The Density of Bounded Diffusions.[1]

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Abstract

This paper presents a methodology for deriving the closed-form density of diffusions restricted to finite intervals with reflecting or absorbing barriers. Bounded diffusions are useful, for example, in finance, resource economics, or industrial organization. Results are derived for popular diffusions.

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I. Introduction

Many economic time series may be usefully modeled with bounded diffusions. Reflecting barriers are warranted for price floors or price ceilings, or for limits on a population density; exchange rates or interest rates appear bounded by reflecting barriers. Alternatively, a lower absorbing barrier is needed when an investment opportunity vanishes at low values or for a renewable resource that could become extinct.

Unfortunately, explicit expressions for the density of bounded diffusions are unavailable in the econometrics literature, which may constrain modeling and precludes using maximum likelihood techniques. This may also create some econometric difficulties. For exchange rates, for example, it is difficult to reject the unit root hypothesis (see Perron 1989 or Caporale, Pittis, and Sakellis 2003), although some time series exhibit weak mean-reversion (Aït-Sahalia 1996).

This paper presents a methodology for finding the density of diffusions restricted to intervals with reflecting or absorbing barriers. It relies on separation of variables techniques and on the classical solution of homogenous, parabolic partial differential equations. Results are provided for several popular diffusions.

II. Methodology

Consider the diffusion $\{X(t),t\geq 0\}$ defined over $[L,R]$, where both L and R are finite. L and R are either reflecting or absorbing barriers:

$$dX = \mu(X)dt + \sigma(x)dw. \qquad (1)$$

The infinitesimal drift $\mu(x)$ and volatility $\sigma(x)>0$ are assumed to be continuous; and dw is an increment of a standard Wiener process (Karlin and Taylor 1981).

For $(x,y)\in[L,R]^2$, let $p(t,y;x)$ be the density of $X(t)$ at y given that $X(0)=x$. Karlin and Taylor (1981) show that $u(t,x)\equiv p(t,y;x)$ verifies the Kolmogorov backward equation

$$\frac{\partial u(t,x)}{\partial t}=\frac{\sigma^2(x)}{2}\frac{\partial^2 u(t,x)}{\partial x^2}+\mu(x)\frac{\partial u(t,x)}{\partial x}, \tag{2}$$

which is a second order parabolic differential equation. Since $X=x$ at time 0, the initial condition for this problem is

$$u(0^+,x)=\delta_y(x), \tag{3}$$

where $\delta_y(x)$ is the Dirac delta function. To have a well-defined problem, we need boundary conditions at L and R. Let ξ designate either L or R. From Karlin and Taylor (1981), if ξ is reflecting, then

$$\forall t\geq 0,\ \frac{\partial u(t,\xi)}{\partial x}=0. \tag{4}$$

If instead ξ is absorbing we have,

$$\forall t\geq 0,\ u(t,\xi)=0. \tag{5}$$

Equations (2), (3), and boundary conditions at L and R uniquely define $u(t,x)$, if a solution exists (Lieberman 1996). Since this problem is homogeneous, a solution by separation of variables can be attempted: let $u(t,x)=g(t)f(x)$. We see that $g(t)$ is proportional to $e^{-\lambda t}$, where $\lambda>0$ is an unknown constant (if $\lambda<0$, the solution explodes), and $f(x)$ verifies

$$\frac{\sigma^2(x)}{2}f''(x)+\mu(x)f'(x)+\lambda f(x)=0. \tag{6}$$

Multiplying (6) by

$$r(x)=\frac{2}{\sigma^2(x)}\exp\left(\int_L^x\frac{2\mu(z)}{\sigma^2(z)}dz\right) \tag{7}$$

gives the Sturm-Liouville equation (Boyce and DiPrima 1969)

$$\left[p(x)f'(x)\right]'+[\lambda r(x)-q(x)]f(x)=0, \tag{8}$$

with

$$p(x)=\exp\left(\int_L^x\frac{2\mu(z)}{\sigma^2(z)}dz\right) \text{ and } q(x)=0. \tag{9}$$

