65-11, 090

KASSOUF, Sheen Thomas, 1928-
A THEORY AND AN ECONOMETRIC MODEL FOR COMMON STOCK PURCHASE WARRANTS.

Columbia University, Ph. D., 1965
Economics, finance

University Microfilms, Inc., Ann Arbor, Michigan

A THEORY AND AN ECONOMETRIC MODEL FOR COMMON STOCK PURCHASE WARRANTS

Sheen Thomas Kassouf

1965

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy, in the Faculty
of Political Science, Columbia University

ABSTRACT

A THEORY AND AN ECONOMETRIC MODEL FOR
COMMON STOCK PURCHASE WARRANTS

Sheen Thomas Kassouf

In the never-never land of Complete Certainty, a free securities market serves the important function of channeling investment funds into those industries where marginal product is highest thus assuring an optimal allocation of resources. So it is often claimed that the individual (investor) seeking his self interest concomitantly serves the public interest. But in the imperfect world of reality, investors are saddled with uncertainty and it is far from clear if their activities always result in an "optimum." This is because decisions are formed with the aid of expectational-utility functions and very little is known about how these subjective phenomena affect or are affected by economic forces.

Chapter I describes the decision-making process of an investor who attempts to maximize his expected utility. Then, because of the intimate connection between a security and an option on that security, a reconstruction of the decision-making process is attempted from observed prices. It is then shown that many different combinations of

risk-attitudes and expectation distributions can give rise to the same observed result. Nevertheless, by describing the decision-making procedure quantitatively, the role of expectations and risk-attitudes is made explicit and tractable.

Using a priori and empirical considerations, the relationship of the price of a common stock and its associated warrant is defined in Chapter II as a function of investor expectations, risk-attitudes, and a set of other variables. By specifying an econometric model and examining the computed relationship, it is suggested that something may be inferred about investors' expectations and risk-attitudes.

Chapter III tests the specified models. It is found that common stock price, length of option period, dilution ratio, common stock dividend yield, and common stock price performance are all significant variables influencing warrant price. It is also found that the "better" the price performance of the common stock in the immediately preceding eleven months, the less investors were willing to pay for the warrant. Unfortunately, because the universe under study was not large (warrants listed on the American Stock Exchange between 1945 and 1964) no conclusive findings could be made as to how and whether risk-attitudes change.

The study concludes by pointing out how the models developed can be extended to cover a broader spectrum of securities thereby leading to the possibility of more conclusive findings. This may provide economists and econometricians a key to the house now occupied almost solely by psychologists and sociologists. Answers to some intriguing questions may now be possible: do investors' expectations lead or lag security prices (and thus aggregate economic activity); does caution or risk-aversion cause lower security prices, or if there is a relation is it the other way around; do the expectations of subsets of the investor population (e.g., insiders, professionals, "the public," etc.) differ and in what manner? Definitive answers would perhaps put some flesh on the invisible hand ruling a market economy and carry important implications for the control of potentially dangerous business fluctuations.

PREFACE

In 1961 I was called upon to evaluate the common stock of Textron, Inc. After a favorable review, I noted that the company had a warrant listed on the American Stock Exchange and was thus faced with an intrusive alternative that could not be ignored. I assumed that the choice between the common stock and the warrant could be made routinely if only I knew something about the nature of warrants. In searching the literature, I was dismayed to discover that too often financial analysts resolved this problem by either ignoring warrants or embracing them on questionable premises. Later that year I became more frustrated when I thought the Molybdenum warrant was at an unexplainable level and could be the vehicle for an ideal investment. But again I was disappointed in finding confirmation for my suspicions. Consequently, I decided to study warrant behavior. I was surprised that my exploration led to a theory rich in economic significance. This work is the product of that exploration.

It is a pleasure to acknowledge my debts: To Professor Arthur F. Burns whose incisive comments often cut through my cloudy reflections, exposing the kernel of the situation. His careful use of language and lucid exposition set an (unattainable) goal for my writing. To Professor

Jon Cunnyngham for the kind use of the versatile regression program he developed for the National Bureau of Economic Research and for his helpful discussions. To my wife, Gloria, who typed the first draft and who provided the encouragement and atmosphere conducive to study. To my brother, Ned, who helped gather the data and whose faith was extraordinary--he invested funds on the basis of my earliest hunches.

CONTENTS

LIST OF ILLUSTRATIONS

Figure | Page

INTRODUCTION

In 1911 the American Power and Light Company issued notes with common stock purchase warrants attached, thereby enriching investment alternatives in an intriguing way. The introduction of the warrant obviously added another security to the growing number of outstanding issues, but less obviously, it permitted sophisticated investors to acquire portfolios that promised substantially higher returns than were previously available. This characteristic alone makes a detailed study of warrants appropriate.

More importantly, a theory of warrant prices can be extended, mutatis mutandis, to cover any convertible security; and because of the intimate connection between the price of a convertible and its associated common stock, light can be shed on investors' expectations and risk-attitudes, i. e. , their behavior under conditions of uncertainty. (Theories of investor behavior under complete certainty are sterile and of little value in describing reality.)

Chapter I will develop a general theory of the investment decision-making procedure under the realistic constraint of uncertainty. The theory will apply to investors who are insignificant relative to the market--the counterpart of atomistic competition in product and factor markets--and thus will not encompass game-theoretic principles. In the highly developed securities markets of today, this theory will thus cover all but a few of the giant investors.

Chapter II discusses a priori considerations for an econometric model for warrant price and specifies two models. Chapter III tests them against the data of the preceding two decades.

CHAPTER I

A GENERAL THEORY OF WARRANT PRICE

Definition

A common stock purchase warrant is a security which the holder may exchange, at his option, into equity capital. The exchange may be effected by surrendering the warrant and a prespecified sum of money before a prespecified date to the corporation issuing the common stock. Warrants are therefore convertible securities and because they are options on common stock it is customary to call the act of conversion the _exercise_ of the warrant and the accompanying money the _exercise price_. There are essential differences between the various convertible securities[1] but the theory that will be outlined here can be adapted to cover convertible bonds, stock rights, call options, stock options, and other convertible forms.

---

[1]A definition of a convertible security, if it is not to become unweildy, and if it is not to be reduced to a tautology, e.g., a convertible security is one that can be converted, must exclude many special cases. For instance, it may be said that a convertible security may under certain conditions be converted into common stock. But this definition would exclude the Universal American 1955 warrants which can be converted into common stock _and_ Universal American 1962 warrants; when defining the exercise price, it is usual to state the amount of money required with each warrant for conversion; but this would not define the exercise price of the Central States Electric warrant which expired in 1934 and whose exercise price varied with the number of

warrants exercised--the greater the number of warrants exercised, the greater the exercise price of the remaining outstanding warrants. With the knowledge then, that no satisfactory inclusive definition is possible, it may still be helpful to outline the general characteristics of several important types of convertible securities.

Security	How Exercised	Length of Option Period	Issued By	Transferable
Warrants	Surrendered with cash (exercise price)	Usually more than one year; some extend indefinitely	Corporation, from treasury or authorized stock	Yes
Convertible bonds	Surrendered usually without cash	Usually more than five years; if bond is called option expires with redemption	"	"
Convertible preferred stock	"	Usually for life of preferred stock; if called option expires with retirement of preferred	"	"
Executive and employee options	Surrendered with cash	Usually less than five years	"	No
Stock rights	Surrendered with cash	Usually less than three months	"	Yes
Call options	Surrendered with cash (striking price)	Usually not more than one year	Any individual or organization (called a writer)	"

As an example, the McCrory Corporation warrant, when accompanied with \$20, may be converted into one share of McCrory common stock anytime before March 15, 1976. This warrant is said to be _exercisable_ at \$20 during the entire length of its option period. In many cases, the exercise price varies during the option period: the Trans World Airlines warrant is exercisable at \$20 on or before June 1, 1965 and at \$22 on or before December 1, 1973 after which time it expires.

Very often because of stock dividends or mergers a warrant is convertible into more than one share of common stock:[1] the Sperry Rand warrant, with \$28, is convertible into 1.08 shares of common at any time on or before September 15, 1967. One Sperry warrant can be considered the equivalent of 1.08 _adjusted_ warrants, and each adjustable warrant is exercisable at \$28/1.08 = \$25.93. Throughout the remainder of this study, unless otherwise specified, the term _warrant_ will refer to _adjusted warrants_.

---

[1]Most warrants are protected against any dilution that may occur should common stock be issued for a consideration per share less than the exercise price of the warrants. For instance, if a 100% stock dividend is declared, the exercise price of a warrant will be halved and the number of shares that it may be converted into is doubled. This retains the original position of the warrant holder vis-a-vis the common stockholders. If the warrant holder were not so protected, his position could be seriously worsened by merely declaring stock dividends or splitting the stock. In general, for full protection, if the number of outstanding shares of common is z, and if x shares are issued for a consideration of \$y per share where y is less than a, the exercise price of the common, then the warrant should become the right to purchase $1 + (a-y)x/az$ shares for a total consideration of \$a. Virtually all warrants today are fully protected against dilution. A few, however, are protected only if $(a-y)x/az$ is greater than a specified amount, usually if this ratio is greater than 0.10.

Warrant Price as a Function of Common Stock Price

The market price of a security reflects the anticipations of investors and speculators. (An investor might be defined as one who anticipates the future income stream of a security, whereas a speculator anticipates the anticipations of investors and perhaps other speculators.) For this reason it is impossible to place a priori bounds on the market price of a common stock other than to say that it may extend from zero to infinity. If, for instance, it were suddenly apparent that XYZ common would never yield a dividend, its price might fall to zero. On the other hand, if it were believed that the dividend yield of XYZ would grow indefinitely at a rate equal to the market rate of interest, an investor attempting to maximize money gain could not pay too much for a share of XYZ because its present value, the sum of the discounted expected future dividends, would equal infinity.[1]

In contrast, common stock purchase warrants are constrained to move, a priori, in narrower channels. A warrant cannot sell for less than the difference between the price of its associated common stock and its exercise price, except momentarily, nor for more than the price of the common stock. The lower bound of this interval is defended by

[1]The theory of present value is explored thoroughly by John Burr Williams in The Theory of Investment Value (Cambridge: Harvard University Press, 1938). For a more explicit reference to the possibility of an infinite valuation see David Durand, "Growth Stocks and the Petersburg Paradox," Journal of Finance, XII (September, 1957), 348-363 and also Philip Kotler, "Elements in a Theory of Growth Stock Valuation," Financial Analysts Journal, XVIII (May-June, 1962), 35-44.

arbitrageurs: if the warrant could be purchased for less, a simultaneous short sale of the common would yield an instantaneous and riskless profit. For example, if the McCrory common stock were selling for \$30, the warrant will not sell for very long for less than \$10; otherwise, an alert trader would sell the stock short for \$30, convert the warrant at a cost of \$20 plus the price of the warrant, and thereby net an immediate gain. (This example has abstracted from buying and selling costs to simplify the exposition.) Thus, arbitrage activity tends to raise the price of the warrant and depress the price of the common until the price of the warrant is above or at the lower bound of the interval defined above.

The upper bound is dictated by reason: the warrant is inferior to the common with respect to dividends, voting power, and length of life. Since the warrant has no offsetting characteristics, its value cannot exceed the value of the common.[1]

---

[1]Williams, 173, asserts that the difference between the price of a warrant and its associated common stock must be less than the exercise price but more than the capitalized value of the dividend yield of the common. In symbols, (different from his) $d/i \leq x-y \leq a$, where d is the annual dividend payment, i the interest rate an investor considers "normal", x the price of the common, y the price of the warrant, and a the exercise price. He claims that d/i cannot exceed x-y, for if it did, "warrant holders could sell their warrants, buy the same number of shares of common stock, and receive a better-than-normal return on the additional cash invested." If his claim is correct, then the region in Fig. 1, below, is unnecessarily wide. But it can be shown that Williams' contention does not conform to rational behavior. His inequality is equivalent to: $x-d/i \geq y \geq x-a$. To show that $x-d/i \geq y$ is incorrect, consider the Hilton Hotels warrant, which in the Fall of 1962 was exercisable at \$42 and was selling for \$6.50 with the common at \$24. The common was yielding annually \$1.50 per share. If invest-

If the price of a warrant is considered a function of the price of the associated common stock, a region is carved out of the quadrant of positive prices--a region to which this functional relation is restricted. In Fig. 1, this region is the unshaded portion of the quadrant, where a is the exercise price, y the price of the warrant, and x the price of the common.

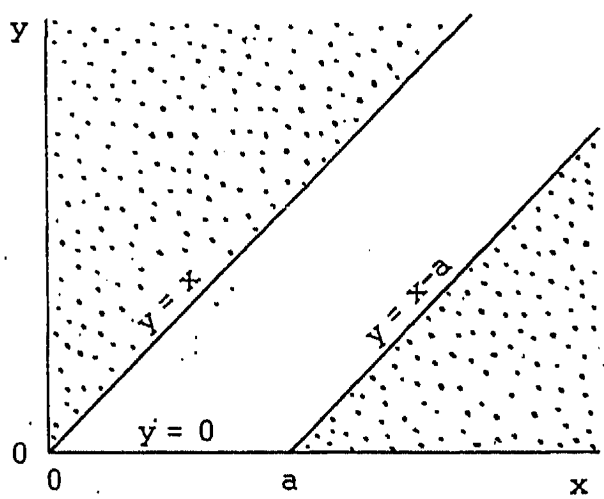


Fig. 1.--The region to which warrant prices are restricted.

---

ors considered 6.25% a "normal" return, Williams' inequality becomes: 24-1.50/.0625 $\geqslant$ y $\geqslant$ 24-42, i.e., 0 $\geqslant$ y $\geqslant$ -18, so that at best, he is asserting that the warrant is worthless. A rational investor, in his view, would sell the warrant, add $17.50 to the proceeds and buy one share of Hilton common. In this way, he argues, the investor would receive 8.6% on the additional $17.50. The $17.50 presumably came from an investment yielding 6.25%, so that on his original $24, ($6.50 in the warrant and $17.50 at 6.25% interest), this investor was receiving a yield of 4.55%. Thus, in essence, Williams contends that a rational investor will switch from his original investment to one yielding 6.25%. This is unreasonable behavior if the investor anticipated a rise in the price of the common and warrant such that his original investment would be more profitable than an investment in the common alone. Investors are not (primarily) concerned with the interest yield of an investment, they are concerned with the combined interest and capital appreciation that an investment offers. Therefore, it seems inappropriate to narrow the interval in which a warrant must sell.

Although the function is restricted to this region, it cannot be completely defined without the knowledge of some other variables. One of the variables determining this function is the length of the option period. For instance, if the length of the option period were so short that only one trade could take place with each security, then the function would coincide with the lower bound of this region.

Other variables define this function but for Chapter I they need not be specified. It will be sufficient here to consider the price of a warrant as a function of the price of its associated common stock plus a set of as yet unspecified variables, S. Because S contains, as a variable, the length of the option period, it will be convenient to denote this function as $y=Q_t(x, S)$. (At any moment in time, the price of the warrant depends upon the price of the common and the set S.) In Chapter II the set S will be examined and some models proposing the explicit form of Q_t will be constructed. For the remainder of Chapter I it will be assumed that Q_t is continuous and "sufficiently smooth."

Formation of Investors' Price Expectations

Investors (and other mortals) base their decisions partly upon expectations--at least "rational" investors do. A controversy reaching into philosophy and the foundations of mathematics exists concerning subjective expectations. It will be <u>assumed</u> in this study, without necessarily implying the validity of the doctrine of the subjective probabilists, that investors behave as <u>if</u> their expectations take the form of probability distributions.[1] Although many, including Keynes, Carnap, and Shackle, do not believe in the existence or meaningfulness of subjective probability distributions, it has been demonstrated[2] that from the acceptance of a few "plausible" axioms, one can deduce that "rational" behavior is equivalent to behavior based upon the formation of subjective probability distributions.

An example may clarify the distinction between subjective and objective probability distributions. After examining the physical attributes of a coin and observing the outcome of a number of tosses, an individual may conclude that the coin is "fair", i. e., the probability of a head (or tail) on the next toss is 1/2. This specification of a quantitative probability for each possible outcome is an objective probability distribution. On the basis of this distribution, an individual might decide how to behave when offered a wager concerning the outcome of the next toss. The intuitive significance that is attached to the distribu-

---

[1]Advocates of the view that subjective probabilities exist, i. e., that individuals behave as if they existed, are Frank Plumpton Ramsey, The Foundations of Mathematics and Other Logical Essays (New York: Harcourt Brace and Company, 1931) and Leonard J. Savage, The Foundations of Statistics (New York: John Wiley and Sons, 1954). Among those opposed to this concept are John Maynard Keynes, A Treatise on Probability (London: Macmillan and Company, 1921); G. L. S. Shackle, Expectations in Economics (Cambridge, England: University Press, 1949) and Uncertainty in Economics and Other Reflections (Cambridge, England, 1955); Rudolph Carnap, Logical Foundations of Probability (Chicago: University of Chicago Press, 1950).

[2]In addition to Ramsey and Savage, see Harry M. Markowitz, Portfolio Selection: Efficient Diversification of Investments (New York: John Wiley and Sons, 1959), Chapter xii.

tion is expressed in the so-called "law of large numbers": the frequency of any outcome will approach the probability associated with that outcome. In the case of the fair coin, the frequency of the number of heads will approach 1/2 as the number of trials is increased.

Now consider General Motors common stock. On December 31, 1975, the price of this stock could range between zero and infinity. Anyone who is certain that the price on that date will be $100 (say) per share would be viewed with suspicion. However it seems reasonable and natural for many to express their beliefs in terms of probabilities. For example, an investor might believe that the probability that General Motors will be less than $50 in 1975 is 1/5; between $50 and $100, 2/5; and over $100, 2/5. Such an assignment of probabilities to all possible outcomes is a subjective probability distribution, and an investor might very well make decisions based on this distribution. Such distributions, of course, may vary enormously among investors. Furthermore, the justification for the assignment of probabilities cannot be based upon an objective examination of previous "trials". The underlying intuitive basis for subjective probabilities, however, is similar to the intuitive expectation regarding objective probability distributions: the subjective probability of 1/5 that General Motors will be less than $50 is equivalent to the belief that given a sufficiently large number of psychologically similar (and independent) situations, the frequency with which the price will be below $50 can be made as close to 1/5 as desired.

Specifically, then, it will be assumed that each investor, for every stock within his range of interest, has a subjective probability distribution for every future time period. This probability distribution will be denoted f_t, where the subscript refers to the time period. For convenience, let t=0 represent the present, so that f_0 degenerates into the point function assigning 1 to the present price of the stock and 0 to every other price.

Decision Making; A First Approximation

Armed with the family of functions (Q_t) and (f_t), it is possible to show how a certain investor evaluates a warrant. (The notation (Q_t) represents the sequence $Q_1, Q_2, Q_3, \ldots$. In general, a subscripted variable enclosed in parenthesis will represent the sequence of variables whose subscript runs through the natural numbers.) The individual considered in this Section is one who seeks to maximize his money gains. It is conceivable that no such individual exists, for it will be shown that the desire to maximize monetary gain in the face of uncertainty leads to paradoxical behavior. Nevertheless, this simplifying assumption, which will be modified to accord with more realistic behavior in the next Section, will give rise to a special theory that will prove helpful in understanding some of the implications of the general theory. Few theories are born full-blown in all their generality. Evolving a general theory from its more restrictive antecedents is not only an heuristic device--it also reveals many ramifications that are not always apparent in the final product.

