}{\partial \omega}\left(\frac{\partial G}{\partial x^*}\right)^{-1}. \tag{A.3}$$

ℓ is reflecting.

Let $z_R = x_R^* / \ell$. Here, $\frac{\partial G_R}{\partial z} = F_R^{'}\left(\frac{x_R^*}{\ell}\right) < 0$ from Table 1. Now let $v \equiv \sigma^2$. We have

$$\frac{\partial G_R}{\partial v} = \left[\frac{d\theta_1}{dv} - \frac{d\theta_2}{dv}\right] \ln(z_R) z_R^{\theta_1 - \theta_2} \left\{[1-\theta_1] z_R + J\theta_1\right\} +$$
$$\frac{d\theta_1}{dv}\left\{-z_R^{\theta_1-\theta_2}\left[z_R - J\right] + \frac{\theta_2 - 1}{\theta_2} z_R - J\right\} + \frac{d\theta_2}{dv}\frac{\theta_1}{\theta_2^2} z_R.$$

From (30), $(1-\theta_1)z_R + J\theta_1 < 0$, and since $\frac{\theta_2 - 1}{\theta_2} z_R - J = \left[1 - \frac{1}{\theta_1}\right] z_R^{\theta_1 - \theta_2 + 1} - J z_R^{\theta_1 - \theta_2}$,

$\left\{-z_R^{\theta_1-\theta_2}\left[z_R - J\right] + \frac{\theta_2 - 1}{\theta_2} z_R - J\right\} = -\frac{1}{\theta_1} z_R^{\theta_1 - \theta_2 + 1} < 0$. With (A.1), (A.2), and (A.3), we conclude

$$\frac{dx_R^*}{dv} > 0. \tag{A.4}$$

Let us now consider the impact of ℓ. In this case, $\frac{\partial G_R}{\partial x_R^*} = \frac{\partial G_R}{\partial z}\frac{\partial z_R}{\partial x_R^*} < 0$ since

$\frac{\partial G_R}{\partial z} = F_R^{'}\left(\frac{x_R^*}{\ell}\right) < 0$ and $\frac{\partial z_R}{\partial x_R^*} > 0$. Moreover, after using that $F_R(z_R) = 0$,

$$\frac{\partial G_R}{\partial \ell} = \frac{1}{\ell}\left[\theta_2 - \theta_1\right]\left[-\frac{\theta_1}{\theta_2}\right]\left\{[\theta_2 - 1] z_R - \theta_2 J\right\} > 0.$$

Hence,

$$\frac{dx_R^*}{d\ell} > 0. \tag{A.5}$$

Finally, let us consider $\frac{dx_R^*}{dJ}$. As above, $\frac{\partial G_R}{\partial x_R^*} < 0$ and now $\frac{\partial G_R}{\partial J} = \theta_1 \left[z_R^{\theta_1 - \theta_2} - 1 \right] > 0$, so

$$\frac{dx_R^*}{dJ} > 0. \tag{A.6}$$

<u>ℓ is absorbing.</u>

Let $z_A = x_A^* / \ell$ and $v \equiv \sigma^2$. As above, $\frac{\partial G_A}{\partial z} = F_A'\left(\frac{x_A^*}{\ell}\right) < 0$ from Table 1. We now have

$$\begin{aligned}\frac{\partial G_A}{\partial v} = & \left[\frac{d\theta_1}{dv} - \frac{d\theta_2}{dv}\right] \ln(z_A) z_A^{\theta_1 - \theta_2} \left\{ [1-\theta_1] z_A + J\theta_1 \right\} + \\ & \left\{ -\frac{d\theta_1}{dv} z_A^{\theta_1 - \theta_2} + \frac{d\theta_2}{dv} \right\} [z_A - J].\end{aligned}$$

This time, however, $[1-\theta_1] z_A + J\theta_1 > 0$ (see (30)), so the right hand side terms in factor of $\frac{d\theta_1}{dv} - \frac{d\theta_2}{dv}$ are negative, while the terms in factor of $z_A - J$ are positive. It is difficult to compare the two because of the term in $\ln(z_A)$.