Here $p(x)$, $p'(x)$, $q(x)$, and $r(x)$ are continuous on $[L,R]$, $p(x)>0$ and $r(x)>0$ on $[L,R]$, so we know, from the Sturm-Liouville theorem (Boyce and DiPrima 1969), that:

1) There exists an infinite, countable sequence of real, strictly increasing, non-negative eigenvalues λ_n, and associated nonzero eigenfunctions $F_n(x)$ that verify $F_n(\xi)=0$ or $F_n'(\xi)=0$ ($\xi\in\{L, R\}$. Note that if we had L=-∞ or R=+∞, the spectrum of eigenvalues might be a mixture of discrete and continuous values; if (L, R) = (-∞, +∞), the spectrum of eigenvalues would likely be continuous;

2) The $F_n(x)$s are orthogonal with respect to $r(x)$, i.e. if $m\neq n$ then $\int_L^R r(z)F_n(z)F_m(z)dz=0$; and

3) The $F_n(x)$s forms a complete orthogonal set of functions defined on $[L,R]$, so any piecewise continuous function $g(x)$ can be expressed as a linear combination of the $F_n(x)$s:

$$\sum_{n=0}^{+\infty} c_n F_n(x) = \begin{cases} g(x), & \text{if } g(.) \text{ is continuous at } x, \\ \dfrac{g(x^-)+g(x^+)}{2}, & \text{if } g(.) \text{ is discontinuous at } x, \end{cases}$$

where $c_n = \int_L^R r(y)g(y)F_n(y)dy \left(\int_L^R r(y)F_n^2(y)dy \right)^{-1}$.

If problem (2)-(3) with appropriate boundary conditions at L and R has a well-behaved solution, it may then be written

$$u(t,x) = \sum_{n=0}^{+\infty} A_n e^{-\lambda_n t} F_n(x), \tag{10}$$

where

$$A_m = r(y)F_m(y)\left(\int_L^R r(\xi)F_m^2(\xi)d\xi \right)^{-1}, \tag{11}$$

after imposing $\sum_{n=0}^{+\infty} A_n F_n(x) = \delta_y(x)$ at t=0^+ (from (3)). Hence,

$$p(t,y;x) = \sum_{n=0}^{+\infty} \left\{ e^{-\lambda_n t} r(y)F_n(y)F_n(x)\left(\int_L^R r(\xi)F_n^2(\xi)d\xi \right)^{-1} \right\}. \tag{12}$$

This powerful approach is well known in applied mathematics but it seems to have been overlooked in econometrics. It is especially useful when an explicit expression of the eigenfunctions $F_n(x)$ can be found, which we show is the case for a number of common diffusions. Otherwise, Aït-Sahalia's method (2002), which approximates densities with Hermite polynomials, may be used.

III. Applications

To start with, let us consider the trendless Brownian motion (BM) over [L,R], with $\sigma > 0$:

$$dX = \sigma dw. \tag{13}$$

Let us first assume that L and R are reflecting. A generic solution of (6) that verifies $f'(L) = f'(R) = 0$ is $F_n(x) = \cos(n\pi \frac{x-L}{R-L})$, n integer, so that $\lambda_n = \frac{1}{2}\left(\frac{n\pi\sigma}{R-L}\right)^2$. The weight function here is $r(x) = \frac{2}{\sigma^2}$, so $A_0 = \frac{1}{R-L}$ and $A_n = \frac{2}{R-L}$ for n>0. Hence,

$$p(t,y;x) = \frac{1}{R-L}\left\{1 + 2\sum_{n=1}^{+\infty}\exp\left(-\frac{t}{2}\left(\frac{n\pi\sigma}{R-L}\right)^2\right)\cos\left(n\pi\frac{x-L}{R-L}\right)\cos\left(n\pi\frac{y-L}{R-L}\right)\right\}. \tag{14}$$

As expected, as $t \to +\infty$, $p(t,y;x)$ tends towards the uniform stationary density $\psi(y) = (R-L)^{-1}$.