Consider first that rare individual who is certain that at time t, XYZ common will sell for x_t. His family of functions (f_t) consists solely of point functions, so that for any future time period t, he is certain the XYZ warrant will sell for $Q_t(x_t)$. Let P_t represent the point (x_t, y_t). The locus of all such points will trace a path, P, indicating his subjective estimate of the temporal relationship between the price of XYZ common and its warrant. The path P can be described parametrically:

$$\begin{cases} x = g_1(t) \\ y = g_2(t) \end{cases}$$

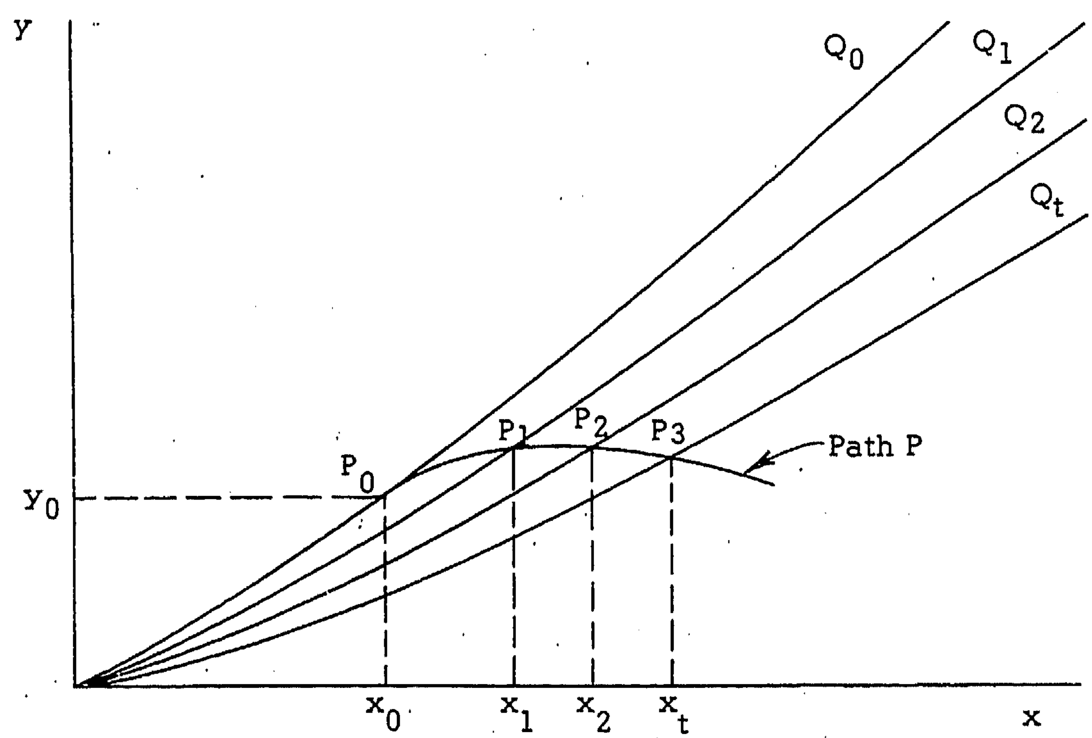


Fig. 2.--The path, P, for an individual whose family of functions (f_t) are all point functions.

Consider next the case where (f_t) is not a family of point functions. Suppose f_1, the subjective probability distribution for the price of the common at time period 1 is defined:

$$f_1(x) = \begin{cases} 1/2 \text{ for } x=20 \\ 1/4 \text{ for } x=16 \text{ and } x=24 \\ 0 \text{ for all other } x \end{cases}$$

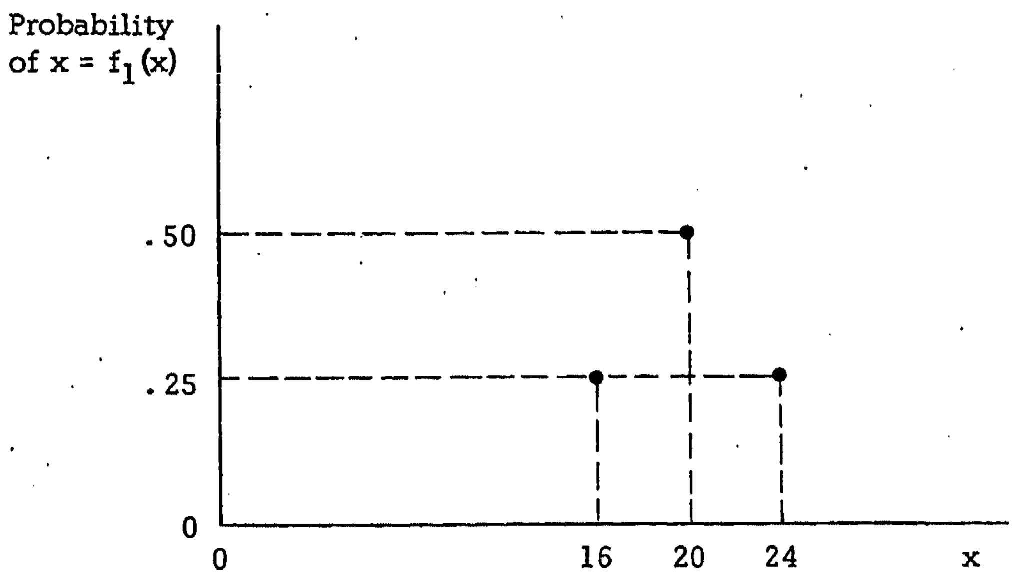


Fig. 3.--A subjective probability distribution, f_1

Fig. 3 is a graph of f_1. The mathematical expectation of x, (the expected value of x), is $1/2(20) + 1/4(16) + 1/4(24) = 20$. Thus given a sufficiently large number of psychologically similar and independent situations, i. e., subjective probability distributions, this investor believes that "on average", the price of the common stock will be 20. Since y, the price of the warrant, is a function of x, f_1 implicitly defines the mathematical expectation of y. Suppose $Q_1(16) = 7$, $Q_1(20) = 8$, and $Q_1(24) = 10$. Then the expected value of $y = \sum_x Q_1(x)\, f_1(x) = 7(1/4) + 8(1/2) + 10(1/4) = 8\text{-}1/4$. (In general, when the f_t are defined over a continuum, the expected value of $y = \int_{o}^{\infty} Q_t f_t \, dx$.) The expected value of y has the same significance as the expected value of x: over a number of similar situations, it is expected that the warrant will "on average" equal 8-1/4. Notationally, let the expected value of x in time period t be denoted u_{x_t} and the expected value of y in time period t be u_{y_t}. In Fig. 4, the path P is the locus of all points (u_{x_t}, u_{y_t}), and this graphs the subjective temporal relationship between the expected values of the common and the warrant. It is the shape of this path that determines investor behavior.

An investor interested in maximizing monetary gain is interested in comparing the percentage gains of various investments. The ratio of

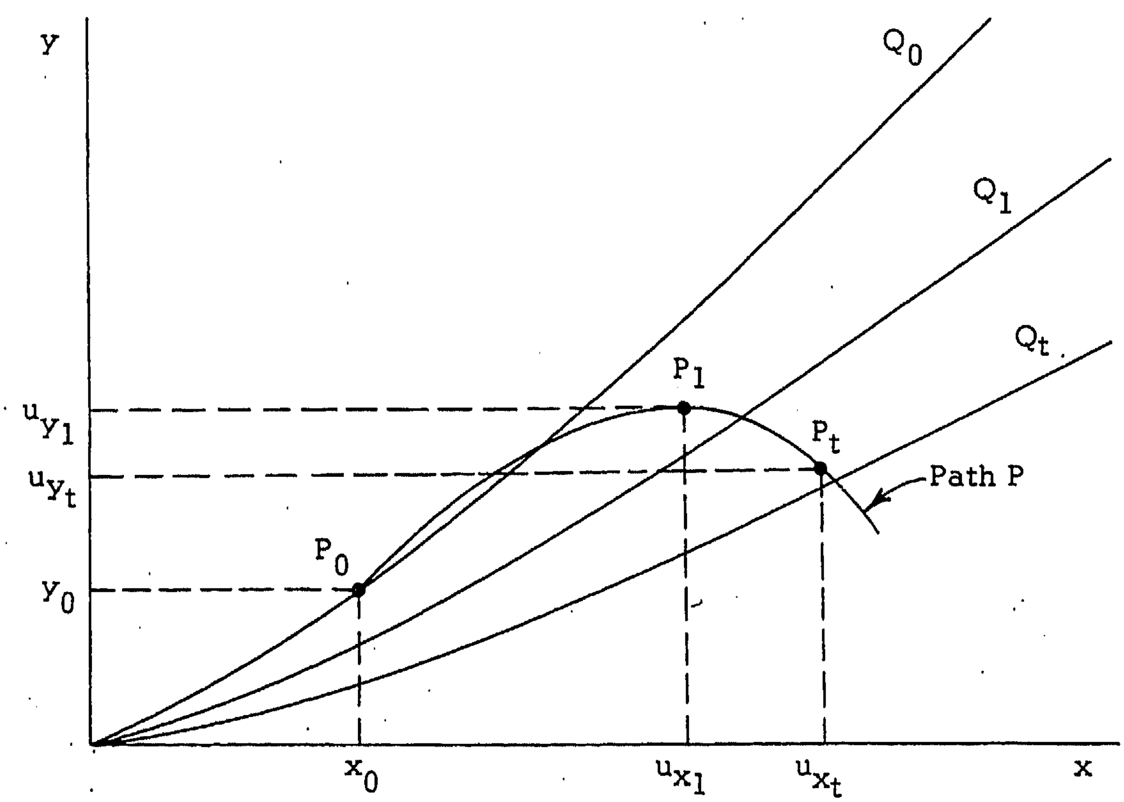


Fig. 4.--The path P of the subjective temporal relationship between the expected values of the common and the warrant.

the percentage gain of the warrant to the percentage gain of the common can be called, in accordance with the fundamental economic concept, the elasticity of the warrant. In ranking the warrant and the common, this investor will prefer the warrant if and only if the warrant's elasticity is greater than unity. This can be neatly interpreted geometrically. In Fig. 5, the path P indicates that this investor believes the expected price of the common will rise through time. The ray through the origin and P_0 describes a path that has unitary elasticity, so that if P rises above this ray, the expected value of the warrant will be appreciating faster, in percentage terms, than the expected value of the common. Now any ray through the origin has unitary elasticity, so that this investor will prefer the warrant through time period n, at which time the warrant's elasticity will be less than unity. (P_n is the point of tangency that a ray through the origin makes with the path P.) It is important

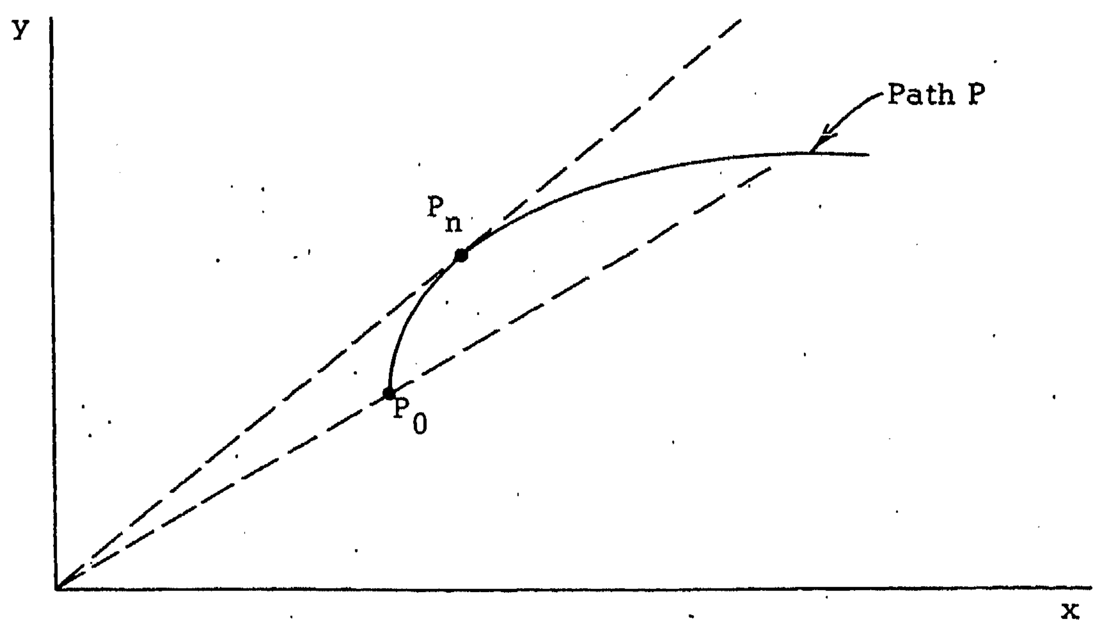


Fig. 5.--Determining the time through which a security is preferred.

to note that this investor's decision, which must be based on his ranking of available investments, will not necessarily commit him to wait until time period n before he considers the warrant inferior to the common. His ranking at time period 0 is conditional upon the invariance of his family of subjective probability distributions. If, for instance, (f_t) changes because of an unexpected outbreak of war, this investor's ranking of various investments will also change. In fact, the discrepancy between the actual market price at some future time period and his previous expectation will ordinarily cause him to change his expectations for subsequent time periods.

It is not enough that this investor prefer the warrant to the common for him to decide to purchase the warrant. The expected return of the warrant must exceed any alternative expected return available to him. If the maximum of all alternative rates available to him is r, then he will purchase the warrant only if $(dy/dt)(1/y) \geqslant \log(1+r)$. Geometrically, with the log of u_{y_t} graphed as a function of t, this investor will purchase the warrant if the slope of this function at time period 0 is greater than $\log(1+r)$, and he will conditionally plan to hold the warrant until time period m (see Fig. 6) when he will switch to the investment promising a yield of r per cent. If the path P indicated that the common was preferable to the warrant then it is not sufficient to consider u_{x_t} as a function of t--the dividend yield of the common must be taken into account. (It has been tacitly assumed that this investor is indifferent between capital gain and dividend return and that all securi-

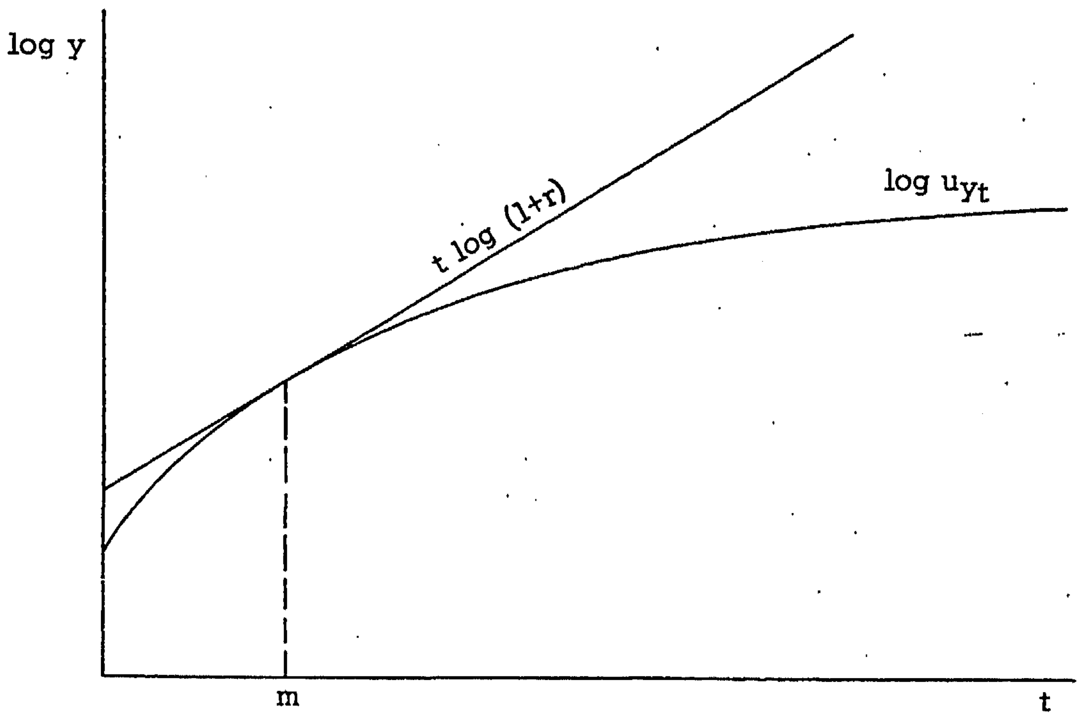


Fig. 6.--The time period through which an investment is planned to be held.

ties under consideration are perfectly liquid in that they can be bought and sold at exactly the same price.)

Assuming now that this investor wishes to maximize monetary gain, observing his behavior will yield some clues about his expectations. Consider, for instance, the case of a warrant with an option period of infinite length, i.e., a warrant with no expiration date. If this investor believes that the set S will remain constant through some time period t, then $Q_1=Q_2=Q_3=\ldots=Q_t$, so that the subscript may be dropped and $y=Q(x)$. Faced with the present prices x_0 and y_0, suppose he purchases the common. With the aid of the following lemma, the paradoxical conclusion results that this investor believes the price of the common will fall!

<u>Lemma</u>: If $Q(x)$ is monotonic increasing and convex (and anticipating the empirical findings of Chapter III this is a reasonable assumption), i. e., Q' and Q'' are both positive, then $u_{y_t} > Q(u_{x_t})$ for all x. This lemma thus proves that the expected value of the warrant must be greater than the <u>price</u> of the warrant no matter what the expected price of the common is.

<u>Proof</u>: By Taylor[1] it is known that for every x there exists an x^* interior to the interval joining x and u_{x_t} such that

$$Q(x) = Q(u_{x_t}) + Q'(u_{x_t})(x-u_{x_t}) + Q''(x^*)/2\,(x-u_{x_t})^2.$$

The last term on the right is necessarily positive, so taking expectations, $E(Q(x)) > Q(u_{x_t}) + Q'(u_{x_t})E(x-u_{x_t}) = Q(u_{x_t}) + Q'(u_{x_t})(u_{x_t}-u_{x_t}) = Q(u_{x_t})$. But $E(Q(x)) = u_{y_t}$ so $u_{y_t} > Q(u_{x_t})$. This can be seen geometrically in Fig. 7, where $Q^*(x)$ is the tangent to $Q(x)$ at u_{x_t}, for then

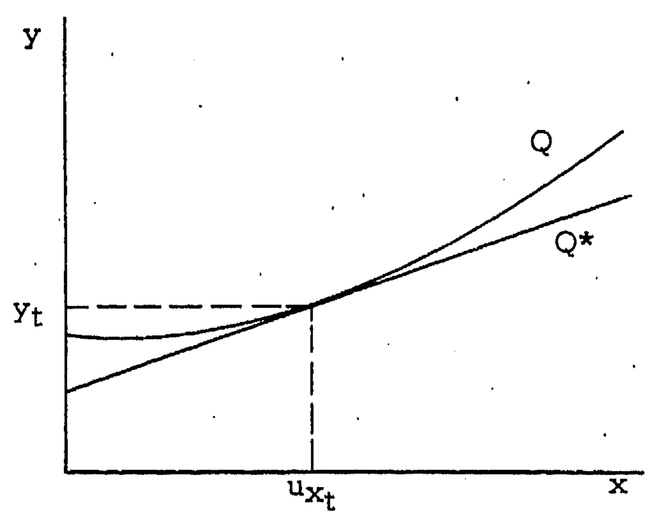


Fig. 7.--The expected value of the warrant can be greater than the price.

[1]See Tom M. Apostol, <u>Mathematical Analysis</u> (Reading, Mass.: Addison-Wesley Publishing Company, 1957), 96-97.

$u_{y_t} = E(Q(x)) = \int Q(x)f(x)dx > \int Q^*(x)f(x)dx = Q^*(u_{x_t}) = Q(u_{x_t})$. The first equality to the right of the inequality sign follows from the fact that the expectation of a linear function of a random variable is equal to the linear function of the expectation of the random variable.[1]

Since the path P must lie below the ray through the origin and P_0 for this investor to have chosen the common, (see Fig. 8), this path must lie to the left of x_0 through time period t, for only then can it lie above Q(x), as required by the lemma, and below the unitary ray. But such a path indicates an expectation of lower prices for the common through period t. A decision to purchase the common, given all the assumptions of this Section, then cannot be "rational", for there exists an alternative that will yield greater monetary gain, i. e., less loss, viz., the abstention of any purchases (or even selling the common short).

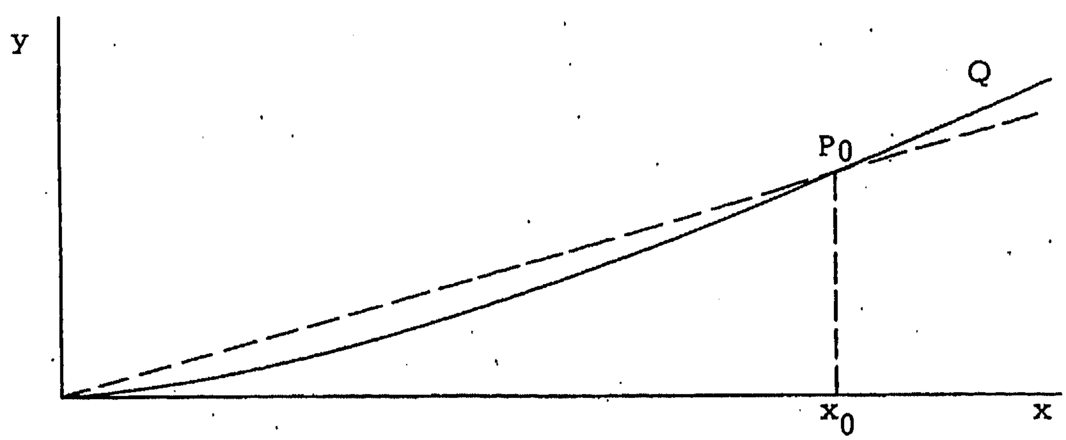


Fig. 8.--The expected value of the warrant greater than the price.