Let us now consider the impact of ℓ. Using the same arguments as before, $\frac{\partial G_A}{\partial x_A^*} = \frac{\partial G_A}{\partial z} \frac{\partial z_A}{\partial x_A^*} < 0$ since again $\frac{\partial G_A}{\partial z} = F_A'\left(\frac{x_A^*}{\ell}\right) < 0$ and $\frac{\partial z_A}{\partial x_A^*} > 0$. Moreover,

$$\frac{\partial G_A}{\partial \ell} = \frac{1}{\ell} [\theta_2 - \theta_1] z^{\theta_1 - \theta_2} \left\{ [1-\theta_1] z_A + \theta_1 J \right\} < 0,$$

which leads to,

$$\frac{dx_A^*}{d\ell} < 0. \qquad \text{(A.7)}$$

Finally, let us consider $\frac{dx_A^*}{dJ}$. As above, $\frac{\partial G_A}{\partial x_A^*} < 0$ and now $\frac{\partial G_A}{\partial J} = \theta_1 z_R^{\theta_1 - \theta_2} - \theta_2 > 0,$ so

$$\frac{dx_A^*}{dJ} > 0. \qquad \text{(A.8)}$$

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Table 1: Variations for $F_A(z)$ and $F_R(z)$.

$\alpha<0$ (so $\theta_1>1$)

Z	J		$+\infty$
$F''(z)$	-	**always -**	**-∞**
$F'(z)$	+	↘	**-∞**
$F(z)$	+	↗ ↘	**-∞**

$0\leq\alpha<\rho$ (so $\theta_1>1$)

Z	J		$+\infty$
$F''(z)$	+	**+ then -**	**-∞**
$F'(z)$	+	↗ ↘	**-∞**
$F(z)$	+	↗ ↘	**-∞**

$\alpha\geq\rho$ (so $\theta_1\leq1$)

Z	J		$+\infty$
$F''(z)$	+	**always +**	+∞
$F'(z)$	+	↗	**-∞**
$F(z)$	+	↗	**-∞**

Note: these results apply to both $F_A(z)$ and $F_R(z)$ so the relevant subscript is omitted.