If instead L and R are absorbing ($f(L) = f(R) = 0$), $F_n(x) = \sin(n\pi\frac{x-L}{R-L})$, with $\lambda_n = \frac{1}{2}\left(\frac{n\pi\sigma}{R-L}\right)^2$, $A_n = \frac{2}{R-L}$, all for n>0, and $A_0 = \frac{1}{R-L}$. Hence,

$$p(t,y;x) = \frac{2}{R-L}\sum_{n=1}^{+\infty}\exp\left(-\frac{t}{2}\left(\frac{n\pi\sigma}{R-L}\right)^2\right)\sin\left(n\pi\frac{x-L}{R-L}\right)\sin\left(n\pi\frac{y-L}{R-L}\right). \tag{15}$$

This time, $p(t,y;x)$ tends towards 0 as $t \to +\infty$ as X is progressively "captured" either by L or by R, so over time $p(t,y;x)$ tends towards a uniformly null function on (L, R).

Figures 1a and 1b illustrate respectively the reflecting and the absorbing cases. They correspond to classical heat diffusion problems in a laterally insulated rod with an initial temperature distribution. In the former, ends are insulated; in the latter, they are at temperature 0. Alternatively, we could analyze "mixed" boundary conditions, where one barrier is reflecting and the other absorbing, using the same approach. Note that ten terms were used in the summation of $p(t,y;1)$; there was little difference between 8 and 12 terms.

Now consider the Ornstein-Uhlenbeck (OU) process

$$dX = \nu(K - X) + \sigma dw, \tag{16}$$

where $\nu>0$ drives how fast X reverts to $K \in (L,R)$ and $\sigma > 0$. A series expansion shows that $\varphi_{1n}(x) = \phi\left(\frac{-\lambda_n}{2\nu}, \frac{1}{2}, \frac{\nu(x-K)^2}{\sigma^2}\right)$ and $\varphi_{2n}(x) = (x-K)\phi\left(\frac{1}{2} - \frac{\lambda_n}{2\nu}, \frac{3}{2}, \frac{\nu(x-K)^2}{\sigma^2}\right)$ are two independent solutions of (6). The function

$$\phi(a,c;z) = \sum_{k=0}^{+\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!}, \tag{17}$$

with $(a)_0=1$ and $(a)_k=a\cdot(a+1)\cdots(a+k\text{-}1)$, designates the confluent hypergeometric function of the first kind (Luke 1969).

Looking first at the reflected case, we find that a generic solution of (6) is given by

$$F_n(x) = \varphi_{1n}(x) - \frac{\varphi_{1n}'(L)}{\varphi_{2n}'(L)}\varphi_{2n}(x), \tag{18}$$

The λ_ns are the roots of $\varphi_{1n}'(L)\varphi_{2n}'(R) - \varphi_{1n}'(R)\varphi_{2n}'(L) = 0$; they need to be obtained numerically. Here, $r(x) = \frac{2}{\sigma^2}\exp\left(-\frac{\nu}{\sigma^2}(x-K)^2\right)$, so we calculate A_m from (11) and (18), and then $p(t,y;x)$ from (12). As $t \to +\infty$, $p(t,y;x)$ tends towards the stationary density $\psi(y) \equiv r(y)\left(\int_L^R r(\xi)d\xi\right)^{-1}$.

We proceed similarly for the absorbing case. The λ_ns are now the roots of $\varphi_{1n}(L)\varphi_{2n}(R) - \varphi_{1n}(R)\varphi_{2n}(L) = 0$, and a generic solution $f(x)$ of (6) is

$$F_n(x) = \varphi_{1n}(x) - \frac{\varphi_{1n}(L)}{\varphi_{2n}(L)}\varphi_{2n}(x). \tag{19}$$

Figures 2a and 2b illustrate these results. Ten terms were used in the summation of $p(t,y;1)$; the difference between 8 and 12 terms was visually insignificant. Since the λ_is depend on the model parameters (K, ν, and σ), using the proposed framework for maximum likelihood requires iterating.