---

[1]See Arthur S. Goldberger, Econometric Theory (New York: John Wiley and Sons, 1964), 85.

The conditions imposed upon the warrant were not unrealistic--three famous warrants satisfy the requirements of the preceding example, viz., the Atlas, Alleghany, and Tri-Continental warrants. Thus for the purchasers of these common stocks it is necessary to conclude that they were not attempting to maximize money gain. Before leaving the assumptions of this Section it may be interesting to point out a corollary of the above lemma: given the warrant described above, an investor whose expectations for the common are neutral, i.e., his expected price for the common through some time period t is equal to its present price, cannot be neutral toward the warrant, for its expected value must be greater than its present price.

The Subjective Value of Money

The paradox of the preceding Section suggests that investors do not invariably attempt to maximize money gains (or that Q is not convex). And there are additional reasons for discarding this assumption. Daniel Bernoulli was perhaps the first to question the "principle of mathematical expectation" and his celebrated St. Petersburg paradox convinced virtually everyone that maximizing monetary gain is not the *sine qua non* for rational decision-making.[1] More appropriate evidence with regard to the subject of this paper is the observed fact that investors almost always diversify their commitments: if an investor attempted

[1]In addition to Durand, see R. Duncan Luce and Howard Raiffa, *Games and Decisions: Introduction and Critical Survey* (New York: John Wiley and Sons, 1957), Chapter ii.

to maximize monetary gain, he would rank all possible alternatives and then plunge his entire stock of capital into the investment with the highest expected return. Except for those with relatively small amounts of capital, this behavior is rarely observable, so again it seems necessary to change the assumption that investors seek to maximize their monetary gain.

It will now be assumed that "rational" behavior is that which accords with the Expected Utility Hypothesis. This assumption is closely linked to the assumption concerning the formation of subjective probability distributions and is just as controversial. Nevertheless, since first Ramsey and then von Neumann and Morgenstern[1] have shown that this hypothesis can be deduced from a few "reasonable" axioms, and since no alternative intuitively appealing hypothesis has been proposed, it has become widely accepted.

Briefly, this hypothesis asserts that individuals behave as if they attach a subjective cardinal value, called utility, to different amounts of money income and that they attempt to maximize their <u>subjective</u> value of money, or utility, not the cash value. A rule which attaches cardinal values to money income is called a <u>utility function</u>. This function, when consistent with the axioms, is unique to within a linear transformation. This concept of utility carries none of the stigma

---

[1] John von Neumann and Oskar Morgenstern, <u>Theory of Games and Economic Behavior</u> (3rd ed.; Princeton: Princeton University Press, 1954). A slightly different axiomatization can be found in Savage, Markowitz, and Luce and Raiffa.

attached to the classical marginal utility theory. The older notion is meaningless in the sense that any monotonic transformation of a "utility" function is again a "utility" function so that defining it cardinally is superfluous and leads to the erroneous implication that "utility" can be interpersonally compared. The newer concept states only that if an individual's behavior is consistent with a few axioms, he will behave as _if_ he were attempting to maximize his expected utility--there is no implication that interpersonal comparisons are possible. Furthermore, behavior will be identical if the expected utility of any linear transformation of his utility function is maximized.

As an example of a utility function consider:

$$\text{Utility} = \log(x + 1), \quad x = \text{monetary gain.}$$

Consider now an investor with such a utility function who is faced with the following alternatives: (1) a 50% probability of \$10 gain and a 50% probability of \$20 gain, (2) a 50% probability of \$30 gain and a 50% probability of 0 gain.

His mathematical expectation in either case is \$15, but his expected utility in case (1) is $\frac{1}{2}(\log(10+1)) + \frac{1}{2}(\log(20+1)) = 1.1818$, and in case (2), $\frac{1}{2}(\log(0+1)) + \frac{1}{2}(\log(30+1)) = 0.7457$, so that he will prefer case (1) to case (2). (The probabilities in these alternatives may be either objective or subjective.)

It can now be shown that the existence of a non-linear utility function would account for portfolio diversification. (A linear utility function is equivalent to a preference for maximizing money gain.)

When confronted with various "lotteries," i.e., probability distributions, an investor will choose that distribution which maximizes his utility. His utility function, then, completely defines his attitude toward risk. To say that something has been learned about his risk-attitude, therefore, is to say that something has been learned about his utility function.

Decision Making; An Optimum Procedure

An investor whose subjective price expectations are formed quantitatively with probability distributions and whose behavior is consistent with the Expected Utility Hypothesis is seeking that distribution which will maximize his utility. A subjective expectation is linked to every security that interests him. It is important to note that a mixture, or portfolio, of these securities will give rise to a possibly different distribution than associated with any of the individual securities. Thus an investor is seeking that portfolio which maximizes his utility.[1]

As an example, consider this relatively simple situation: an investor whose utility function is $\log(x+1)$ is faced with a "universe" consisting of only Security A and Security B. Assume that he cannot borrow as much as he desires, so that his

---

[1]Markowitz has shown, in Portfolio Selection, that if an investor's utility function can be approximated by a quadratic function, then this investor will choose an "efficient" portfolio, i.e., there will be no other portfolio with the same expected monetary return whose probability distribution has a smaller variance. By assuming a quadratic utility function, he shows how it is operationally simpler to choose portfolios than it would be in the general case.

"budget" is finite. Suppose his expectations and his expected utility for each security can be tabulated:

	Profit Outcome	Probability	Expected Utility
Security A	0 +30	$\frac{3}{4}$ $\frac{1}{4}$	$\frac{1}{4}\log(31)=0.3729$
Security B	0 +5	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}\log(6)=0.3891$

If he were forced to make a choice between these two securities, he would choose Security B. But ordinarily he is not so constrained: he may invest a fraction of his capital in each security, or perhaps just one security leaving the remainder in cash. In this instance, since he does not believe there is any possibility of loss, he will not wish to hold cash, but he must decide how to apportion his capital. Let p be the proportion of his capital invested in Security A, so that 1-p is the proportion invested in Security B. If it is further assumed that the outcomes are independent, then a probability distribution of a portfolio can be tabulated:

Profit Outcome	Probability	Expected Utility of Portfolio
$p(0)+(1-p)(0)=0$	$\frac{3}{8}$	$\frac{3}{8}(U(0))+\frac{3}{8}(U(5-5p))+$
$p(0)+(1-p)(5)$	$\frac{3}{8}$	$\frac{1}{8}(U(30p))+\frac{1}{8}(U(25p+5))=$
$p(30)+(1-p)(0)$	$\frac{1}{8}$	$\frac{3}{8}\log(6-5p)+\frac{1}{8}\log(30p+1)+$
$p(30)+(1-p)(5)$	$\frac{1}{8}$	$\frac{1}{8}\log(25p+6)$

In order to maximize expected utility, set $dE(U)/dp =$

$$-15/8(6-5p) + 30/8(30p+1) + 25/8(25p+6) = -3750p^2 + 980p + 228 = 0,$$

so that p is approximately 0.4. This investor would thus invest 40% of his capital in Security A and 60% in Security B and his probability distribution and expected utility would be:

Profit Outcome	Probability	Expected Utility of Portfolio
0	$\frac{3}{8}$	$\frac{3}{8}\log 4 + \frac{1}{8}\log 13 + \frac{1}{8}\log 16 =$ 0.5155
3	$\frac{3}{8}$	
12	$\frac{1}{8}$	
15	$\frac{1}{8}$	

By considering portfolio investments the range of investment alternatives has been enlarged. By allowing short sales, opportunities are increasingly enriched, and not only in the obvious way. (A short sale allows an investor to profit from a fall in price in precisely the same way he can profit from a rise in price, so that by considering the totality of expected price changes, not merely price gains, an investor's universe is expanded.) To illustrate the subtle range of opportunity open to an investor able to engage in short sales[1], consider the uni-

---

[1]A self-imposed restriction prohibits most regulated investment companies from engaging in short sales. Presumably this provision in their charter is designed to bolster confidence in their operations because many investors consider short-selling a speculative and dangerous device. Section 12 (a) (3) of the Investment Company Act of 1940 states in part: "It shall be unlawful for any registered investment company in contravention of such rules or regulations or orders as the Securities and Exchange Commission may prescribe as necessary or appropriate in the public interest or for the protection of investors--to effect a short sale of any security. . . ." The Commission has not to date issued any rules, regulations, or orders of general application

verse consisting of only one common stock and its associated warrant. By buying various amounts of common and warrants, an investor will have a large number of possible portfolios. But by buying common and shorting warrants, the set of possibilities is extended and it is very likely that in this extension lies a portfolio that maximizes expected utility beyond the supremum of the subset of portfolios consisting only of purchases.

For concreteness, consider how an investor whose utility function is log(x+1) chooses a portfolio from a universe consisting only of the ABC common and the ABC warrant. The present price of the common is $10, and the warrant, exercisable at $20, can be bought or sold for $5. This investor's capital is $1500 and he wishes to select that portfolio which will maximize his utility at time t (say), when the warrant expires. His price expectations can be summarized:

$$f_t(x) = \begin{cases} \frac{1}{4} \text{ for } x=10 \\ \frac{1}{4} \text{ for } x=15 \\ \frac{1}{4} \text{ for } x=20 \\ \frac{1}{4} \text{ for } x=25 \\ 0 \text{ otherwise} \end{cases} \quad \text{so that} \quad f_t^*(y) = \begin{cases} \frac{3}{4} \text{ for } y=0 \\ \frac{1}{4} \text{ for } y=5 \\ 0 \text{ otherwise} \end{cases}$$

where f* is his subjective expectation for the price of the warrant, which follows directly from f since $Q_t(x) = y_t = x_t - 20$.

Assume first that this investor cannot sell short. Then he must decide how to apportion his $1500 among common stock, warrants, and

under this section. Most fiduciaries also refrain from short selling since the courts may not consider this a tool for "prudent" men. Thus private individuals account for most of the short selling and the extended set of portfolio possibilities is open only to them.

cash. Since his expectations are that the warrant will not be above 5 at time t, he will not purchase any warrants for at best he will show no profit. On the other hand, since he believes that the common will not be below its present price at that time, he will not wish to hold cash for the worst that can happen (in his view) if he holds only common is that he will show no profit. (For convenience it is now assumed that the rate of interest is zero.) Thus he will purchase 150 shares of common and the expected possible outcomes, their probability, and his expected utility become:

Profit Outcome	Probability	Expected Utility of Portfolio
\$2250	$\frac{1}{4}$	$\frac{1}{4}\log 751+\frac{1}{4}\log 1501+$
1500	$\frac{1}{4}$	$\frac{1}{4}\log 2251 = 2.3511$
750	$\frac{1}{4}$	
0	$\frac{1}{4}$	

Assume now that this investor is able to sell short so that he may apportion his \$1500 between buying common and shorting warrants. Let p be the proportion of his capital devoted to purchasing common and 1-p the proportion devoted to shorting warrants. His expected profit at time t is a function of the price of the common at time t:

$$\text{Profit} = \begin{cases} (1-p)1500 + 150p(x-10), & \text{when } x \leq 20, \\ (1-p)300(5-x+20) + 150p(x-10), & \text{when } x > 20, \text{ i.e.,} \end{cases}$$

$$\begin{cases} 1500 - 3000p + 150px, & \text{when } x \leq 20, \\ 7500 - 9000p + 450px - 300x, & \text{when } x > 20. \end{cases}$$

Given f_t, the possible profit outcomes, their probability and his expected utility become:

Profit Outcome	Probability	Expected Utility of Portfolio
(x=10) 1500-1500p	$\frac{1}{4}$	$\frac{1}{4}\log(1501-1500p)+$
(x=15) 1500-750p	$\frac{1}{4}$	$\frac{1}{4}\log(1501-750p)+\frac{1}{4}\log 1500$
(x=20) 1500	$\frac{1}{4}$	
(x=25) 2250p	$\frac{1}{4}$	

Therefore, by setting $dE(U)/dp = 0$, it can be seen that his utility will be maximized when p is approximately 0.42, so that this investor will purchase 63 shares of common and sell short 174 warrants. With this portfolio the possible outcomes, their probability and the expected utility of the portfolio become:

Profit Outcome	Probability	Expected Utility of Portfolio
(x=10) 870	$\frac{1}{4}$	$\frac{1}{4}\log 871+\frac{1}{4}\log 1186+$
(x=15) 1185	$\frac{1}{4}$	$\frac{1}{4}\log 1501+\frac{1}{4}\log 946=$
(x=20) 1500	$\frac{1}{4}$	3.0416
(x=25) 945	$\frac{1}{4}$	

Shorting warrants against purchases of common resembles an arbitrage operation--instead of a guaranteed profit regardless of the future course of prices, an investor who "hedges" by selling warrants and purchasing common usually extends the range in which future prices may profitably lie. This extension, coupled with a suitable subjective probability distribution, may yield a larger expected utility than otherwise possible. Warrants and other convertible securities appeal to many investors for precisely this reason. In the above example, by "hedging", this investor was able to increase his expected utility from

2.3511 to 3.0416.

In the above example the expected utility of a specific portfolio for time t was calculated with the help of the investor's utility function and his subjective expectations. This algorithm will produce his expected utility for any future time period, so that associated with any portfolio available to him is a function of time that relates expected utility with future time periods.

Fig. 9 graphs the relation between expected utility and time for three hypothetical portfolios (Portfolio 1 consists of 100% cash). Faced with these portfolio functions, the idealized investor of this Section will choose that one which maximizes dE(U)/dt at time period 0, i. e., that one which has maximum slope at the origin.

Fig. 9 graphs the portfolios available to this investor at time period 0 only, so that having made his choice at that time, he will not be committed to remain with the same or even another portfolio pictured there for any length of time. With the passage of time, other alternatives will present themselves depending upon the actual outcome of previous investment decisions--if he experiences a large gain by time

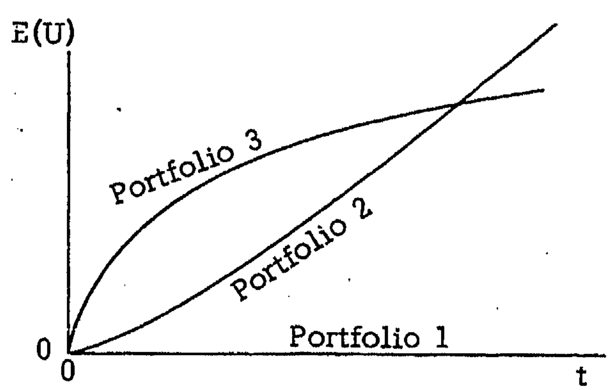


Fig. 9.--Expected utility from three portfolios.

period m (say), a number of possible portfolios will then exist that were not possible at time 0. Similarly, a large loss may restrict the number of choices open to him. It is convenient to describe his decision-making process at discrete time intervals: each business day at 9:45 A.M. New York time, this investor graphs the relation between expected utility and time for every portfolio available to him and selects that portfolio with greatest slope at the origin.

Before leaving this idealized investor, it is interesting to note that his choice process is general enough to cope with different utility functions for different time periods. For instance, an investor aged 25 may project to the time when he will be 65, a different utility function than his present one. Whether or not his actual function at that time will agree with this projection is immaterial to his present decision which must be based upon the best available evidence he has <u>now</u>.

Individual and Aggregate Demand Curves; The Equilibrium Price of a Warrant

A rational investor, by the assumptions of this Chapter, will select that portfolio which maximizes his expected utility. It is relatively easy to see how his demand curve for a warrant (or any security) is defined. Consider the A warrant: holding all other security prices constant, varying the price of the A warrant will result in a varying set of possible portfolios. If the A warrant can be purchased or sold for $5, the portfolio that maximizes his utility may include 100 warrants, so that at $5 this investor will demand 100 warrants. At a price of $10, the optimum portfolio may be short 100 A warrants, so that at $10, this investor will supply 100 warrants. Thus for each price of the A warrant is associated an optimum portfolio which indicates his supply or demand for the warrant. If the positive half of the horizontal axis denotes his demand and the negative half his supply, and if the vertical axis denotes the price of the A warrant, his supply-demand function will usually be represented by a negatively sloped line. Where it intersects the vertical axis, he is indifferent to the warrant, i. e., he neither demands nor supplies it at this price. By horizontally adding all individual demand curves for warrant A, an aggregate demand curve is obtained, and where it intersects the vertical axis denotes the equilibrium price for the A warrant, for only then will demand equal supply.

It may be interesting to note that the supply of a security is not necessarily fixed. When an investor sells shares short, he has actually created new shares, and theoretically, there is no limit to the number of such shares that may be created. In March 1963, 176,000 Molybdenum Corporation warrants were outstanding, but there was a short position of 101,000 warrants[1], so that in effect, there existed 277,000 warrants. Usually, however, the short position in a security is a small fraction of the original issue, so that the supply is virtually fixed. The reasons cited in the footnote, page 26, probably account for this.

---

[1]American Stock Exchange, Monthly Short Interest Report (New York: April, 1963).

Inferences Concerning Utility and Expectations

It has been conjectured that prevailing market prices imply some information about investors' expectations and risk-attitudes.[1] Unfortunately, the implication is tenuous and cannot be drawn in the manner suggested. Consider an investor whose universe consists of one common stock and its associated perpetual warrant. If he purchases the warrant, what can be inferred about his risk-attitude and expectation? His behavior could be the result of (1) an indifference to risk, i.e., a linear utility function that dictates he attempt to maximize money gain, and an expectation that the common will fall in price through some time period, or (2) a convex utility function and an expectation that the common will rise through time. But (1) and (2) are completely opposed.

A numerical example will illustrate these possibilities. Let the present prices of the common and the warrant be \$10 and \$5 respectively, and $Q(x)$, constant through time, with $Q(5) = 2.5$, $Q(10) = 5$, and $Q(15) = 10$. Case (1): an investor indifferent to risk, expects the common in time period t to be \$5 with probability 0.35, \$10 with probability 0.40, and \$15 with probability 0.25. His expected value for the common is thus \$9.50, or less than the current price. His expected value for the warrant is $0.35(2.5) + 0.40(5) + 0.25(10) = 5.375$, or more than the current price. Even though he believes the price of the com-

[1] Case M. Sprenkle, "Warrant Prices as Indicators of Expectations and Preferences," Yale Economic Essays, I (Fall, 1961), 179-231 and A. James Boness, "Elements of a Theory of Stock Option Value," Journal of Political Economy, LXXII (April, 1964), 163-75.

mon will "on average" be less than its current price, he believes that the price of the warrant, "on average", will be greater than the present price. Seeking to maximize monetary gain, he will purchase the warrant. Case (2): an investor's utility function is z^3, where z is the percentage gain on an investment. His function Q is the same as in the previous Case, but his expectation for the common in time period t is that the price will be \$5 with probability of 0.25, \$10 with probability of 0.40, and \$15 with probability of 0.35. His expected value for the common is \$10.50, or more than the current price. His expected utility from an investment in the common is $0.25(-50)^3 + 0.35(50)^3 =$ 12,500, whereas his expected utility from an investment in the warrant is $0.25(-50)^3 + 0.35(100)^3 = 968{,}750$. Seeking to maximize his utility, this investor will also purchase the warrant.

Thus the observed behavior of the purchase of the warrant could be the result of a higher or lower expectation for the common and a linear or non-linear utility function. Without further knowledge or assumptions, little can be inferred.

Sprenkle[1] optimistically believed that by imposing some conditions on the <u>form</u> of investors' subjective probability distributions, viz., that they are lognormal, he would be able to infer investor expectations, risk-attitudes, and "time horizons". His results were disappointing:

> It was originally intended to estimate [risk-attitudes, expectations, and the standard deviation of the future subjective probability dis-

[1]Ibid., 179-182.