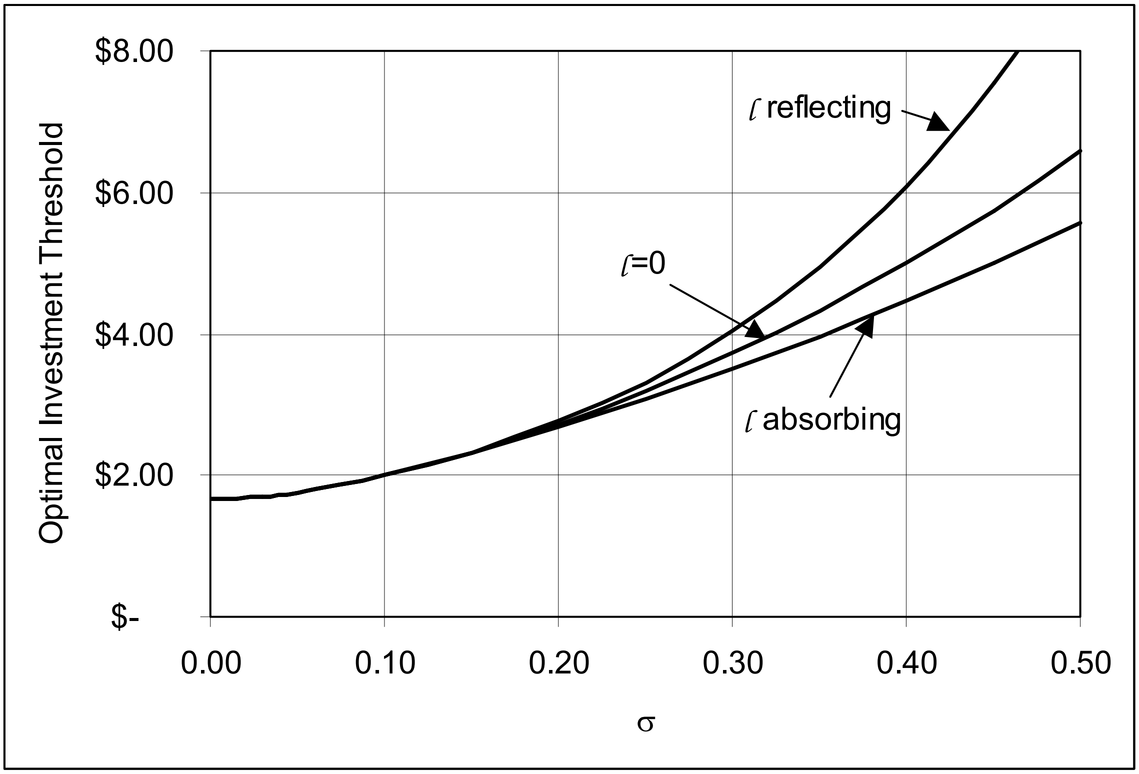

Figure 1A. Optimal Investment Threshold x^* as a function of σ.

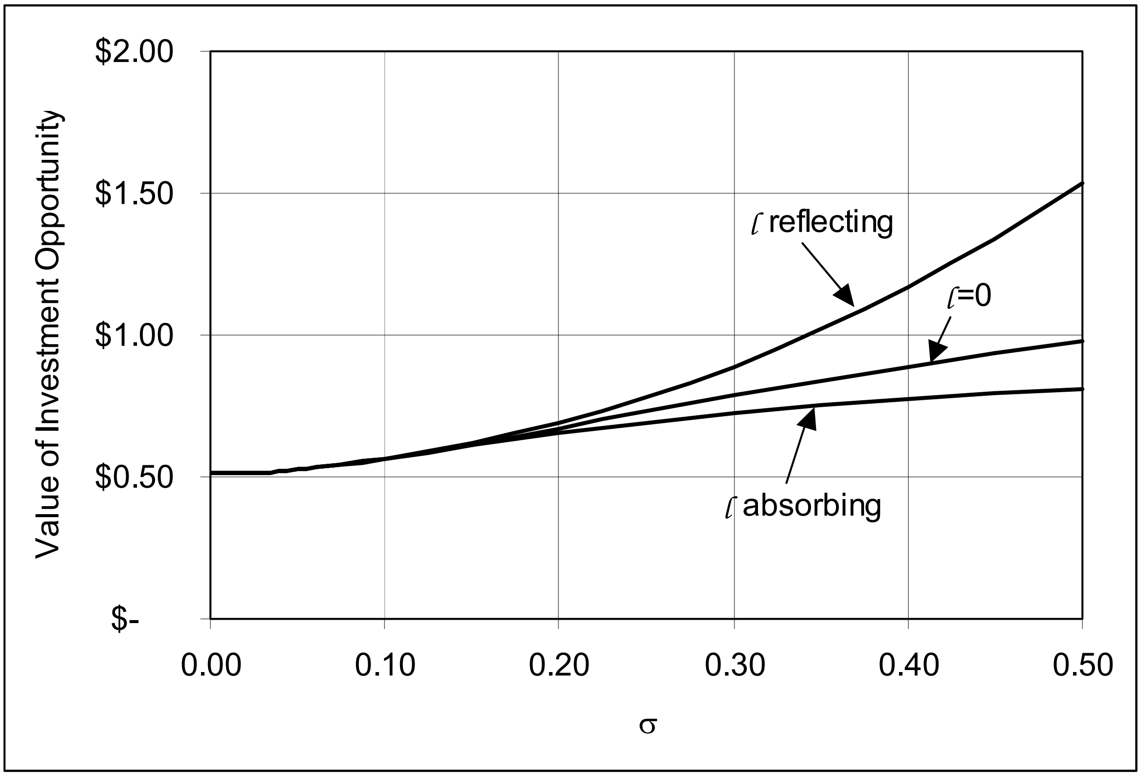

Figure 1B. Value of Investment Opportunity as a function of σ.