An explicit expression of the eigenfunctions $F_n(x)$ can also be found for several other popular diffusions. The key is to derive 2 independent solutions of (6) and to combine them with the relevant boundary conditions. In the following, let $\varphi_{1n}(x)$ and $\varphi_{2n}(x)$ denote 2 independent solutions of (6).

- For the square root process (Cox, Ingersoll, and Ross 1985) with $\nu>0$, $\sigma>0$, and $L<K<R$,

$$dX = \nu(K - X) + \sigma\sqrt{X}dw, \tag{20}$$

we have $\varphi_{1n}(x) = \phi\left(\frac{-\lambda_n}{\nu}, \frac{2\nu K}{\sigma^2}, \frac{2\nu x}{\sigma^2}\right)$ and $\varphi_{2n}(x) = \psi\left(\frac{-\lambda_n}{\nu}, \frac{2\nu K}{\sigma^2}, \frac{2\nu x}{\sigma^2}\right)$, where,

$$\Psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)}\Phi(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\Phi(1+a-c,2-c;z) \tag{21}$$

is the confluent hypergeometric function of the second kind and $\Gamma(y) \equiv \int_0^{+\infty} e^{-t} t^{y-1} dt$ is the gamma function (Luke 1969).

- For the Gompertz Brownian motion process (ν>0, σ>0, and L<K<R)

$$dX = \nu X(Ln(K) - Ln(X))dt + \sigma X dz, \tag{22}$$

we find $\varphi_{1n}(x) = \phi\left(\frac{-\lambda_n}{2\nu}, \frac{1}{2}, f^2(x)\right)$, $\varphi_{2n}(x) = f(x)\phi\left(\frac{1}{2} - \frac{\lambda_n}{2\nu}, \frac{3}{2}, f^2(x)\right)$, with $f(x) = \frac{\sqrt{\nu}}{\sigma}\left[\ln\left(\frac{x}{K}\right) + \frac{\sigma^2}{2\nu}\right]$.

- Finally, for the Brownian motion (σ>0)

$$dX = \mu dt + \sigma dz, \tag{23}$$

we obtain $\varphi_{1n}(x) = \mathrm{e}^{\frac{-\mu}{\sigma^2}x} \cos\left(\sqrt{\frac{2\lambda_n}{\sigma^2} - \left(\frac{\mu}{\sigma^2}\right)^2}\, x\right)$, $\varphi_{2n}(x) = \mathrm{e}^{\frac{-\mu}{\sigma^2}x} \sin\left(\sqrt{\frac{2\lambda_n}{\sigma^2} - \left(\frac{\mu}{\sigma^2}\right)^2}\, x\right)$ where necessarily $\mu^2 - 2\lambda_n\sigma^2 < 0$. Note, however, that $p(t, y; x)$ cannot be found by this approach if either R is reflecting when μ>0 or L is reflecting when μ<0. This suggests the presence of a singularity at the boundary that cannot be handled by this approach. Similar results can be derived for the geometric Brownian motion.