> tribution,] but it was found impossible to obtain these estimates . . . . The next step was to assume the value of one of the parameters and iterate for the values of the other two . . . but the results from using this method showed no improvement over the method which attempts to estimate all three parameters. Because of this it was necessary to assume values of two of the parameters and iterate for the value of the third.[1]

Even if Sprenkle's calculations had yielded "results", they would have been false and misleading. His theory of warrant prices embraces an awkward measure of risk-attitude[2] which has no apparent relation to a utility function.[3] But the fatal error in his theory is the assumption that Q is invariant through time <u>and equal to the minimum value of the region in Fig. 1.</u> The bias of this unfortunate assumption is compounded by his

> second major consideration . . . that . . . the warrants used [for empirical testing] must have expiration dates which are in the quite distant future . . . . This consideration alone rules out the use of all but a few warrants.[4]

But it is precisely those warrants that he chooses to study for which Q <u>cannot</u> be equal to x-a.

Boness[5] developed a theory of price for call options and adroitly

---

[1]Sprenkle, 212.

[2]Ibid., 199.

[3]Markowitz has justified his concept of "efficient" portfolios and has shown their connection to utility functions. Since Sprenkle does not give explicit expression to the variance, the connection between his measure of risk and utility functions is not clear. See Sprenkle, 199.

[4]Sprenkle, 205.

[5]Boness, 163.

avoided Sprenkle's error by considering only relatively short-term options (usually of no more than one year duration), so that the assumption that Q = x-a is not too biased. (He claims that his theory can be applied to warrants, convertible bonds, and other convertible securities without expressly mentioning the difficulty that would be encountered, viz., that the family of functions (Q_t) is not identical to the function Q = x-a.)[1] He also assumes that expectations are formed with lognormal distributions, and further, that investors are risk-indifferent. Aside from this questionable latter assumption, the major shortcoming of his theory is that no provision is made for the possibility that an investor's "time horizon" may not be equal to the length of the option period, i. e., the purchaser of the option may be anticipating the sale of the option before it expires, not intending to exercise it. This possibility then requires the knowledge of (Q_t), which will be distinct from x-a, except for the last member of the family.

Summary

The major assumptions of this Chapter are: (A1) Anyone who buys and sells securities attempts to maximize his expected utility and his expected utility is calculated with the aid of subjective probability distributions. (A2) There exists a warrant-common price relationship that is continuous and possesses derivatives of the second order. With these assumptions, the procedure for determining an individual's de-

[1]Boness, 163.

mand-supply curve for a security was detailed using a partial equilibrium analysis. Aggregating individual curves yielded the market equilibrium price. Because a convertible security is closely linked to its associated common stock, the decision-making procedure was examined to see if observed investor behavior implied anything about risk-attitudes and price expectations. Unfortunately, unless some further (questionable) assumptions are made, it was shown that little can be inferred.

This Chapter has developed a theory of price for warrants, i. e., it has explained how the market equilibrium price is attained. It was not necessary to know the exact warrant-common price relationship. The remainder of this study will attempt to discover the form and parameters of this relationship using statistical inference and it will be seen that inferences about expectations and risk-attitudes may be made on firmer ground.

CHAPTER II

A PRIORI CONSIDERATIONS FOR AN ECONOMETRIC MODEL FOR WARRANT PRICES

Purpose of the Model

Chapter I revealed that the equilibrium price for a warrant can be determined by aggregating individual demand curves; and an individual demand curve is determined by price-expectations for the common, risk-attitudes, and an estimate of the family of functions defining the relation between the price of the common and its associated warrant through future time periods. Expectations and risk-attitudes may vary enormously among investors, whereas estimates of (Q_T), the true warrant-common price relationship, will probably not vary greatly among investors. (The capital subscript will refer to the true relationship, as opposed to the lower-case subscript which refers to an individual's estimate of the true function.) For instance, a substantial number of investors may believe that Sperry Rand common will be double its present price one year from today while a similar number may believe that it will be one-half its present price. But if asked what will the price of the warrant be, given the price of the common one year from today, opinions will probably not vary as greatly. In fact, if certain other information is given, viz., the elements of S, it is conceivable that estimates of the price of the warrant will range over a relatively narrow interval.

Chapter II and III will attempt to provide an objective estimate of Q_T, given the price of the common and certain other variables. That is, an attempt will be made to explicate the precise relationship between the price of a warrant and a set of "explanatory" variables by means of statistical inference.

Expectations and Risk-Attitudes as Determinants of Q_T

The present price of a warrant is a function of (E_i), (R_i), and (Q_i), the totality of investors' expectations, risk-attitudes, and estimates for the warrant-common price relationship, respectively. (The index i runs through the total number of investors interested in the warrant.) It was assumed that Q_t, for any individual, was a function of x and S. Since the task is now to specify Q_T, the true relationship, it is necessary to lift the lid on the S box.

It is interesting that (E_i) and (R_i) are also determinants of Q_T. For then Q_i, investor i's estimate of Q_T, is a function of, among other things, investor i's estimate of the entire families (E_i) and (R_i), i.e., his estimates of everyone else's expectations and risk-attitudes. This hauntingly resembles Keynes' description of the professional investor "anticipating what average opinion expects average opinion to be."

To see why (E_i) is an element of S, consider two common stocks, A and B, equal in current price. Assume that each company wishes to issue warrants and that the warrants will be identical with respect to exercise terms. If warrant investors, en masse, had different expectations for A and B, the present price of the A warrant could be different from the present price of the B warrant; if, for instance, the expected value of the A warrant is greater than that for the B warrant, aggregate demand for the A warrant could exceed the demand for the B warrant resulting in a higher price for the A warrant.

On the other hand, if warrant purchasers "on average" had the same expectations for A and B stock, the present price of the warrants would be the same. (This is not to say that the average of all warrant purhcasers' expectations is equal for both stocks--it is only to say that the effect of aggregate expectation is the same for both warrants. The aggregate demand curve cannot be derived by first aggregating expectations and risk-attitudes--it must be derived by summing individual demand curves.)

Let the effect that the totality of investors' expectations exerts on the price of the warrant be denoted by E. Then E is an element of S. It might seem reasonable that, for a warrant with a large audience, the cross-currents in expectations may tend to randomize the total effect, in which case the variable could be ignored in an econometric model by relegating it to the domain of the stochastic variable. Nevertheless, in the sequel a variable will be introduced which will proxy for E.

Similarly, if warrant investors en masse had the same risk-attitude at one period of time, and a different risk-attitude at another, all else equal, the price of a warrant could differ in both periods. Thus R, the effect that the totality of investors' utility functions exerts on the price of a warrant, will also be an element of S. At any moment in time, R is a constant, so that if at different time periods the parameters of the true relation Q_T are significantly different, the difference might be attributed to a change in R. For instance, near the peak of a stock market cycle utility functions in general may be characterized by less concave (or even convex) curves than at a period near the trough, i. e., investors in general may be more averse to risk-taking at the trough than at the peak. So by studying the parameters of the true relation at different periods of time, inferences may be drawn about R. This possibility is discussed in Chapter III.

The Exercise Price, a

For notational convenience, let y* and x* represent the prices of a warrant and associated common stock, respectively. (In Chapter I, y and x were used.) The exercise price of the warrant clearly affects the price of a warrant: <u>ceterus paribus</u>, the lower the exercise price, a, the more valuable the warrant. But the dependence of y* on a is illusory if it is assumed that

(A.3) The function $y^* = Q(x^*, a, E, R, \ldots)$ is homogeneous of degree one with respect to x* and a, i. e., if x* and a are multiplied by any constant, k, then $Q(kx^*, ka, E, R, \ldots) = ky^*$.

As an intuitive justification for this assumption, consider this situation: ABC common is at \$20 and its warrant, exercisable at \$20, is selling at $10. If the common is split 2 for 1, its price per share will become $10, and if the warrant is protected against dilution, it will become the right to purchase 2 shares of the split stock for a total cash outlay of $20, i.e., the adjusted exercise price is $10, or half the original exercise price. Now the stock split has not changed anything in real terms and (A3) asserts that investors in the warrant market will behave as they did prior to the split, i.e., the warrant will still sell for $10, or at an adjusted price of $5. Thus a proportional change in the exercise price and the price of the common will result in the same proportional change in the price of the warrant. This assumption is thus equivalent to asserting that money illusion does not exist in the warrant market.

By "normalizing" the price of the warrant and the common, i.e., by dividing these prices by a, the exercise price will no longer be a determinant of the normalized warrant price. Let $y = y^*/a$, $x = x^*/a$. Then $y = Q(x, E, R, \dots)$ and a is no longer an element of S.

By considering only normalized prices, different warrant prices in any sample become comparable. Common prices that are at equal percentages above or below their exercise price are considered equal and their normalized warrant prices can be compared. This is again tantamount to assuming no money illusion. If, for instance, ABC stock is at $40, and its warrant, exercisable at $40, is selling at $20, then ceteris paribus, if XYZ stock is at $20 and has a warrant exercisable at $20, the XYZ warrant may be expected to sell for $10, or half the exercise price.

This normalizing procedure seems more reasonable than subtracting the exercise price from the unadjusted prices. In the immediately preceding example, if subtraction instead of division were used to adjust prices, both stocks would be at an adjusted price of 0, but the ABC warrant would be at an adjusted price of -20, and the XYZ warrant at -10, implying that the XYZ warrant is more valuable. Note also that with subtraction, adjusted prices can take negative values, but with division, adjusted prices are restricted to positive values.

For the remainder of Chapter II all prices will be understood to be normalized by division, so that "price" is now a pure number.

The Length of the Option Period, t

It is clear that the longer the option period the more valuable the warrant--a warrant that expires tomorrow must be worth less than a warrant that expires next year, all else equal. If the option period is so short, for instance, that only one transaction can take place with the warrant before it expires, then the warrant will be worth the difference between the price of the common and the exercise price, if the common is greater than the exercise price, and zero otherwise. In general, it is reasonable to expect that the price of the warrant will be a monotonically increasing function of the length of the option period, i.e., y increases with t.

The Dividend Yield on the Common, r

In choosing between a warrant and its associated common stock, an investor is attempting to maximize his return (utility, or in an extreme case, his monetary gain). The dividend yield of the common will add to the return promised by the stock, so that the higher this yield, all else equal, the higher the value of the stock <u>relative</u> to the value of the warrant. In other words, the higher the dividend yield, the less valuable the warrant, so it is reasonable to expect that y will be a monotonically decreasing function of r, i.e., y will decrease for increasing r.

The Potential Dilution, d

The number of outstanding warrants, relative to the number of common shares issued, is also a variable that affects the value of a warrant. Outstanding warrants represent potential dilution of ownership for the present stockholders. Consider this example: just prior to expiration, with <u>one</u> outstanding warrant exercisable at \$20, the common is at \$30. There are 100 outstanding shares of common. The warrant will be worth about \$10 since it may purchase a share of common, worth \$30, for \$20. But now assume there are <u>100</u> warrants outstanding. Everything else equal, the price of the common will be less than \$30, for after conversion, an additional 100 shares of common will be issued for a consideration (\$20) less than the current price of the stock. With only one warrant outstanding, the corporation may (naively) be valued at about \$30 X 100 = \$3000. But with the 100 warrants outstanding, after conversion, there will be <u>200</u> shares outstanding, and only \$2000 (the sum of the exercise price of the warrants) is "added" to the value of the corporation, so that a naive first approximation will value the corporation at \$5000, or \$25 per share. Anticipating this dilution, in the case of 100 outstanding warrants, investors will be willing to pay only \$25, not \$30 per share. Thus the warrant will only be worth \$25-\$20 = \$5, not \$10. (Of course this method of valuation is crude for consideration was not given to the changed earning power of the corporation resulting from the \$2000 received from the exercise of the warrants.) Other things equal, therefore, the greater the number of outstanding warrants, the less the potential price of the common and hence of the warrant. Let the number of outstanding warrants as a ratio of the number of outstanding shares of common stock be d. (For instance, with 100 shares and 100 warrants outstanding, this ratio would equal 1.) Then it is reasonable to expect that y will be a decreasing monotonic function of d, i.e., y decreases for increasing d.[1]

The Form for a Model

Leaning on the foregoing, it will now be assumed that:

(A4) $y = Q(x, t, d, r, E, R, e)$, where e is a stochastic variable.

Having specified the determinants of the warrant price, it is now

---

[1]For a specific example of the effect of dilution, see Benjamin Graham and David L. Dodd, Security Analysis (3rd ed.; New York: McGraw-Hill Book Co., 1951), 572. Their discussion indicates strong disapproval of warrants as a method of financing. Their hostility erupts, 570, where they reflect: "Beginning about 1925 their possibilities--mainly for evil, we are certain--began to be exploited." Their view undoubtedly affected fiduciaries.

necessary to specify the form of the functional relationship. Fortunately, from the empirical data of the last twenty years, it is possible to select examples such that many of the variables are constant through a period of time. For example, consider the Electric Power and Light Corporation warrant. This was a perpetual warrant (it is not unusual for a "perpetual" warrant to expire due to the demise of the corporation or a reorganization) whose common stock yielded no dividend from January 1945 through its expiration in July 1949. Furthermore, d was also constant through this period, with the common ranging from a low of $3\frac{7}{8}$ to a high of $29\frac{1}{2}$. Fig. 10 plots the relation between the price of the common and the warrant on both arithmetic and log-log grids. These graphs suggest that the relation between the log of prices is linear, i. e., $\log y = k_1 + k_2 \log x + f_3(t, d, r, E, R) + e$.

Unfortunately, the data do not contain examples with subsets of other variables constant. Therefore, the form of f_3 will have to be hypothesized without the aid of empirical evidence. For a classical least squares regression analysis, it would have to be assumed that the function is, or can be well approximated by a function that is, linear in its parameters, as for instance:

$$\log y = k_1 + k_2 \log x + k_3 f_3(t) + k_4 f_4(d) + k_5 f_5(r) + k_6 f_6(E) + k_7 f_7(R) + e.$$

But even this is insufficient, for since the function must be restricted, a priori, to the unshaded region of Fig. 1, the above form will not allow the <u>slope</u> of the partial with respect to log x to vary with any of the variables. For instance, as t approaches 0, it is reasonable to assume

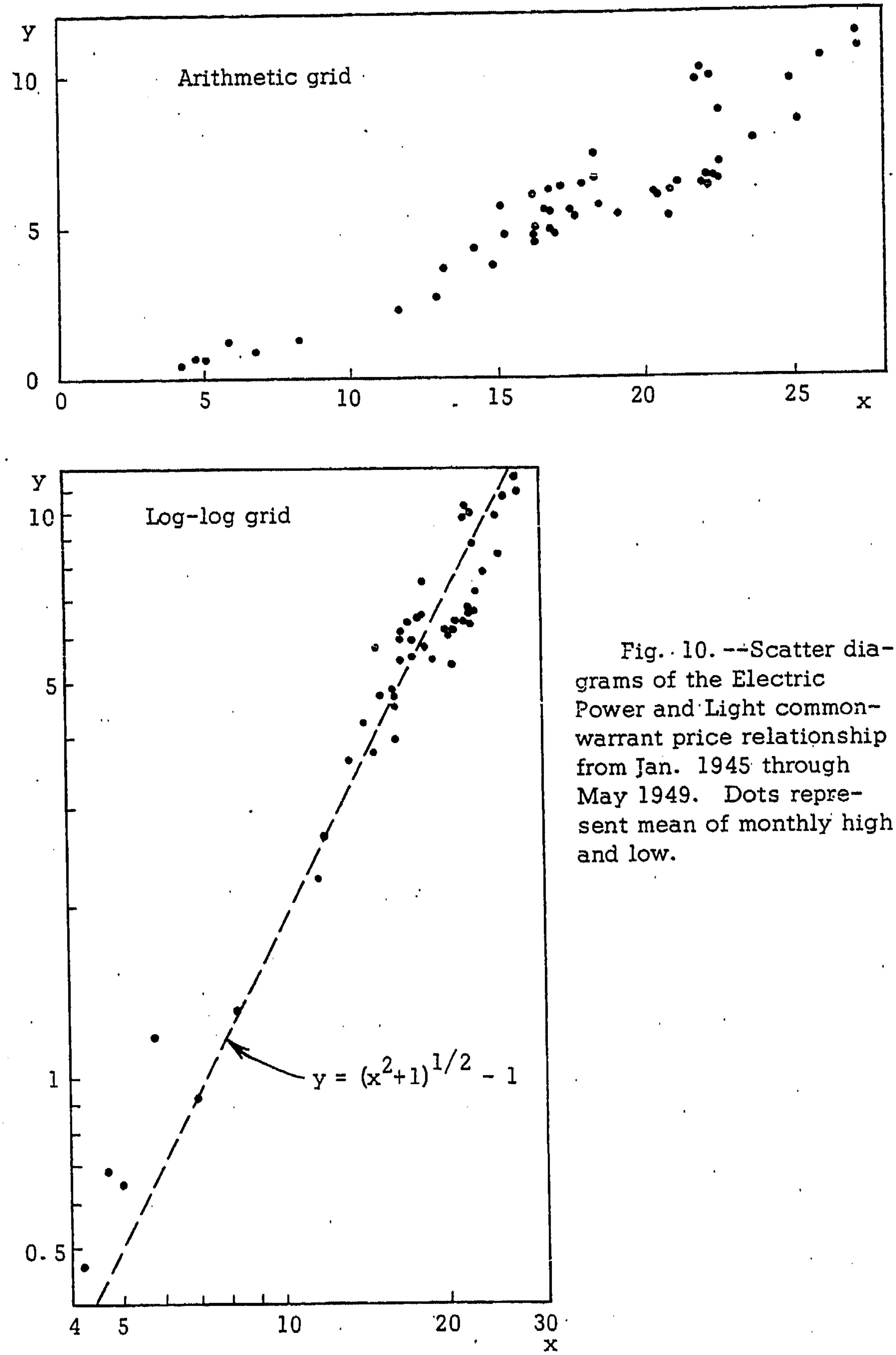


Fig. 10. --Scatter diagrams of the Electric Power and Light common-warrant price relationship from Jan. 1945 through May 1949. Dots represent mean of monthly high and low.

that the partial function will approach the lower boundary of the unshaded region of Fig. 1. In order for the slope of the partial to vary, the model will have to admit interaction between log x and the other independent variables, i. e.,

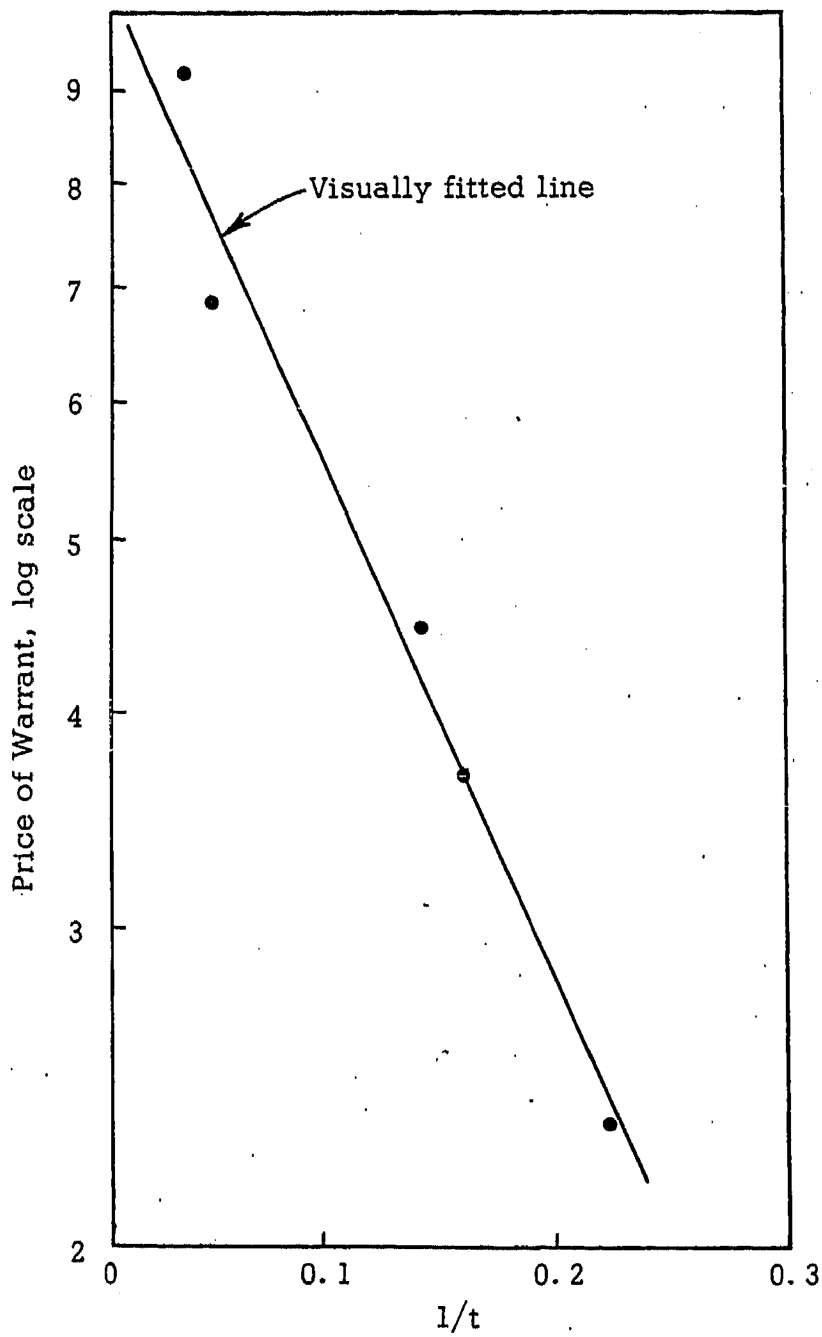
$$\log y = k_1 + k_2 \log x + k_3 f_3(t) + k_4 f_4(d) + k_5 f_5(r) + k_6 f_6(E) + k_7 f_7(R) + k_8 \log x f_8(t) + k_9 \log x f_9(d) + k_{10} \log x f_{10}(r) + k_{11} \log x f_{11}(E) + k_{12} \log x f_{12}(R) + e.$$

This model will not only allow the intercept term, k_1, to vary, but also the slope of log x, k_2, may vary.