Notes for Figures 1A and 1B: these results were generated using I=\$1.0 (the cost of the investment), α=2% per year (the expected rate of growth of the investment), ℓ=\$0.5 (for the absorbing and reflecting cases), and ρ=5% per year (the discount rate). The volatility coefficient, σ, is in $(\text{year})^{-0.5}$. The value of the investment opportunity is calculated at y=X(0)=\$1.5.

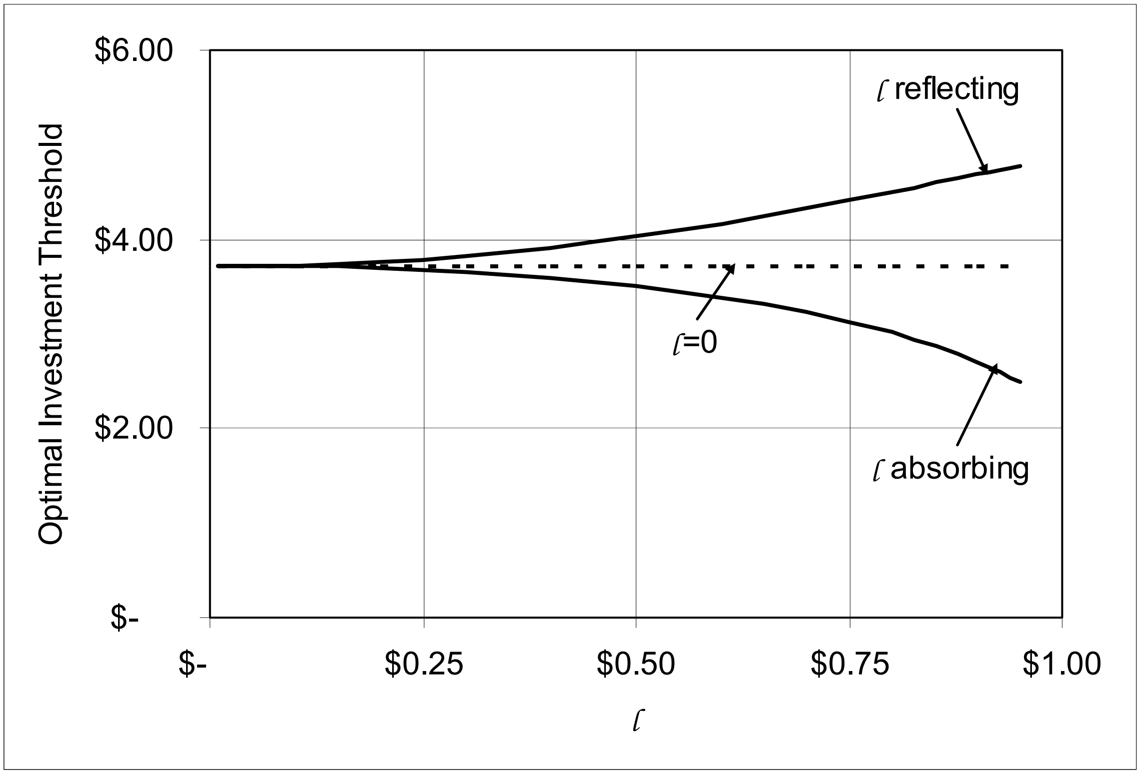

Figure 2A. Optimal Investment Threshold x^* as a function of ℓ.