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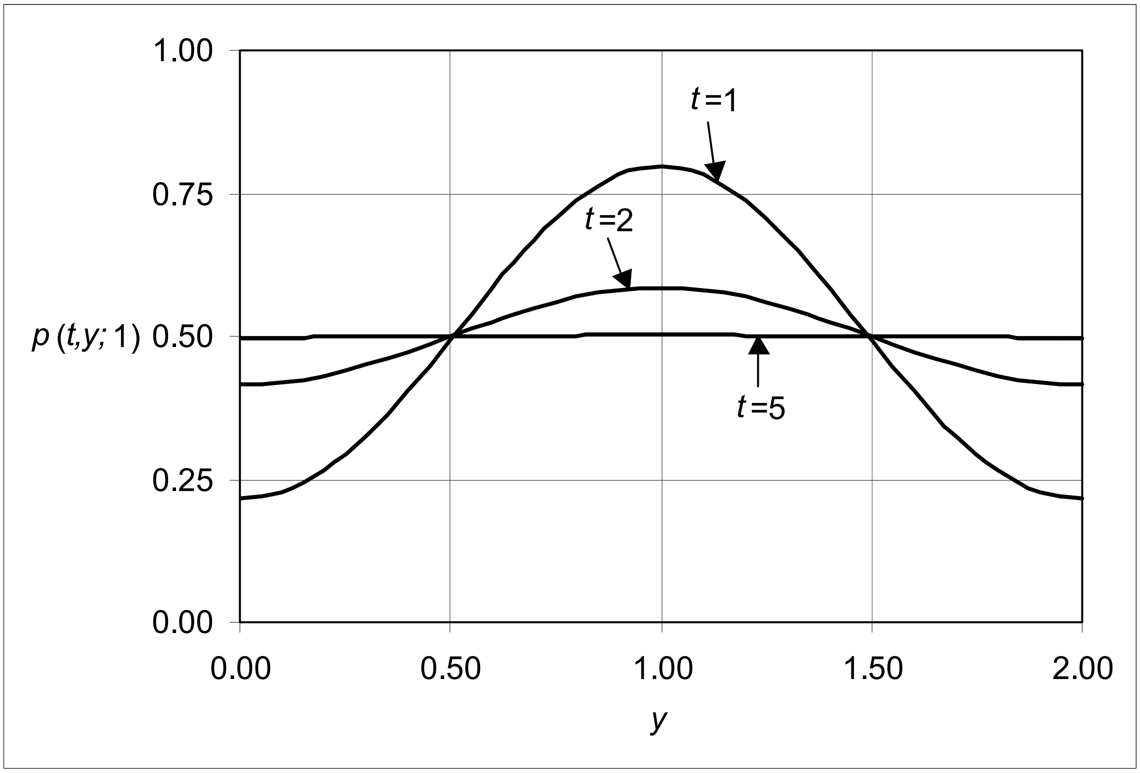

Figure 1a. $p(t,y;1)$ when X follows a reflected, trendless Brownian Motion.

Notes. These results were generated with L=0, R=2, σ=0.5, and x_0=1.