If t is measured in months (say), this model would not be able to utilize perpetual warrants with infinite t unless f_3 is not the identity function. But if $f_3(t) = 1/t$, it may adequately describe perpetual warrants. To show that this is not an unrealistic assumption, Fig. 11 graphs the relation between 1/t and log y for the Pennsylvania-Dixie Cement warrant for the period January 1946 to April 1949 after which they expired. This scatter diagram is not a rigorous demonstration concerning the nature of f_3--its purpose is to intuitively indicate that the assumption that $f_3 = 1/t$ is plausible. Unfortunately, little direct evidence can be brought to bear on the nature of f_4, f_5, . . . , f_{12}, but Occam's razor suggests that, unless further embellishments prove more satisfactory,

$$\log y = k_1 + k_2 \log x + k_3(1/t) + k_4 d + k_5 r + k_6 E + k_7 R + k_8 (1/t) \log x + k_9 d \log x + k_{10} r \log x + k_{11} E \log x + k_{12} R \log x + e.$$

E, the effect of investors' expectations for the common on the price



Date	Common Price	Warrant Price	1/t
Oct. 1946	19.6875	9.2500	.0317
Aug. 1947	19.6250	6.8750	.0426
Oct. 1948	19.7500	4.4375	.1333
Nov. 1948	19.7500	3.6875	.1538
Jan. 1949	19.1250	2.3125	.2222

Fig. 11.--Scatter diagram of 1/t-log Y relationship for Pennsylvania Dixie Cement warrant. Observations were chosen in an attempt to hold all the explanatory variables of (A4) constant except for 1/t.

of the common, has yet to be quantified. Many possibilities, all incomplete, exist. First, it may be hypothesized that expectations are intimately connected with "fundamental" data such as earnings, dividends, etc. Then by allowing such data to enter as determinants of log y, it may be possible to infer something about expectations. Second, it may be hypothesized that expectations are closely related to "technical" data, particularly the price behavior of the common stock. Technical analysis, so-called, is based upon chart patterns and has recently entered the respectable domain of the academicians.[1] It seems likely that warrant buyers tend toward technical rather than fundamental analysis--warrant investors are usually characterized as speculators, and speculators are attracted more to price changes than to fundamentals. The second hypothesis will be adopted for this study and as a measure of price change, the ratio of the present price to the average of the year's high and low will be used. (The observations to be used in Chapter III occurred in the November months so the year's high and low will be taken to mean the high and low made by the common in the period from January through November.) If the present price, for example, is \$20, and the average of the year's high and low is \$15, the ratio 20/15 roughly indicates that the stock is presently "acting well", i.e., the price movement is in an uptrend. Similarly, a ratio of less than one indicates the price is in a downtrend. Since this ratio is

---

[1] See, e.g., Harry V. Roberts, "Stock Market 'Patterns' and Financial Analysis," Journal of Finance, XIV (March, 1959), 1-10, and C.W.J. Granger and O. Morgenstern, "Spectral Analysis of New York Stock Exchange Prices," Kyklos, XVI (Fasc. 1, 1963), 1-27.

limited below to zero and is unlimited above, it seems proper to take the log of the ratio. Thus a ratio of 2 and a ratio of $\frac{1}{2}$ will have the same absolute effect (but different in sign) upon the price of a warrant. (This is analogous to an investor who compounds his moves--a 50% loss followed by a 100% gain, or vice versa, will leave a portfolio unchanged in value, so that a 50% loss and a 100% gain have the same absolute effect on the value of the portfolio.)

As regards R, for a cross-sectional study, its effect on each warrant price will be constant, so that it may be excluded from the model. But even if the sample included observations from different time periods, the effect of R may well be random, so that again it may be excluded with immunity. This would be likely if the observations spanned a period of many stock market cycles, for then it could be presumed that risk-attitudes fluctuated randomly. Accordingly, letting log of $X/\bar{X}$ proxy for E, where X is the present unadjusted price of the common and $\bar{X}$ is the yearly mean of the unadjusted high and low, and ignoring or relegating R to the stochastic domain, the following econometric model is suggested:

$$\log y = k_1 + k_2 \log x + k_3(1/t) + k_4 d + k_5 r + k_6 \log (X/\bar{X}) + k_7(1/t) \log x + k_8 d \log x + k_9 r \log x + k_{10} \log (X/\bar{X}) \log x + e. \quad \text{(I)}$$

An Indirect Model

With ten parameters to estimate, this model clearly requires a large number of observations. Furthermore, as the partial function approaches the lower bound of the region defined in Fig. 1, the linearity between log x and log y becomes untenable. Of course, this could be overcome by introducing powers of log x into the model, but then the number of parameters to be estimated would increase substantially and the model would become unwieldy. There is a more elegant solution. Consider the family of functions (y_z), with $y_z = (x^z + 1)^{1/z} - 1$, with z taking all real positive values greater than or equal to 1. Note that y_1 is the function y = x, the upper bound of the region to which the partial function is restricted, and y_∞ is the function y = 0 if x is less than or equal to 1 and y = x-1 if x is greater than 1, i.e., y_∞ is the lower bound of this region. As z increases without bound, the entire region is spanned. Fig. 12 graphs selected values of y_z. Now a model for z implicitly defines a model for y. Ideally, (from an econometric viewpoint), z might not be dependent upon x, so that given 1/t, d, r, log of $X/\bar{X}$, a unique z can be determined that will trace out a path in the a priori region describing the relation between x and y. In Fig. 10, the path corresponding to z = 2 has been traced out in a broken line. Visually, it fits the data reasonably well. But more rigorous tests will be applied in Chapter III to determine whether or not z varies differently for different values of x.

For a classical least squares regression on z, it would be felicitous if $z = K_1 + K_2(1/t) + K_3r + K_4d + K_5\log X/\bar{X} + e$. Fig. 13 plots the observed z against the observed (1/t) for the Pennsylvania-Dixie Cement warrant, and it can be seen that the assumption that z is linear

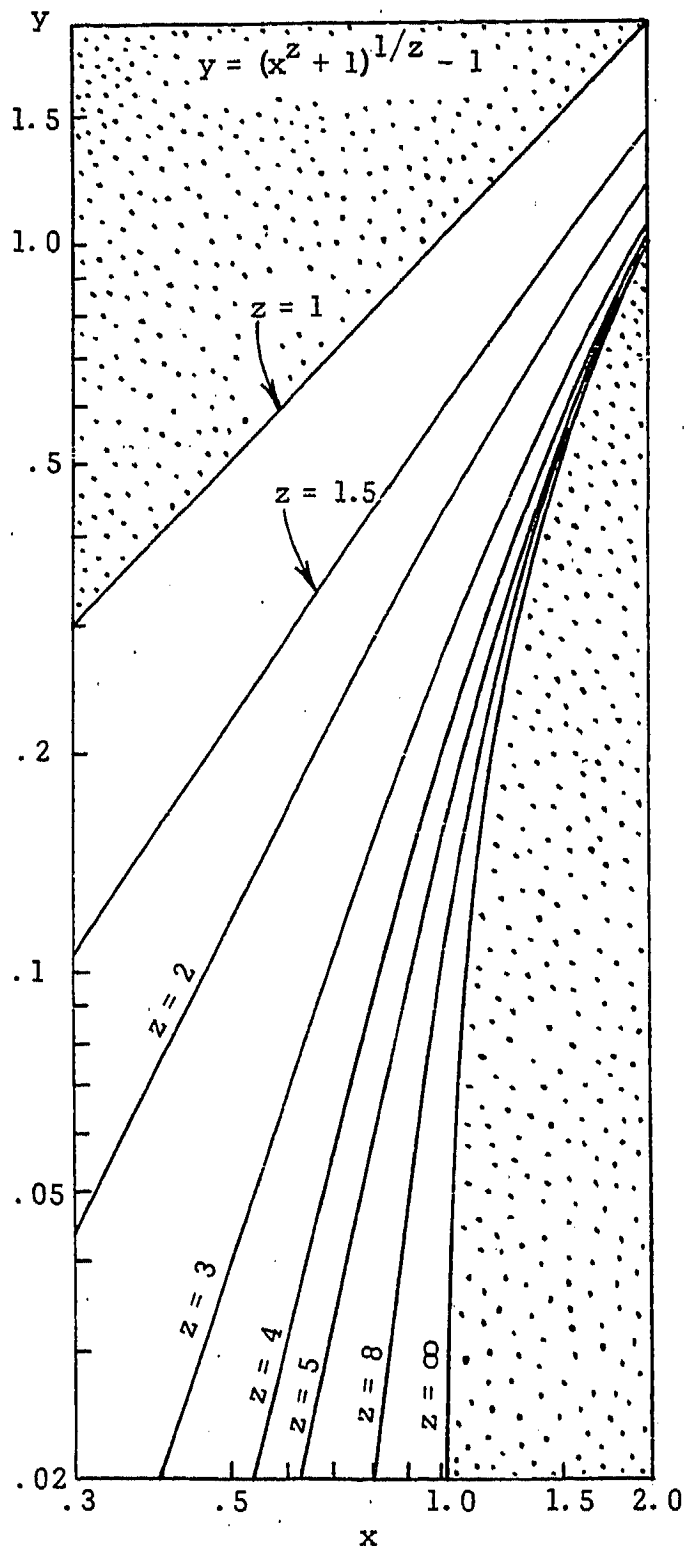


Fig. 12.--Selected graphs of y_z

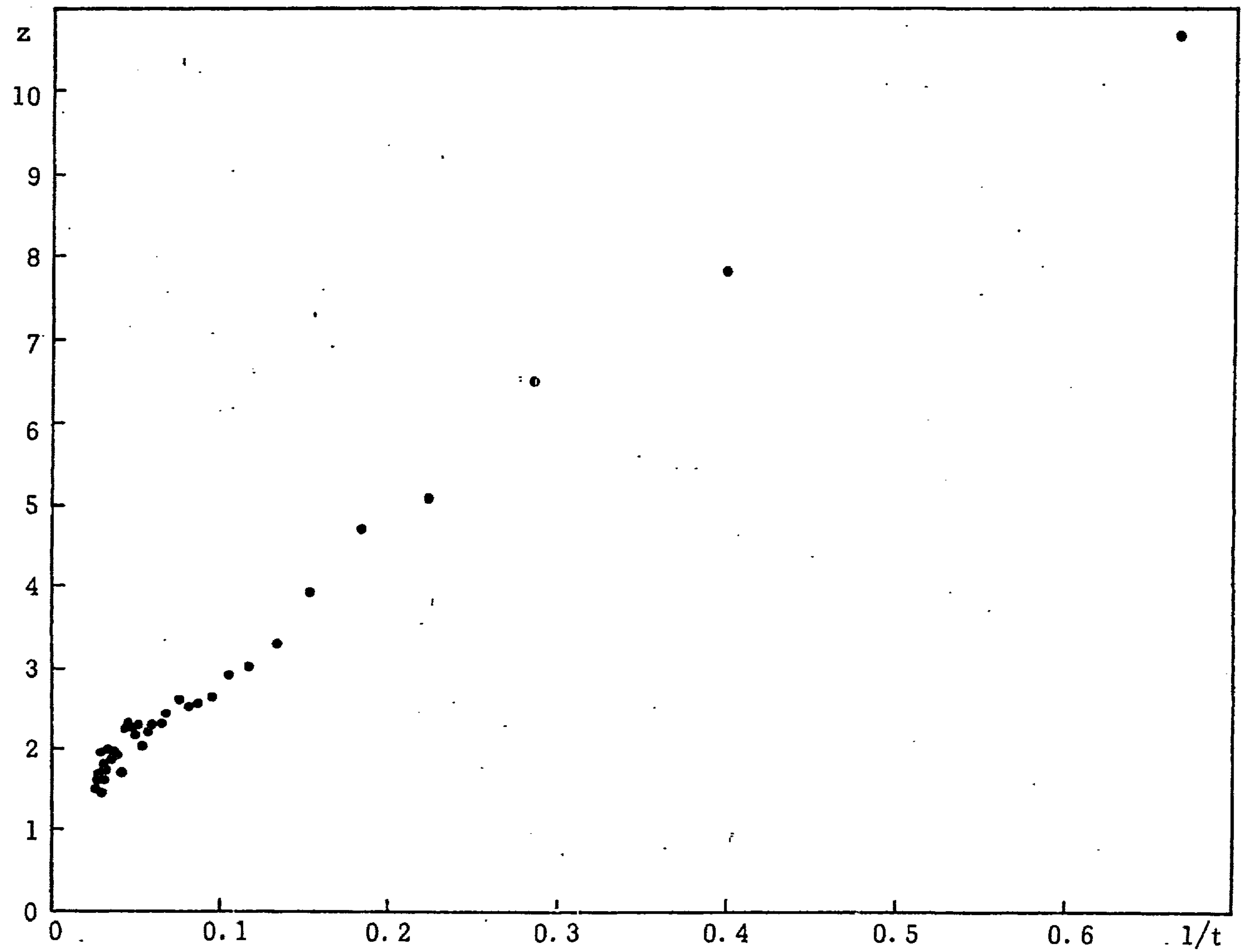


Fig. 13. --Relation between z and 1/t, Pennsylvania Dixie Cement warrant. Jan. 1946-Apr. 1949.

in 1/t is not wildly inconsistent with the evidence. Unfortunately, it is impossible to select data to similarly test the effect of the other variables on z. Unless a more complicated relation seems indicated,

$$z = K_1 + K_2(1/t) + K_3(r) + K_4(d) + K_5 \log X/\overline{X} + K_6 x + e \qquad \text{(II)}$$

will be used to test the data of the past twenty years.

CHAPTER III

TESTING THE ECONOMETRIC MODELS

The Warrants Under Study

In Chapter I, the elasticity of a warrant was defined as the percentage increase in the price of the warrant as a ratio of the percentage increase in the price of the associated common stock, so that in the immediate neighborhood of present prices, the elasticity is $(dy/dx)(x/y)$. This is actually a measure of leverage that an investment in the warrant has relative to an investment in the common. For instance, if the elasticity is 2, then a 1% rise (or fall) in the price of the common will be associated with a 2% rise (or fall) in the price of the warrant. It is well known that warrants possess elasticities greater than unity--in fact, this is their raison d'etre. (If a warrant's elasticity were unity, no investor would prefer the warrant to the common because the investor's expected return with either security would be equal; but the common possesses desirable characteristics that the warrant does not, viz., voting rights, dividend participation, and indefinitely long life (shared only by perpetual warrants). If the elasticity were less than unity, the warrant's price would rise and fall at a slower rate than the common; but the same effect can be achieved by purchasing less of the common.

For instance, if the elasticity were $\frac{1}{2}$ and an investor's budget were $100, a portfolio of $50 in the common and $50 in cash would be equivalent to $100 in the warrant; again, the common's superior characteristics would obliterate the demand for the warrant.) The elasticity, or leverage, can then be considered compensation for the inherent inferior qualities of the warrant.

Leverage can also be obtained by purchasing a security on margin. If a $10 security could be purchased on 50% margin, an investor need only commit $5, borrowing the balance from his broker. In this way, a 10% rise (say) in the price of the security to $11, would result in a <u>20%</u> rise in his investment. Thus, when the rate of margin is 50%, the elasticitity of a fully margined investment is 2. Now the warrants traded over-the-counter are not marginable and are not comparable to those that are, for investors seeking leverage would treat a marginable warrant differently from a non-marginable warrant. Primarily for this reason, this study will be restricted to warrants traded on the American Stock Exchange. (Warrants are not traded on the New York Stock Exchange and with rare exceptions, warrants traded on the regional exchanges are also listed on the American Stock Exchange. Also, many of the important warrants listed on the Canadian exchanges are listed on the American. The primary market for these dually listed warrants, as evidenced by the volume of transactions, is usually the American Stock Exchange.) Another reason for excluding the over-the-counter warrants is the unreliability of the available price data. Quotations in the "pink sheets" often vary considerably from "inside quotes" and transaction prices.

The time period under study extends from January 1945 through December 1964. The last issue of Barron's for each year in the study was searched for warrants listed on the American Stock Exchange, so this study includes every warrant listed there for a period of at least one year. (It is possible that some warrant listed early in any year and delisted before December of the same year escaped the net used in this study.) During this period, 41 companies had 43 warrants of diverse characteristics listed. Appendix A includes a chart of the price movements of these companies' common stocks during the time their warrants were listed.

The elements of the sample to be tested were chosen from the November prices of every year. This month was chosen arbitrarily. In an attempt to avoid autocorrelated disturbance terms, it was decided to use the data for only one month of every year. The mean monthly price for the securities was used. For instance, in November 1945, 15 warrants were listed, so 1945 contributed 15 observations to the sample. In November of that year, the Atlas Corporation warrant had a high and low, respectively, of 8 and $6\frac{5}{8}$, so that its price for that observation was 7-5/16. The common had a high and low, respectively, of $24\frac{1}{4}$ and 20, so that its price for that observation was $22\frac{1}{8}$.

In this way, the twenty year period yielded 240 observations. Six of these observations were judged not to belong to the universe under study so they were deleted from the sample. Appendix B is a compilation of these deleted observations.

Models I and II were therefore tested with 234 observations, compiled in Appendix A. The universe of this study can therefore be defined as all listed warrants that could conceivably have existed during the twenty year study period. Whether the results of these tests can be applied to warrants of a different universe, e.g., warrants of 1965, is a matter that requires separate consideration. If any of the assumptions or conditions of the test period are significantly altered, then clearly, the findings of these tests may not apply. The difficulty, of course, is recognizing when the test conditions have been violated.

The Results

Model I. Using Professor Jon Cunnyngham's step-wise regression program, a classical least-squares regression on the 234 observations comprising the test sample was performed on an IBM 7094. The program was allowed to run "completely unconstrained," so that at each step, the variable added to the multiple regression had "the highest partial correlation of all the unentered variables with the dependent variable (holding all entered variables constant), the highest F-ratio and the largest contribution to the multiple correlation."[1] Not surprisingly, the first variable to enter the regression was log x, the price of

[1] Jon Cunnyngham, "Generalized Step-Wise Regression Program," Columbia University Computer Center, 1964, page 2. (Mimeographed.)

the common:

Step 1. Variable entering: log x (Natural logs in this Chapter.)
Standard error of estimate: 0.566
Adjusted R^2: 0.798

Variable entered	Regression coefficient	Standard error	Partial correlation	T value	Mean square
Intercept	-1.02	0.04		-26.73	
log x	1.57	0.05	0.89	30.31	294.81

Clearly, the influence of the price of the common on the price of the warrant is paramount. This justifies the foregoing theoretical analysis where the price of the warrant was usually graphed as a function of the price of the common. If all the other variables are ignored, Step 1 indicates that for an "average" warrant, y = .36 when x = 1.0 and its elasticity is 1.57, i.e., when the common stock is selling at the exercise price, the price of an "average" warrant will be 36% of the price of the common and it will tend to appreciate and depreciate 1.57 times as fast as the common.