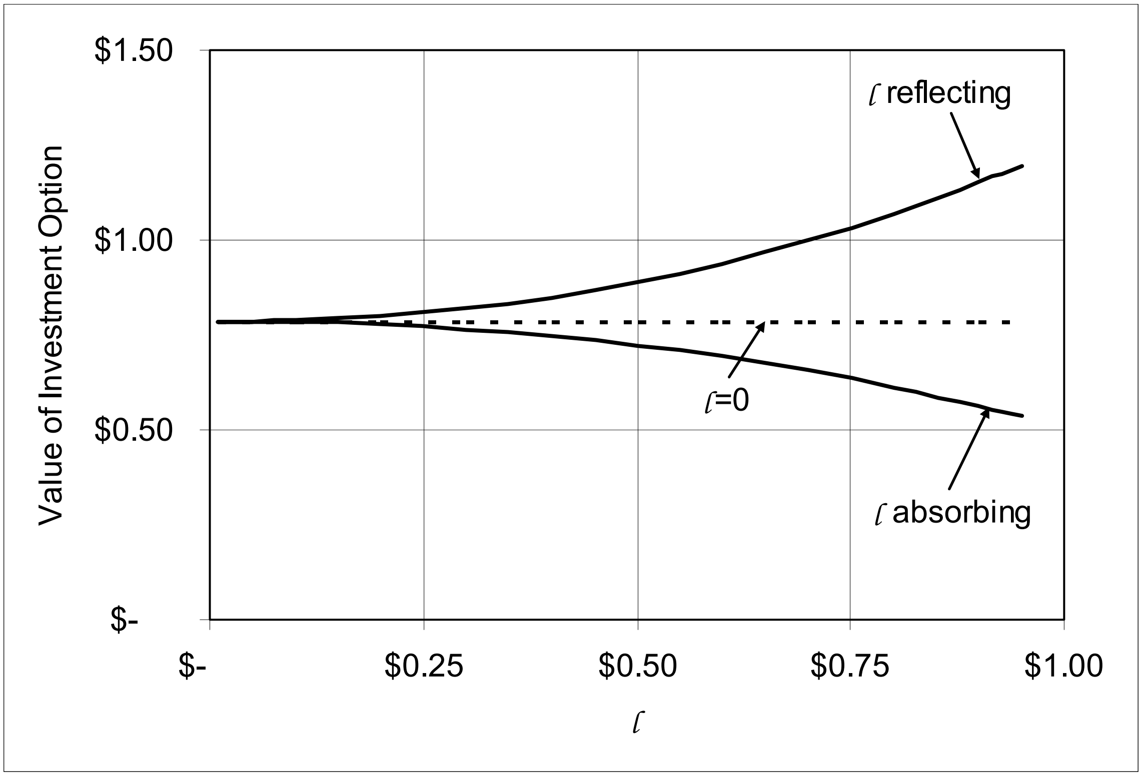

Figure 2B. Value of Investment Opportunity as a function of ℓ.

Notes for Figures 2A and 2B: these results were generated using I=\$1.0 (the cost of the investment), α=2% per year (the expected growth rate of the investment), σ=0.3 (in $(\text{year})^{-0.5}$), and ρ=5% per year (the discount rate). The value of the investment opportunity is calculated at y=X(0)=\$1.5.

[1] Brock, Rothschild, and Stiglitz (1982) analyze what the standard "tree cutting problem". Their comprehensive analysis applies to general stochastic processes but it relies on advanced mathematics and on what has become the real options methodology; in addition, they assume the existence of a single stopping value and they do not analyze the GBM case. By contrast, the analysis herein presents an alternate approach that relies only on elementary mathematical tools, and it proves the existence of a single stopping value for the GBM with barriers.

[2] For other types of barriers, see Dumas 1991 or Dixit 1993.

[3] A parenthesis means that an interval is open at that end, while a square bracket means that it is closed. Thus $(a, b]$ includes b but not a.

[4] This is clearly a strong assumption but the presence of an attainable barrier makes it difficult to use the Capital Asset Pricing Model to find the nondiversifiable risk of the investment opportunity. Technical difficulties may detract us from our goal, i.e. analyzing the impact of a barrier on the decision to invest in a simple framework.

[5] Dixit, Pindyck and Sødal (1999) deal only with an unattainable lower barrier.

[6] It corresponds to $-\infty$ for $\ln(X)$, which follows a Brownian motion, and we know that diffusions have only finite variations in finite time.

[7] As discussed in Dixit and Pindyck (1984, page 150), assuming that σ varies independently of other parameters (such as the discount rate ρ or the expected rate of growth α of X) is often not very satisfactory. For simplicity, we adopt this assumption here because we are interested in possible impacts of barriers on the decision to invest. Note, however, that the discount rate is typically mandated exogenously for public projects.