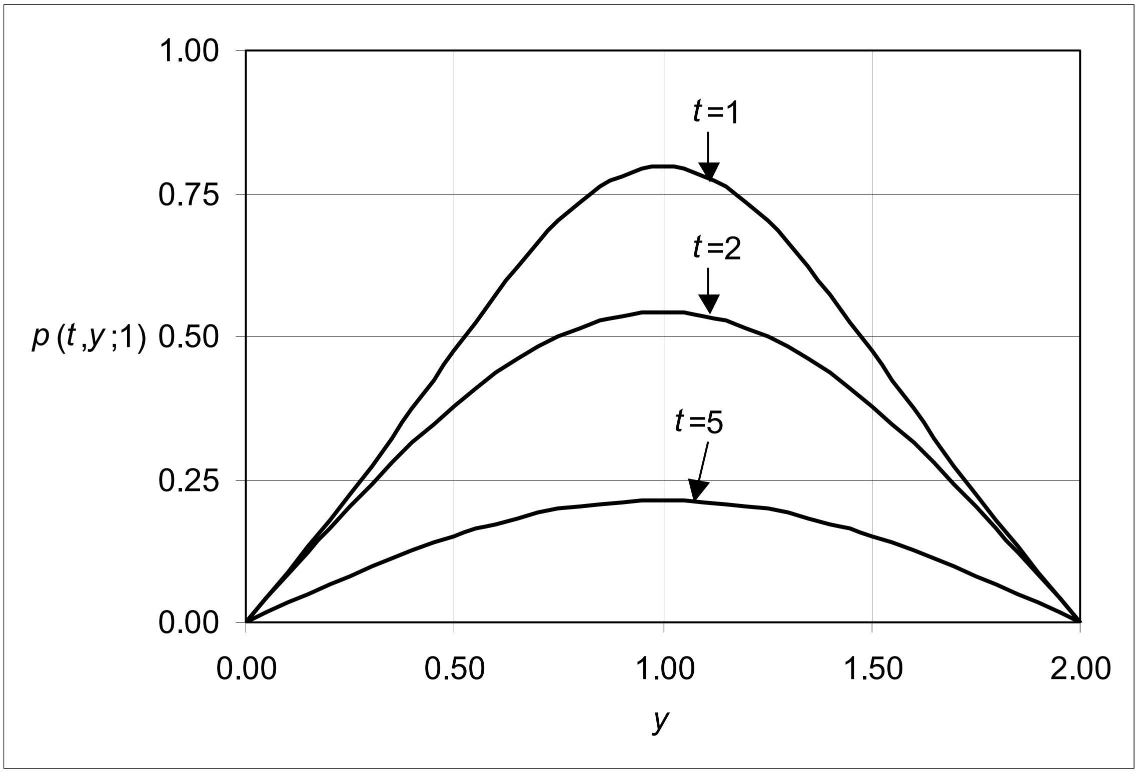

Figure 1b. $p(t,y;1)$ when X follows an absorbing, trendless Brownian Motion.

Notes. These results were generated with L=0, R=2, σ=0.5, and x_0=1. As t increases, the probability that X never hits L or R goes to zero so $p(t,y;1)$ vanishes to 0 on (L,R).

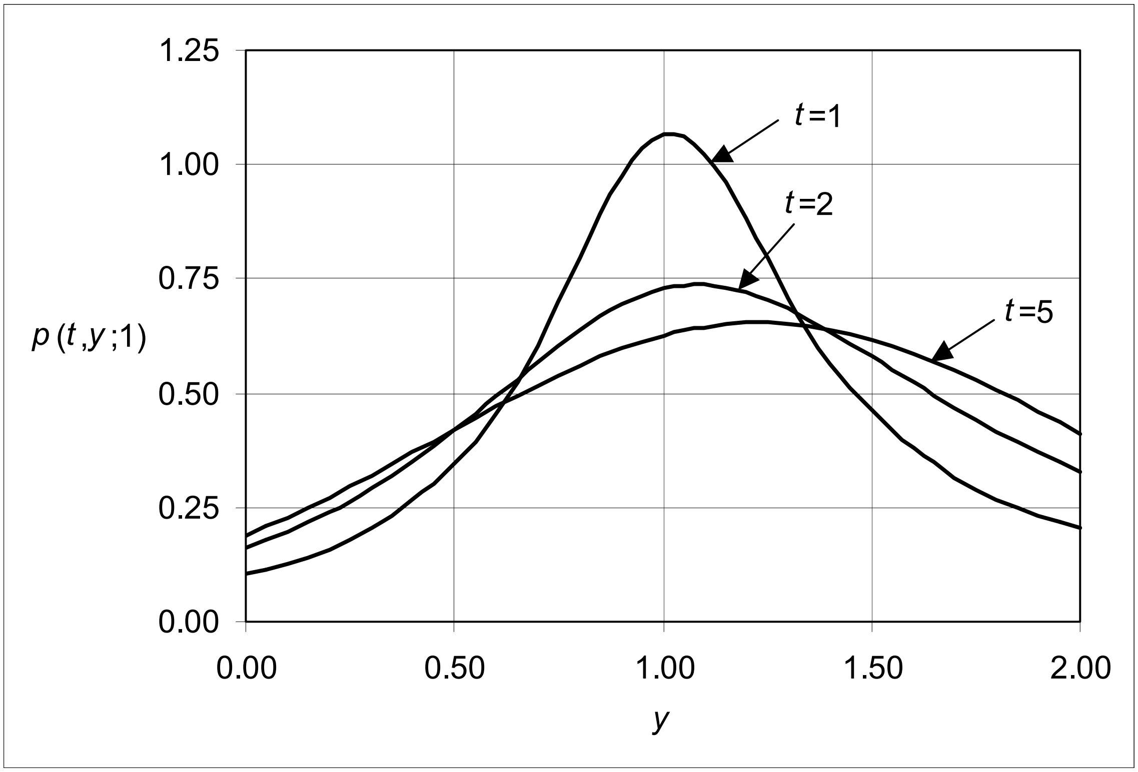

Figure 2a. $p(t,y;1)$ when X follows a reflected Ornstein-Uhlenbeck process.