Step 2. Variable entering: 1/t
Standard error of estimate: 0.452
Adjusted R^2: 0.871

Variable entered	Regression coefficient	Standard error	Partial correlation	T value	Mean square
Intercept	-0.89	0.03		-27.44	
log x	1.56	0.04	0.93	37.63	294.38
1/t	-3.88	0.34	-0.60	-11.52	27.60

Considering log y a function of two variables, log x and 1/t, has raised R^2 to 0.871 from 0.798. This sample indicates that the length of the option period is the second major influence on the price of the warrant, and this is not contrary to what one may have guessed.

Step 3. Variable entering: r
Standard error of estimate: 0.410
Adjusted R^2: 0.894

Variable entered	Regression coefficient	Standard error	Partial correleation	T value	Mean square
Intercept	-0.67	0.04		-15.70	
log x	1.64	0.04	0.94	41.78	295.01
1/t	-3.84	0.30	-0.64	-12.62	26.90
r	-6.07	0.84	-0.43	- 7.20	8.75

The sample indicates that the next most significant variable is r, the yield on the common stock. Although the addition of r has only raised R^2 to 0.894 from 0.871, there can be little doubt that it is statistically significant. Having achieved a relatively high coefficient of determination with only three explanatory variables, the results so far may be sufficient for some purposes. Clearly, R^2 cannot be expected to increase very much from this level. But if the significant influences are sought, the search for explanatory variables should continue until a low T-value is encountered.

Step 4. Variable entering: $\log X/\bar{X}$
Standard error of estimate: 0.398
Adjusted R^2: 0.900

Variable entered	Regression coefficient	Standard error	Partial correleation	T value	Mean square
Intercept	-0.69	0.04		-16.49	
log x	1.68	0.03	0.94	42.61	293.78
1/t	-3.94	0.30	-0.66	-13.24	28.38
r	-5.93	0.82	-0.43	- 7.22	8.45
$\log X/\bar{X}$	-0.55	0.14	-0.24	- 3.78	2.31

The next variable that enters the multiple regression is $\log X/\bar{X}$.

Although its influence has barely changed the regression coefficients, the T-value of -3.78 indicates that the hypothesis that log $X/\bar{X}$ has no influence would be rejected at the 1% significance level. Arithmetically it has contributed little to the functional relation, but the fact of its statistical significance suggests some interesting hypotheses about investor' expectations. The sign of the coefficient indicates that the higher the price of the common relative to the mean value of the year's high and low, the lower the price of the warrant. Now $X/\bar{X}$ is admitedly a crude measure of price trend and a more refined statistic may invalidate this conclusion. But a quantitative measure of price trend must in any case be arbitrary. For instance, the time period considered may change the results. (Are warrant purchasers more influenced by short-term price trends than long-term?) With these reservations in mind, the results are consistent with at least two hypotheses: (1) There is a time lag in the adjustment of warrant price to common price, or (2) Warrant purchasers, who as a class are probably distinct from common stock buyers, become sceptical of a stock that has risen in price and expect a "correction." Other explanations can probably be advanced, and the possibility of extending this study to a broad spectrum of securities, to be discussed below, may lead to conclusive findings concerning investor expectations. If warrant buyers are repelled by common stocks that have risen in price, they may be characterized as bargain seekers or "anti-chartists." (Chartists are that subset of technicians who believe that price trends tend to continue.)

Step 5. Variable entering: 1/t log x
Standard error of estimate: 0.390
Adjusted R^2: 0.904

Variable entered	Regression coefficient	Standard error	Partial correlation	T value	Mean square
Intercept	-0.70	0.04		-17.09	
log x	1.64	0.04	0.94	40.71	302.69
1/t	-3.33	0.35	-0.54	- 9.64	16.98
r	-6.01	0.80	-0.44	- 7.47	10.18
log $X/\bar{X}$	-0.55	0.14	-0.25	- 3.87	2.74
1/t log x	1.38	0.42	0.21	3.28	1.96

Again, the introduction of a new variable has not altered the results greatly, but the variable, 1/t log x, has proved statistically significant. This is the first variable to enter that affects the slope of the partial, $\partial(\log y)/\partial(\log x)$. Previously, only the intercept term varied for different values of the explanatory variables, so that the partial moved up and down but only to parallel positions. The fact that 1/t was the most significant variable after log x, makes it reasonable that it should be the first variable to show interaction with log x, so that the results so far are not contrary to expectation.

Step 6. Variable entering: d
Standard error of estimate: 0.382
Adjusted R^2: 0.908

Variable entered	Regression coefficient	Standard error	Partial correlation	T value	Mean square
Intercept	-0.63	0.05		-13.97	
log x	1.63	0.04	0.94	41.32	302.05
1/t	-3.37	0.34	-0.55	- 9.96	17.54
r	-5.90	0.79	-0.44	- 7.47	9.88
log $X/\bar{X}$	-0.55	0.14	-0.25	- 3.95	2.77
1/t log X	1.39	0.41	0.22	3.36	2.00
d	-0.16	0.05	-0.21	- 3.23	1.84

The dilution ratio, d, also proved statistically significant, although its presence in the multiple regression does not alter the regression coefficients appreciably. After six steps all the variables, except R, hypothesized to influence warrant price have proven highly significant. (R, the aggregate effect of risk-attitudes was not quantified so that its influence, if any, is yet to be determined.)

Step 7. Variable entering: r log x
Standard error of estimate: 0.380
Adjusted R^2: 0.909

Variable entered	Regression coefficient	Standard error	Partial correlation	T value	Mean square
Intercept	-0.65	0.05		-14.14	
log x	1.59	0.05	0.92	35.28	290.87
1/t	-3.40	0.34	-0.56	-10.10	23.83
r	-5.70	0.79	-0.43	- 7.21	12.17
log $X/\bar{X}$	-0.59	0.14	-0.27	- 4.20	4.13
1/t log x	1.32	0.41	0.21	3.19	2.38
d	-0.16	0.05	-0.21	- 3.17	2.35
r log x	2.61	1.33	0.13	1.96	0.90

The interaction of r with log x is the first variable that is not highly significant--it is a border-line case at the 5% significance level. Since, however, r itself has proven highly significant, and since interaction has proven to be present in the case of 1/t, it seems likely that r log x is an influential variable in determining warrant price. If not, then the effect of r is only to change the intercept term and not the slope of the partial function--an unlikely result if one considers high values of r. As r increases, the graph of the partial function would move down in the unshaded region of Fig. 1--but for some value of r the function would intersect and go below the minimum value line.

Step 8. Variable entering: d log x
Standard error of estimate: 0.378
Adjusted R^2: 0.910

Variable entered	Regression coefficient	Standard error	Partial correlation	T value	Mean square
Intercept	-0.67	0.05		-14.19	
log x	1.54	0.05	0.88	28.26	270.61
1/t	-3.38	0.34	-0.56	-10.06	34.29
r	-5.64	0.79	-0.43	- 7.16	17.38
log X/$\overline{X}$	-0.60	0.14	-0.28	- 4.33	6.35
1/t log x	1.39	0.41	0.22	3.35	3.81
d	-0.13	0.05	-0.16	- 2.38	1.92
r log x	3.05	1.35	0.15	2.25	1.72
d log x	0.08	0.05	0.11	1.66	0.94

The addition of d log x has not altered the relation greatly, however, r log x is no longer a border-line case--it is now statistically significant. But d log x, with a T-value of 1.66 is significant only at the 10% level. But again it seems reasonable to accept it as an influential variable, since if d is significant, it is likely that there is interaction between d and log x.

Step 9. Variable entering: (log X/$\overline{X}$)(log x)
Standard error of estimate: 0.377
Adjusted R^2: 0.910

Variable entered	Regression coefficient	Standard error	Partial correlation	T value	Mean square
Intercept	-0.68	0.05		-14.29	
log x	1.54	0.05	0.88	28.37	273.31
1/t	-3.33	0.34	-0.55	- 9.88	33.12
r	-5.67	0.79	-0.43	- 7.21	17.67
log X/$\overline{X}$	-0.50	0.15	-0.21	- 3.27	3.62
1/t log x	1.51	0.42	0.23	3.59	4.39
d	-0.13	0.05	-0.16	- 2.42	1.98
r log x	2.65	1.38	0.13	1.93	1.26
d log x	0.10	0.05	0.13	1.91	1.24
(log X/$\overline{X}$)(log x)	0.27	0.18	0.10	1.48	0.74

This last step indicates that three of the interaction variables, d log x, r log x, (log $X/\bar{X}$)(log x), approach or are at the border-line of statistical significance, with all the other variables highly significant. From the previous discussion it seems reasonable to accept all of the interaction variables as influencing factors. With a final R^2 of 0.91, there is little doubt that this Model is a good description of the universe under study.

Fig. 14 graphs the warrant-common price relationship with all of the "explanatory" variables (except log x) at their sample mean. In the sample, x varied between 0.1 and 3.9. It is possible that over this interval the relation between log x and log y is not linear, e.g., the true relation may "bend" toward the curved minimum value line. To test for this possibility two additional regression analyses were made: one on all observations for which $x \geqslant 1.1$ and another on all observations for which $x \leqslant 0.65$. Fig. 15 plots the common-warrant price relationship for these two subsets of the sample with the values of the variables at the mean of the original 234 observations. When seen with the relationship inferred from the complete sample, it seems likely that the slope of the partial function increases as log x decreases, i.e., that the elasticity increases for decreasing x, rather than remaining constant as implied by Model I. As a further indication that the relation between the log of prices is not linear, an examination of the residuals, when the observations are ordered by increasing y, reveals a marked tendency for the computed value of log y to be greater than the observed

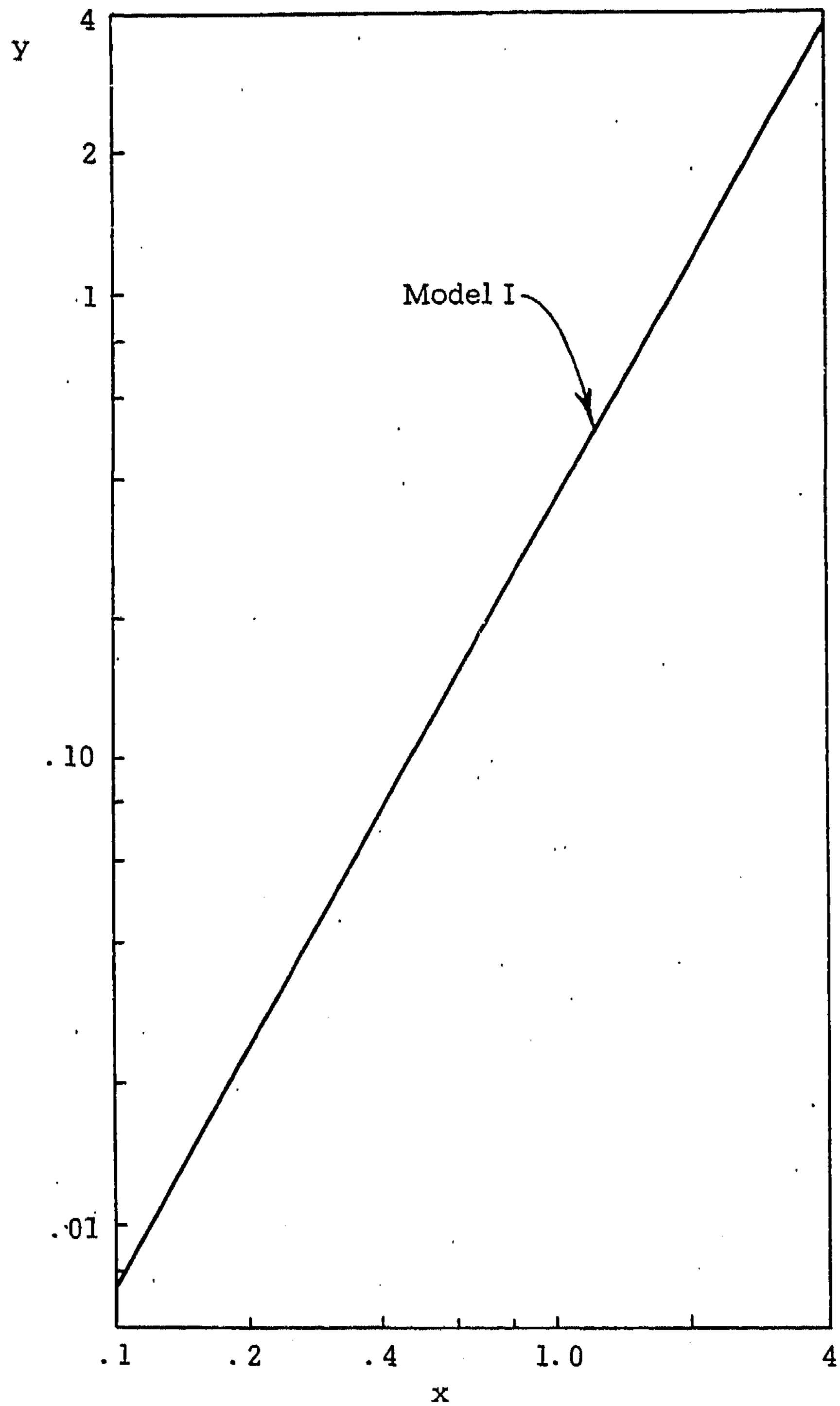


Fig. 14--Warrant-common relationship, Model I. All other variables at sample mean.

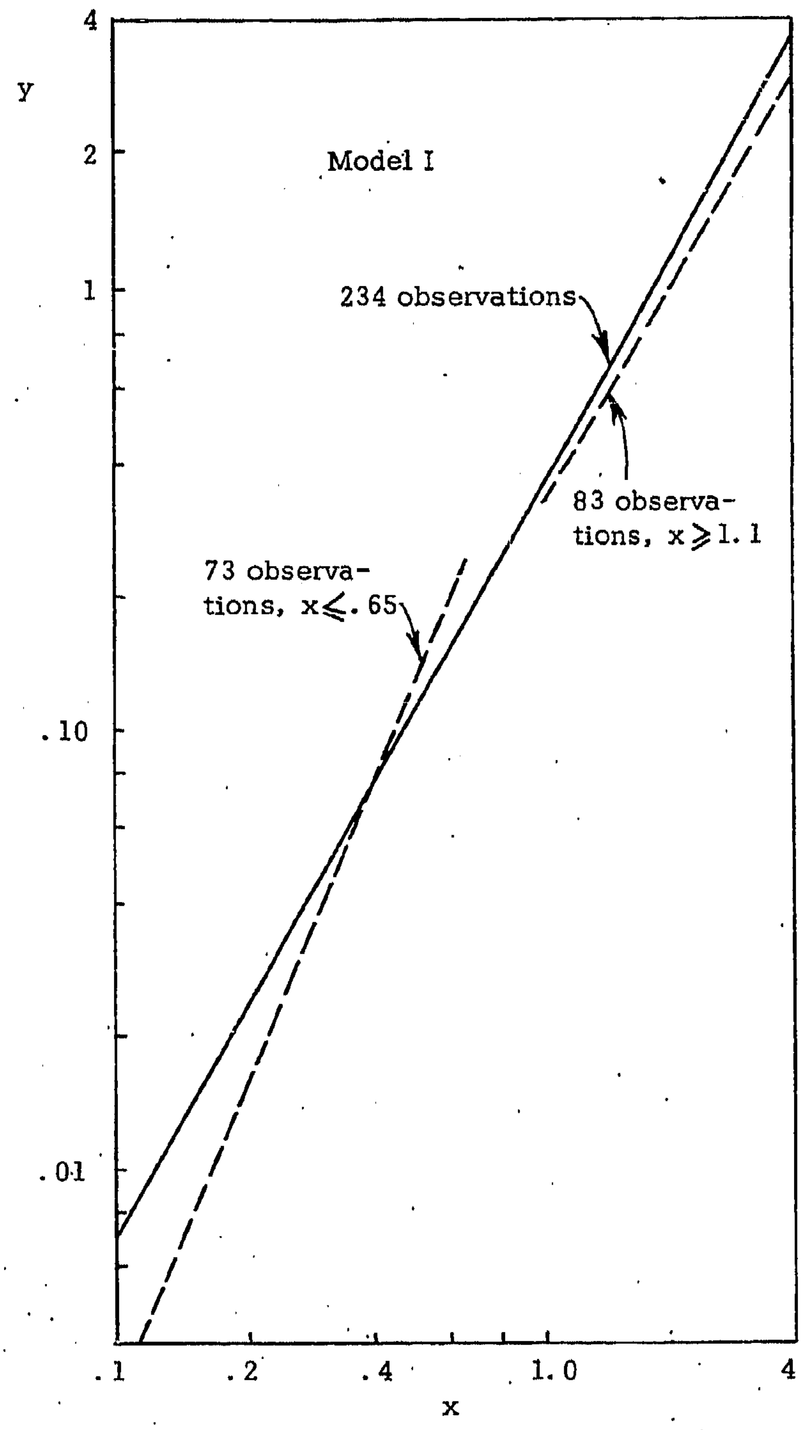


Fig. 15--Model I partitioned for high and low values of x. All variables except x and y at sample mean.

value when $y < 0.6$ and when $y > 1.5$. With this ordering, the Durbin-Watson d-statistic is 1.29 strongly indicating the absence of an explanatory variable. By introducing higher powers of log x and interaction between these powers and the other variables, a curvilinear relation might be found that fits the data even more closely. However, if even only $\log^2 x$ is introduced, this would mean the addition of possibly 5 more variables. This would result in a cumbersome relation consuming degrees of freedom and involving 14 independent variables. Another possibility is to exclude some observations at both ends of the x interval so that a linear fit can more closely approximate a non-linear relation. But then efficiency would be lost and the model would only apply to a smaller universe. As indicated above, pages 51-54, Model II may prove to be a less heavy-footed approach that would allow an estimate of a non-linear relation without loss of efficiency or degrees of freedom.

Ideally, this study should have tested coeval observations. Then, if at different time periods the parameters of the model changed significantly, the difference might be attributed to changes in risk-attitudes. Unfortunately, this was not possible because of the scant number of observations corresponding to each year. As a (poor) substitute for an ideal test, the observations were divided into two periods: 116 observations from the period 1945 through 1957 and 118 observations from the period 1958 through 1964. A multiple regression analysis was made on each subset with the following results:

	1945-1957 116 observations	1958-1964 118 observations	1945-1964 234 observations
Adjusted R^2	0.910	0.934	0.910
Standard error of estim.	0.432	0.243	0.377
Variable	Coefficient	Coefficient	Coefficient
Intercept	-0.74	-0.66	-0.68
log x	1.67	1.25	1.54
1/t	-3.36	-2.50	-3.33
r	-4.78	-6.04	-5.67
log $X/\overline{X}$	-0.49	-0.26	-0.50
1/t log x	2.37	1.42	1.51
d	-0.12	-0.07	-0.13
r log x	0.90	5.48	2.65
d log x	0.03	0.13	0.10
(log $X/\overline{X}$)(log x)	0.06	0.33	0.27

Following Chow[1], a test was made to determine whether the observations in the second set came from the same universe as the first. The F-ratio, $F = (Q_3/10)/(Q_2/224) = 4.72$, with (10, 224) degrees of freedom indicates that both sets of data did not come from the same universe.[2] It is tempting to attribute this to a difference in risk-attitudes in both periods. If, for instance, in the period 1945 through 1957 risk-attitudes were generally more conservative, the warrant-common price relationship for that period may have been closer to the minimum value line than in the period 1958 through 1964. Before accepting this

[1]Gregory C. Chow, "Tests of Equality between Sets of Coefficients in Two Linear Regressions," Econometrica, XXVIII (July, 1960), 591-605. Also described in J. Johnston, Econometric Methods (New York: McGraw-Hill Book Co., 1963), 136-138.

[2]Where Q_1 = sum of squared residuals for the regression on the pooled data, Q_2 = the sum of the squared residuals from the regression on the first partition of the data added to the sum of the squared residuals from the regression on the second partition, and $Q_3 = Q_1 - Q_2$.

conclusion, however, it must be noted that the difference in the results may have been due to the fact that in the first period the values of x were relatively lower than in the second period. In 1945-1957, the average value of x was 0.73 whereas in 1958-1964, the average value of x was 0.96. In the first period, x ranged between 0.10 and 2.30 whereas in the second period x ranged between 0.22 and 3.86. The difference in results may then be attributable to a non-linearity in the relationship over the range $0.10 \leqslant x \leqslant 3.86$: the regression of the first period was made on a lower portion of this interval than the regression of the second period. To help discover if the difference was due to this possible non-linearity, all observations with $x \leqslant 0.3$ and $x \geqslant 2.0$ were excluded from both partitions. This left 94 observations in the period 1945-1957 and 99 in the period 1958-1964. A regression on each partition and the pooled observations yielded:

	1945-1957 94 observations	1958-1964 99 observations	1945-1964 193 observations
Adjusted R^2	0.958	0.919	0.886
Standard error of estim.	0.292	0.264	0.277
Variable	Coefficient	Coefficient	Coefficient
Intercept	-0.77	-0.63	-0.68
log x	1.16	1.22	1.18
1/t	-1.77	-2.39	-2.00
r	-4.25	-6.48	-5.49
log $X/\overline{X}$	-0.45	-0.25	-0.31
1/t log x	6.14	4.17	5.93
d	-0.22	-0.12	-0.18
r log x	1.74	4.25	3.39
d log x	0.36	0.13	0.14
(log $X/\overline{X}$)(log x)	0.09	0.37	0.34

The F-ratio in this case is equal to 0.85 indicating there is little basis for believing that the observations of the second period did not belong to the same universe from which the first set of observations were taken. Therefore, it seems likely that the results of the first partition test, pointing to different universes for each period, are biased due to the possibility that a linear relation is not a perfect approximation to the true relation over an unrestricted portion of the x-axis.

This is a weak and negative finding. It is not to say that the parameters have not changed over time--in the span of each time period, the parameters may have changed from year to year but the total effect in each period may have averaged similarly. A method will be discussed below that will permit extension of this model to a larger list of securities; then a sizable contemporaneous sample may be chosen and perhaps conclusive findings regarding risk-attitudes may be made.

<u>Model II.</u> An observed x and y implicitly define a unique $1 \leq z < \infty$, such that $y = (x^z + 1)^{1/z} - 1$. Using Newton's method, each observation of x and y was iterated for z, i.e., $g(z_{i+1}) = z_i - f(z_i)/f'(z_i) = z_i - (x^{z_i} - y^{z_i} + 1)/(x^{z_i} \log x - y^{z_i} \log y)$ was used to converge on z. Letting $z_i = 2.0$, this iteration was fast and yielded z to four significant decimal places in usually less than four steps. The IBM 7094, using this program, calculated all the z values for the sample under study in less than two minutes.

In 12 cases, observed y was less than or equal to the minimum value prescribed by the a priori conditions discussed in Chapter I, i.e., they were below the minimum value line depicted in Fig. 1. These observations are not paradoxical: the a priori conditions abstracted from buying and selling costs, so that in practice, the interval to which the warrant is restricted is somewhat wider than that shown in Fig. 1. Furthermore, the theory asserts that warrant prices cannot leave the a priori region, except momentarily (see pages 7-8 above).

For these 12 cases z is not defined or equal to ∞ so they were excluded from the test. The remaining 222 observations were used for a regression on z with the following results:

$$y = (x^z + 1)^{1/z} - 1, \text{ where}$$

$$z = \underset{(0.15)}{1.31} + \underset{(0.75)}{5.36}\,(1/t) + \underset{(1.94)}{14.26}\,r + \underset{(0.13)}{0.30}\,d + \underset{(0.35)}{1.01}\log X/\overline{X} + \underset{(0.11)}{0.40}\,x$$

and $R^2 = 0.39$. At first blush this seems to indicate that Model I is superior to Model II. Upon closer examination, this conclusion does not necessarily follow. First, R^2 is not a meaningful statistic for Model II because the dependent variable z has an abnormal distribution. Its mean is 2.4258 and its standard deviation is 1.2175. Since z ranges between 1.1871 and 9.7572, the standard deviation is far from an adequate description of its distribution. Since R^2 measures the amount of "variation" in z, this statistic is meaningful only if the standard deviation is descriptive.[1] Second, a glance at Fig. 12 indicates

[1] See Mordecai Ezekiel and Karl A. Fox, Methods of Correlation and Regression Analysis (3rd ed.; New York: John Wiley and Sons, 1959), 311.

that the disturbance term in Model II cannot be normally distributed. In fact, it is not even symmetrically distributed. Since R^2 is meaningful only for an underlying normally distributed universe with a normally distributed disturbance term, the comparison of R^2 in both models is not appropriate. (Non-normality does not influence the meaningfulness of the regression coefficient estimates if appeal is made to the Central Limit Theorum.)[1] Even if the normality assumptions were not violated in Model II, R^2 would only measure how well the observed z's are explained. To see how well log y is explained, so that a more meaningful comparison can be made with Model I, the computed z's of Model II were converted into the equivalent computed log y's. Then the residuals (the difference between the computed and observed log y) were calculated. The variance of these new residuals is 0.248. The square root of this yields a new "standard error of estimate" and the new "R^2" for Model II becomes 0.832, indicating that Model II fits the observations almost as well as Model I.

Fig. 16 superimposes the warrant-common relation as defined by Model II upon that of Model I. Model II is in fact concave and defines an increasing warrant elasticity as x decreases, thereby satisfying the conditions suggested by the test on the partitioned subsets (see Fig. 15). Model II agrees almost exactly with the upper partition of Model I. Because Model II is not constrained linearly over the x-axis and because

---

[1]See Johnston, 115-16.

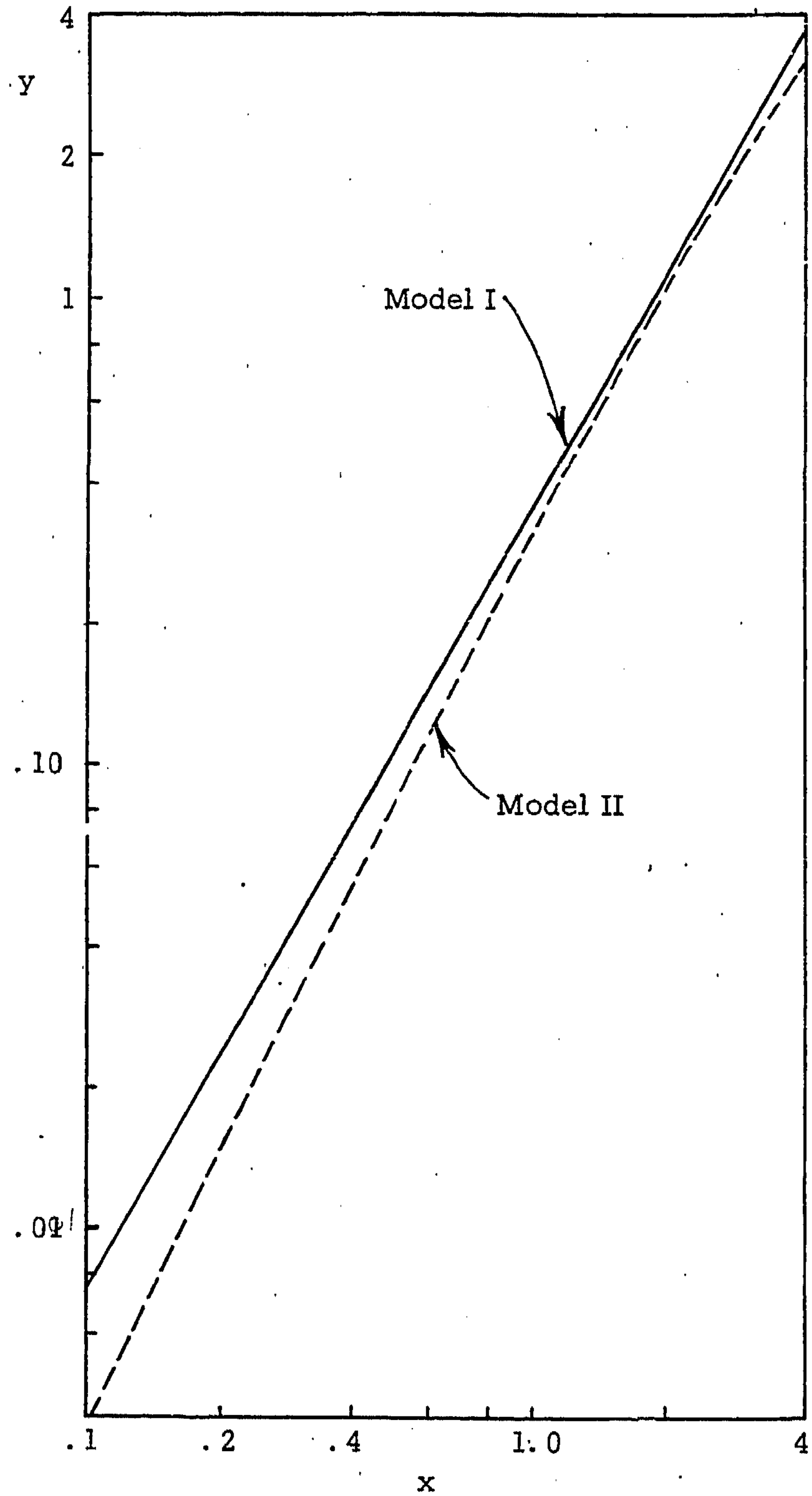


Fig. 16--Warrant-common relationship, Models I and II. All variables except x and y at sample mean.

it involves less variables, it is probably the superior model of this study.

Extensions of the Study

It is possible to consider other forms of convertible securities thereby increasing the sample. Call options, for example, if considered separately, can be tested in precisely the manner indicated by Models I and II. (Although the number of observations might be increased for any time period, calls seldom vary much in the length of the option period, and the common is almost always close to the exercise (striking) price. This may introduce some difficulties in estimating the parameters of the true relationship.)

Another possibility is the convertible bond. In December 1964, there were 134 convertible bonds listed on the New York and American Stock Exchanges. To see how a convertible bond can be incorporated in this study, consider the Philips Petroleum $4\frac{1}{4}$% debenture, due 1987. On December 31, 1964, this bond sold for $1185. It was convertible into 20 shares of common, which on that date was selling at $53. 50 per share. The bond was rated AA by Standard and Poor's Corporation and would probably have sold for $990 without the conversion feature. (The yield to maturity of Standard and Poor's index of AA industrial bonds on that date was 4. 29%.) Thus it might be argued that this bond consisted of an ordinary debenture, worth $990, plus 20 warrants, worth $195, or $9. 75 each. The exercise price of these "warrants" was

$49.50 ($990 divided by 20), the yield on the common was 3.7%, the dilution ratio 0.104, the time remaining on the option 265.5 months, and the ratio of the common's price to its yearly mean was 1.03. Model I computes a price of $21.80 for these hypothetical warrants, and Model II, $19.70. But these "warrants" are not strictly comparable to the warrants of this study, first, because the length of the option period is at the whim of the Corporation: if the bond is called for redemption, the option expires; second, there is the forced tie-in sale of the $990 "debenture" which an investor may not ordinarily wish to buy; third, the exercise price floats with the interest rate. With all of these differences it is not surprising that the market valued them for less than the models of this study. Nevertheless, a study of convertible bonds alone might reach some definitive conclusions concerning risk-attitudes and expectations since their number is more numerous than warrants and relatively large contemporaneous samples can be drawn from this universe.

Concluding Remarks

There can be little doubt that the foregoing tests have identified the most influential variables affecting warrant prices: price of the common, length of option period, dividend yield of the common, dilution ratio, and previous price trend of the common. For practical men the value of the relations defined by Models I and II is clear. But these results may be of even more importance to theorists and policy-makers.

In the never-never land of Complete Certainty, a free securities market serves the important function of channeling investment funds into those industries whose marginal product is greatest thus assuring an optimal allocation of resources. So it is often claimed that the individual (investor) seeking his own self interest concomitantly serves the public interest. But in the imperfect world of reality, investors seek their interest in the face of uncertainty and it is far from clear if their activities always result in an "optimum." This is because decisions are formed with the aid of expectational-utility functions and very little is known about how these subjective phenomena affect or are affected by economic forces. By examining the decision-making process of an individual, Chapter I revealed the futility of attempting to reconstruct the process from observed behavior; but Chapter III outlined a method by which inferences can be made from an existing option-common price relationship. This may provide economists and econometricians with a key to the house now occupied almost solely by psychologists and sociologists interested in motivations and "investor psychology," i. e., investor expectations and risk-attitudes. Answers to some intriguing questions may now be possible: do investors' expectations lead or lag security prices (and thus aggregate economic activity); does caution or risk-aversion cause lower security prices, or if there is a relation is it the other way around; in what way, if any, are expectations connected to risk-attitudes; do the expectations of subsets of the investor population (e. g., insiders, professionals, "the public," etc.) differ and in what manner?

Definitive answers would perhaps put some flesh on the invisible hand ruling a market economy and carry important implications for the control of potentially dangerous business fluctuations. It is hoped that now this dark area can be explored at least with the help of a small flashlight.

APPENDIX A

LISTING OF THE SAMPLE OBSERVATIONS

The 234 observations that comprise the sample used for empirical study are listed below. The price data were obtained from Standard and Poor's monthly Stock Guides. The "observed" price is the mean of the month's high and low prices. Data concerning the other variables were obtained from Moody's annual Manuals. In the few cases where these Manuals were incomplete, the data were taken from the Standard and Poor's annual Corporation Records.

Observation	y	x	1/t	d	r	log X/$\overline{X}$	z
		1945					
1 Am. For. Power	0.0775	0.3000	.0	2.9790	0.0	0.3102	1.6747
2 Atlas	0.2925	0.8850	.0	0.8761	0.0226	0.1231	2.4810
3 Colorado Fuel	0.3571	0.8857	.0198	1.0302	0.0323	0.0753	2.8044
4 Com. South.	0.0083	0.1104	.0	0.5152	0.0	0.1095	1.8850
5 Elec. Power	0.2575	0.7150	.0	0.1548	0.0	0.4411	1.6945
6 Int'l. Mineral	2.8462	3.8615	.0690	0.1004	0.0379	0.2122	*
7 Manati Sugar	0.5000	1.0250	.0422	0.4302	0.0195	0.0919	2.5165
8 Merritt Chap.	0.2625	0.6666	.0	0.1370	0.0	0.1371	1.4324
9 NYC Omnibus	1.1429	2.1536	.0645	0.0008	0.0796	0.1030	*
10 Pan American	0.5382	1.2465	.0392	0.3333	0.0111	0.0056	2.5933
11 RKO	0.4167	1.0083	.0198	0.8527	0.0	0.1856	2.4878
12 Richfield Oil	0.1625	0.8063	.0625	0.3117	0.0465	0.1062	2.6398
13 Tri-Continent.	0.1582	0.5079	.0	0.5068	0.0213	0.1985	1.8963
14 United	0.0250	0.1591	.0	0.2569	0.0	0.3365	1.7101
15 Ward Baking	0.4100	1.0650	.0080	0.4047	0.0113	0.1143	2.2611

Observation	y	x	1/t	d	r	log X/$\overline{X}$	z
1946							
16 Am. For. Power	0.0525	0.2500	.0	2.9790	0.0	-.3800	1.7208
17 Atlas	0.2425	0.9600	.0	0.9443	0.0417	-.1697	2.6694
18 Colorado Fuel	0.2750	0.6821	.0260	1.0302	0.0503	-.3535	2.5716
19 Com. South.	0.0063	0.1188	.0	0.5152	0.0	-.1765	2.0420
20 Elec. Power	0.2425	0.6475	.0	0.1548	0.0	-.2809	1.1473
21 Hussmann	0.7083	1.3125	.0238	0.1705	0.0635	-.1519	3.0570
22 Int'l. Mineral	2.5000	3.6154	.4000	0.0279	0.0340	-.2034	*
23 Manati Sugar	0.3150	0.8300	.0870	0.4302	0.0241	-.2110	2.3973
24Merritt Chap.	0.2250	0.5042	.0	0.1747	0.0331	-.2918	1.4690
25 Pan American	0.2153	0.7361	.0741	0.3320	0.0377	-.3768	2.2138
26 Penn. Dixie	0.4375	0.9781	.0328	0.1501	0.0	-.2044	1.7431
27 RKO	0.3792	1.1083	.0260	0.6576	0.0541	-.3081	2.8783
28 Richfield Oil	0.0594	0.7186	.2500	0.3117	0.0557	-.1417	3.6100
29 Tri-Continent.	0.1221	0.3758	.0	0.5068	0.0432	-.2877	1.7717
30 United	0.0341	0.1432	.0	0.2569	0.0	-.2757	1.5188
31 Ward Baking	0.5700	1.3050	.0089	0.4047	0.0766	.0633	3.4413
1947							
32 ACF Brill	0.2550	0.5800	.0117	0.2911	0.0	-.1737	1.3347
33 Am. For. Power	0.0050	0.1150	.0	2.9790	0.0	-.4480	2.1044
34 Atlas	0.2125	0.9300	.0	0.9507	0.0688	-.0369	3.0726
35 Colorado Fuel	0.3036	1.0429	.0377	1.0302	0.0575	.1961	3.5137
36 Com. South.	0.0026	0.0979	.0	0.5152	0.0	-.0619	2.2179
37 Elec. Power	0.1925	0.6525	.0	0.1548	0.0	.0511	2.0086
38 Hussmann	0.7249	1.5754	.0333	0.0202	0.1127	.0730	2.7638
39 Merritt Chap.	0.2070	0.6446	.0	0.0171	0.0776	.0550	1.9097
40 Niagara Hud.	0.0175	0.1886	.0	0.0519	0.0	-.0209	2.0034
41 Pan American	0.0061	0.5139	.6667	0.3325	0.0270	-.2445	5.1724
42 Penn. Dixie	0.3375	0.9094	.0540	0.1499	0.0687	-.0404	2.0632
43 RKO	0.1833	0.7292	.0377	0.6488	0.1097	-.1823	2.3274
44 Tri-Continent.	0.0999	0.3657	.0	0.5068	0.0741	.0377	1.7176
45 United	0.0182	0.1000	.0	0.2569	0.0	-.1861	1.5480
46 Ward Baking	0.3300	0.9450	.0099	0.4042	0.1143	-.2676	2.2175
1948							
47 ACF Brill	0.1000	0.2700	.0136	0.2911	0.0	-.3417	1.4552
48 Atlas	0.1875	0.8300	.0	0.9888	0.0771	-.0782	2.7369
49 Colorado Fuel	0.2964	1.0893	.0690	1.0302	0.0656	.0820	3.2410
50 Elec. Power	0.2450	0.8375	.0	0.1548	0.0	.1103	2.3202
51 Hussmann	0.6213	1.5533	.0556	0.0191	0.0952	.1001	3.8819
52 Merritt Chap.	0.1833	0.7028	.0	0.1706	0.0785	-.1075	2.2299
53 Niagara Hud.	0.0081	0.1621	.0	0.0519	0.0	-.0938	2.2086
54 Penn. Dixie	0.1844	0.9875	.1538	0.1421	0.0759	.0756	3.9507
55 RKO	0.0833	0.5083	.0690	0.6487	0.0788	-.1796	2.3373
56 Tri-Continent.	0.1193	0.4232	.0	0.4462	0.0717	-.0315	1.7623