Notes. These results were generated with L=0, R=2, K=1.25, ν=0.2, σ=0.5, and x_0=1. The first 10 eigenvalues (λ_is) are 0.0000, 0.6508, 1.1668, 1.7046, 2.2507, 2.8003, 3.3518, 3.9044, 4.4576, and 5.0114.

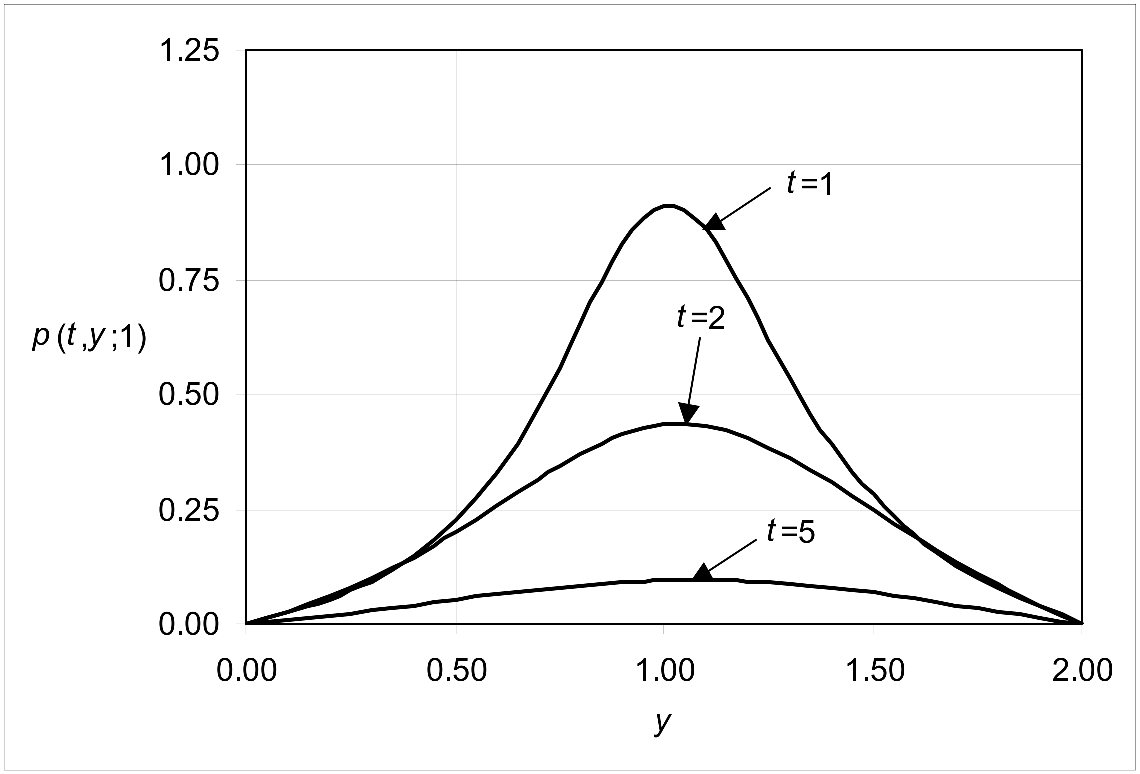

Figure 2b. $p(t,y;1)$ when X follows an absorbed Ornstein-Uhlenbeck process.

Notes. These results were generated with L=0, R=2, K=1.25, ν=0.2, σ=0.5, and x_0=1. The first 10 eigenvalues (λ_is) are 0.4729, 1.0776, 1.6449, 2.2058, 2.7644, 3.3218, 3.8788, 4.4352, 4.9914, and 5.5474. As t increases, the probability that X never hits L or R goes to zero so $p(t,y;1)$ vanishes to 0 on (L,R).