Observation	y	x	1/t	d	r	log X/X̄	z
1948 (Cont'd.)							
57 United	0. 0148	0. 1000	. 0	0. 2569	0. 0	-. 0870	1. 6186
58 Ward Baking	0. 3950	1. 2200	. 0113	0. 4022	0. 1213	. 1081	3. 1794
1949							
59 ACF Brill	0. 0723	0. 1915	. 0163	0. 2911	0. 0	. 1178	1. 3843
60 Atlas	0. 2125	0. 9000	. 0	0. 9960	0. 0711	. 0168	2. 8714
61 Colorado Fuel	0. 0429	0. 8786	. 4000	1. 0140	0. 1301	-. 0121	7. 5799
62 Hussmann	0. 9172	1. 9083	. 1667	0. 0114	0. 0930	. 1685	5. 6189
63 Merritt Chap.	0. 1682	0. 6856	. 0	0. 1588	0. 0805	-. 0431	2. 2734
64 RKO	0. 0036	0. 4917	. 4000	0. 6487	0. 0813	-. 0968	5. 5107
65 Tri-Continent.	0. 1221	0. 4605	. 0	0. 4549	0. 0776	. 1169	1. 8521
66 United	0. 0080	0. 1773	. 0	0. 2569	0. 0205	. 2963	2. 3053
67 Ward Baking	0. 3600	1. 2400	. 0131	0. 3793	0. 1290	. 0797	3. 8932
1950							
68 ACF Brill	0. 0792	0. 2375	. 0202	0. 2911	0. 0	-. 0345	1. 4789
69 Atlas	0. 2075	0. 9950	. 0	0. 9995	0. 0643	. 0332	3. 6282
70 Merritt Chap.	0. 2457	1. 0774	. 0	0. 1300	0. 0574	-. 0884	3. 8561
71 Tri-Continent.	0. 1249	0. 5654	. 0	0. 4549	0. 0929	. 0554	2. 1674
72 United	0. 0040	0. 1455	. 0	0. 2569	0. 0500	. 0480	2. 4073
73 Ward Baking	0. 5350	1. 5200	. 0155	0. 1706	0. 1053	. 0930	6. 4962
1951							
74 ACF Brill	0. 1333	0. 4750	. 0267	0. 2911	0. 0	. 3054	1. 8263
75 Atlas	0. 2700	1. 0750	. 0	1. 0000	0. 0595	. 0093	3. 4553
76 Merritt Chap.	0. 4333	1. 4214	. 0	0. 1281	0. 0136	. 1734	7. 7121
77 Tri-Continent.	0. 1691	0. 7601	. 0	0. 9952	0. 0704	. 1230	2. 5701
1952							
78 ACF Brill	0. 0750	0. 4083	. 0392	0. 2911	0. 0	-. 1512	2. 0482
79 Atlas	0. 2475	1. 0700	. 0	0. 9912	0. 0598	-. 0546	3. 7432
80 Eureka	0. 3500	1. 0000	. 1538	0. 3333	0. 0	-. 3365	2. 3097
81 Merritt Chap.	0. 3342	1. 1758	. 0	0. 1022	0. 0561	-. 0852	3. 5380
82 Tri-Continent.	0. 1939	0. 9326	. 0	0. 9952	0. 0628	. 0663	3. 2990
1953							
83 Alleghany	0. 5583	0. 9833	. 0	0. 6468	0. 0	-. 1272	1. 5336
84 ACF Brill	0. 0292	0. 3250	. 0741	0. 2911	0. 0	-. 0859	2. 3617
85 Atlas	0. 2300	1. 1625	. 0	0. 9936	0. 0642	-. 0149	6. 0152
86 Eureka	0. 2000	0. 5750	. 1538	0. 2730	0. 0	-. 3610	1. 7583
87 Merritt Chap.	0. 8538	1. 5055	. 0	0. 0057	0. 0674	. 1136	1. 8490
88 Tri-Continent.	0. 1496	0. 8833	. 0	0. 9952	0. 0707	. 0040	3. 5608

Observation	y	x	1/t	d	r	log X/$\overline{X}$	z
1954							
89 Alleghany	0.8000	1.3167	.0	0.5407	0.0	.0256	1.5924
90 ACF Brill	0.0083	0.6042	.6667	0.2797	0.0	.1565	5.9347
91 Atlas	0.5226	1.5200	.0	0.8929	0.0500	.0787	9.7572
92 Eureka	0.2500	0.7500	.2857	0.2730	0.0	.1431	2.0000
93 Gen. Accept.	0.7000	1.4688	.0183	0.4169	0.0681	.1030	2.3369
94 Tri-Continent.	0.3547	1.3021	.0	0.9897	0.0549	.1328	5.4174
1955							
95 Alleghany	1.5833	2.3333	.0	0.5407	0.0	-.0420	1.8546
96 Armour	0.5450	1.2350	.0091	0.1230	0.0	.0287	2.1836
97 Atlas	0.7675	1.7000	.0	0.7889	0.0494	-.0290	3.5803
98 Eureka	0.5750	1.4250	.1818	0.2730	0.0	-.1001	2.9821
99 Gen. Accept.	0.6063	1.5500	.0235	2.6824	0.0645	-.0356	4.1735
100 Tri-Continent.	o.4600	1.4499	.0	0.8767	0.0583	.0098	7.7825
101 Van Norman	0.2917	0.9924	.0089	0.2505	0.0611	-.0038	2.6685
1956							
102 Alleghany	1.5000	2.0833	.0	0.5407	0.0	-.1062	1.5374
103 Armour	0.5300	1.3700	.0103	0.1156	0.0	-.1489	2.9856
104 Atlas	0.5800	1.4000	.0	0.6506	0.0343	-.1210	2.7559
105 Gen. Accept.	0.5750	1.5063	.0328	0.2234	0.0664	-.0367	3.9921
106 Tri-Continent.	0.5348	1.5308	.0	0.4935	0.0736	.0139	*
107 Van Norman	0.2311	0.8144	.0099	0.2505	0.0744	-.1142	2.3226
1957							
108 Alleghany	0.8000	1.3000	.0	1.8780	0.0	-.3254	1.5638
109 Armour	0.2917	0.8542	.0117	0.0864	0.1000	-.0523	2.1118
110 Atlas	0.4600	1.1300	.0	0.5320	0.0850	-.2493	2.2131
111 Gen. Accept.	0.4500	1.4188	.0540	2.5186	0.0705	-.0346	*
112 Mack Trucks	0.2265	0.7456	.0095	0.1761	0.0804	-.1597	2.1095
113 Molybdenum	0.1625	0.5479	.0141	0.1775	0.0365	-.3494	1.8685
114 Northspan	0.7813	1.3125	.0091	0.3706	0.0	-.5013	1.6253
115 Tri-Continent.	0.5930	1.5907	.0	0.2387	0.0892	-.0622	*
116 Van Norman	0.1061	0.3220	.0113	0.2126	0.0377	-.5546	1.5611
1958							
117 Alleghany	1.7667	2.4500	.0	1.8354	0.0	.2809	1.6666
118 Armour	0.7000	1.3542	.0136	0.0864	0.0	.1817	1.9414
119 Atlas	0.6300	1.1800	.0	0.5297	0.0703	-.0415	1.7333
120 Gen. Accept.	0.7063	1.7000	.1538	2.4094	0.0588	.0567	*
121 Mack Trucks**	0.3933	1.0735	.0107	0.1700	0.0552	.1199	2.3518
122 Molybdenum	0.5646	0.9511	.0169	0.1750	0.0106	.1788	1.4678
123 Northspan	0.5000	0.7292	.0103	0.3724	0.0	-.4055	1.2656
124 Sperry Rand	0.3950	0.9125	.0094	0.1238	0.0351	.0828	1.8398
125 Symington W.	0.6875	1.1688	.0087	0.2078	0.0257	.1012	1.5736
126 Tri-Continent.	1.3024	2.2241	.0	0.2085	0.1048	.1481	2.8404
127 Van Norman	0.3182	0.6212	.0131	1.2806	0.0	.1726	1.4634

Observation	y	x	1/t	d	r	logX/$\overline{X}$	z
			1959				
128 Alleghany	2.8667	3.6833	.0	1.6522	0.0	.0949	1.8245
129 Armour	1.3792	2.3667	.0163	0.0884	0.1000	.2971	4.3583
130 Atlas	0.4700	0.9600	.0	0.4803	0.0500	-.1630	1.7101
131 Mack Trucks	0.5931	1.4877	.0123	0.2061	0.0394	.1128	3.3844
132 Martin	0.6594	1.1531	.0093	0.1220	0.0347	-.0307	1.6076
133 Molybdenum	1.0812	1.4833	.0213	0.1657	0.0	.0188	1.3599
134 Pacific Pet.	0.5263	0.6974	.0099	0.2232	0.0	-.1406	1.1871
135 Northspan	0.1771	0.3125	.0117	0.0360	0.0	-.6062	1.2661
136 Sperry Rand	0.4650	0.9525	.0106	0.1238	0.0336	-.0361	1.7086
137 Symington W.	0.5688	1.0688	.0098	0.2048	0.0561	-.1365	1.6657
138 Tri-Continent.	1.1444	2.1150	.0	0.1633	0.0753	-.0597	3.8737
139 Van Norman	0.3220	0.6666	.0155	1.2806	0.0	-.0390	1.5375
			1960				
140 Alleghany	1.9167	2.6333	.0	1.3787	0.0	-.1022	1.7079
141 Armour	1.6750	2.6750	.0202	0.0683	0.0256	.2713	*
142 Atlas	0.2200	0.5500	.0	0.4803	0.0500	-.3365	1.6192
143 Gen. Accept.	0.2594	0.8934	.0095	0.3906	0.0761	-.0310	2.4482
144 Guerdon	0.1736	0.5664	.0385	0.2519	0.0865	-.1733	1.8614
145 Hilton	0.1324	0.7366	.0076	0.1664	0.0485	-.0740	2.8275
146 Mack Trucks**	0.3613	1.0380	.0144	0.2000	0.0561	-.2489	2.3953
147 Martin	0.7171	1.5157	.0105	0.1099	0.0294	.1554	2.4609
148 Molybdenum	0.7975	1.2830	.0286	0.1657	0.0	-.1703	1.5406
149 Pacific Pet.	0.3059	0.5099	.0112	0.0899	0.0	-.1214	1.3028
150 Rio Algom	0.1038	0.3454	.0136	0.0456	0.0	.0988	1.6370
151 Sperry Rand	0.3060	0.7650	.0122	0.1237	0.0418	-.1542	1.8008
152 Symington W.	0.5063	1.2250	.0110	0.2048	0.0612	-.0832	2.3397
153 Tri-Continent.	1.0142	1.9883	.0	0.1633	0.0583	-.0262	4.1982
154 Van Norman	0.3485	0.7424	.0190	0.8381	0.0	-.0879	1.6111
			1961				
155 Alleghany	2.1500	2.9167	.0	0.3604	0.0046	-.1235	1.7880
156 Armour	1.6750	2.6750	.0267	0.1505	0.0299	.0243	*
157 Atlas	0.2100	0.4400	.0	0.4803	0.0	-.2048	1.4216
158 Gen. Accept.	0.4375	1.2159	.0107	0.7819	0.0613	.0973	2.7485
159 Guerdon**	0.1340	0.4522	.0714	0.2827	0.0	-.3870	1.7591
160 Hilton	0.2872	0.8110	.0084	0.1638	0.0440	-.0776	2.0015
161 Mack Trucks	0.5431	1.4021	.0174	0.2048	0.0399	.0469	3.1197
162 Martin	1.0095	1.9582	.0120	0.0333	0.0087	.0629	3.5047
163 McCrory	0.4906	1.0813	.0057	0.8272	0.0416	.1427	1.9328
164 Molybdenum	0.5708	1.0753	.0435	0.1139	0.0	-.1870	1.6735
165 Pacific Pet.	0.3816	0.6053	.0130	0.0892	0.0	-.0162	1.2979
166 Rio Algom	0.2185	0.4443	.0163	0.0442	0.0	.1094	1.4047
167 Sperry Rand	0.5300	1.0061	.0143	0.1295	0.0	-.1374	1.6417
168 Symington W.	0.7625	1.5375	.0127	0.1916	0.0520	.0	2.3079

Observation	y	x	1/t	d	r	log X/X̄	z
1961 (Cont'd)							
169 Teleregister	0.4853	1.0417	.0241	0.0432	0.0	-.3972	1.9446
170 Textron	0.4150	1.0075	.0035	0.2867	0.0496	-.0148	2.0186
171 TWA	0.2375	0.5875	.0069	0.4543	0.0	-.2850	1.6394
172 Tri-Continent.	1.8704	2.8822	.0	0.1295	0.0609	.1485	*
173 Van Norman	0.2538	0.7197	.0247	0.8381	0.0	.0	1.8970
1962							
174 Alleghany	1.5833	2.5000	.0	0.3623	0.0	.0408	2.6283
175 Armour	1.1500	1.8781	.0392	0.1452	0.0373	-.1723	1.9248
176 Atlas	0.1500	0.3600	.0	0.5575	0.0	-.1054	1.4566
177 First Nat.R'lty.	0.1876	0.5473	.0091	0.7344	0.0320	-.4447	1.7437
178 Gen. Accept.	0.3250	0.9813	.0123	0.7892	0.0708	-.1142	2.3840
179 Guerdon	0.0099	0.2224	.5000	0.2827	0.0	-.3338	2.5010
180 Hilton**	0.1609	0.5803	.0093	0.1638	0.0611	-.1243	1.9716
181 Jeff.Lake Pet.	0.3170	0.5893	.0098	0.2834	0.0	-.4971	1.4100
182 Mack Trucks	0.3558	1.0735	.0220	0.2045	0.0518	-.0543	2.5931
183 Martin	0.6970	1.5188	.0140	0.0541	0.0449	-.0304	2.6104
184 McCrory	0.2563	0.8688	.0061	0.8289	0.0460	-.1623	2.3684
185 Molybdenum	0.3833	0.8431	.0909	0.1139	0.0	-.2037	1.7175
186 Pacific Pet.	0.3421	0.6579	.0154	0.1367	0.0	-.0488	1.4689
187 Rio Algom	0.1462	0.4362	.0202	0.0442	0.0	.1186	1.6549
188 Realty Eq.	0.2334	1.0154	.0090	0.6537	0.0276	.1189	3.4309
189 Sperry Rand	0.2750	0.5508	.0172	0.1271	0.0	-.3059	1.4481
190 Teleregister**	0.1492	0.3141	.0339	0.0791	0.0	-.5596	1.3571
191 Textron	0.4150	1.0425	.0037	0.2871	0.0480	-.0330	2.1271
192 TWA	0.2000	0.5031	.0075	0.4543	0.0	-.0777	1.5882
193 Tri-Continent.	1.2774	2.2135	.0	0.1249	0.0394	.0838	3.0298
194 Univ. Amer.	0.1818	0.5591	.0190	0.4394	0.0	-.0782	1.8009
1963							
195 Alleghany	1.7333	2.6000	.0	0.3598	0.0113	-.1263	2.2347
196 Armour	1.1438	1.9844	.0741	0.1622	0.0353	-.0888	2.3530
197 Atlas	0.2200	0.5000	.0	0.5589	0.0	.0619	1.5117
198 First Nat.R'lty.	0.1771	0.5513	.0103	0.6403	0.0333	-.1178	1.8030
199 Gen. Accept.	0.2219	0.9156	.0144	0.7131	0.0546	-.0881	2.8676
200 Hilton	0.0774	0.3465	.0105	0.1616	0.0471	-.3641	1.8199
201 Jeff. LakePet.	0.4018	0.9821	.0110	0.2827	0.0182	.1003	1.9992
202 Mack Trucks	0.3539	1.1273	.0298	0.2045	0.0471	.0049	2.9139
203 Martin	0.4751	1.2174	.0168	0.0532	0.0499	-.0124	2.4894
204 McCrory	0.1219	0.5563	.0066	0.8395	0.0719	-.2164	2.1591
205 Pacific Pet.	0.3257	0.5724	.0189	0.0860	0.0	-.1292	1.3614
206 Rio Algom	0.1162	0.5635	.0267	0.0442	0.0	.0543	2.2316
207 Realty Eq.	0.1328	0.8355	.0101	0.6534	0.0330	-.1074	3.4513
208 Sperry Rand	0.2857	0.6535	.0217	0.1267	0.0	.0492	1.6192
209 Teleregister	0.2206	0.4081	.0571	0.0971	0.0	.0274	1.3300

Observation	y	x	1/t	d	r	log X/$\overline{X}$	z
			1963 (Cont'd.)				
210 Textron**	0.5245	1.3444	.0039	0.2036	0.0371	.0435	2.8478
211 TWA	0.6000	1.2813	.0083	0.4543	0.0	.2604	2.0997
212 Tri-Continent.	1.5933	2.5373	.0	0.1207	0.0550	-.0192	2.9216
213 Univ. Amer.	0.1182	0.4818	.0247	0.4356	0.0	-.1487	1.9416
214 Uris	0.8502	1.6341	.0073	0.2442	0.0260	-.0660	2.2770
			1964				
215 Alleghany	2.3500	3.0500	.0	0.3344	0.0096	-.0216	1.6203
216 Armour	1.7313	2.7344	.0667	0.0447	0.0256	.0896	*
217 Atlas	0.1850	0.3900	.0	0.4764	0.0	-.1206	1.3986
218 First Nat. R'lty.	0.1198	0.4912	.0117	0.6655	0.0	-.0476	1.9598
219 Gen. Accept.	0.2719	1.0563	.0174	0.2848	0.0521	.0516	3.2711
220 Hilton	0.1073	0.4688	.0120	0.1270	0.0	.1167	1.9783
221 Jeff. Lake Pet.	0.4139	1.1003	.0127	0.2744	0.0358	-.0148	2.3426
222 Mack Trucks**	0.3738	1.1380	.0465	0.0956	0.0463	-.1267	2.8049
223 Martin**	0.4390	1.1908	.0210	0.0532	0.0510	.0423	2.5978
224 McCrory	0.1969	0.7156	.0071	0.8305	0.0559	.1156	2.1858
225 Pacific Pet.	0.3487	0.6695	.0244	0.0860	0.0	-.0475	1.4731
226 Rio Algom	0.0938	0.4064	.0392	0.0442	0.0427	-.1133	1.8815
227 Realty Eq.**	0.1784	0.9235	.0116	0.4520	0.0323	-.0870	3.4445
228 Sperry Rand	0.2344	0.5474	.0294	0.1267	0.0	-.1955	1.5633
229 Bunker Ramo	0.1250	0.5788	.1818	0.0035	0.0	-.3738	2.2143
230 Textron	0.8542	1.6479	.0041	0.2249	0.0324	.9239	2.3171
231 Tri-Continent.	1.8593	2.8611	.0	0.1248	0.0327	.0466	*
232 TWA	1.4788	2.3530	.0092	0.4543	0.0	.1728	2.3713
233 Univ. Amer.	0.1864	0.5864	.0351	0.4356	0.0	.0807	1.8514
234 Uris	0.5651	1.3635	.0080	0.2442	0.0374	-.0344	2.6458

*z is undefined or equal to ∞.

**These warrants had graduated exercise prices such that the next jump occurred within 12 months of the observation. To give effect to this imminent change the following (arbitrary) method was used to calculate the adjusted exercise price: (1) The difference was taken between the exercise price prevailing at the time of the observation and the price that was to prevail after the next increase. (2) This difference was prorated over the 12 month period prior to the date of the increase. (3) The prorated share of the time that had elapsed at the time of the observation was added to the prevailing exercise price. For example, in November 1963 the exercise price for the Textron warrant was $25.00, but by May 1, 1964 the exercise price was to jump to $30.00. The above procedure would add $2.29 (the prorated share for $5\frac{1}{2}$ months) to the prevailing exercise price, yielding an exercise price of $27.29.

No attempt was made in this study to adjust the exercise price in cases where senior securities, selling below par, could be used in lieu of cash to exercise the warrant. As an example, consider the General Acceptance Corporation: 50 warrants can be exercised by delivering with the warrants one General Acceptance $6\frac{1}{4}$% debenture due 1974 rather than \$1000 in cash. On November 30, 1964, this debenture was selling at \$965.63, so in effect, the exercise price of the warrants was reduced 3.4% to \$19.31 from the stated exercise price of \$20.00. In cases where the senior security is substantially below par it might seem that some adjustment is necessary. But the reduction in the exercise price may be illusory for if the senior security is well below par the common is usually well below the stated exercise price. To calculate a "true" exercise price it is necessary to estimate what the value of the senior security would be if the common advances beyond the stated exercise price for only then will the warrant ever be exercised. If it is expected that the common will advance substantially, then it seems reasonable to expect the senior security to concomitantly approach or exceed par. Thus the discount from the stated exercise price will vanish. (Expectations about interest rates are also involved in this analysis: if an investor believed that interest rates would continually rise, then he might believe that the senior security would remain below par in spite of the improved prospects for the company. Similarly, if he believed interest rates would fall, a discount in the senior security would not be considered a discount in the exercise price.)

In calculating d, not only outstanding warrants but all outstanding options that could give rise to new shares were used in the numerator. This was necessary because in some cases there were only a modest number of outstanding warrants but a relatively large number of other dilutants and it seemed reasonable that investors would consider potential dilution from any source.

In calculating r, stock dividends were ignored unless the warrant was not protected against dilution. When a warrant is protected, the warrant holder "benefits" as much as the common stock holder from a stock dividend.

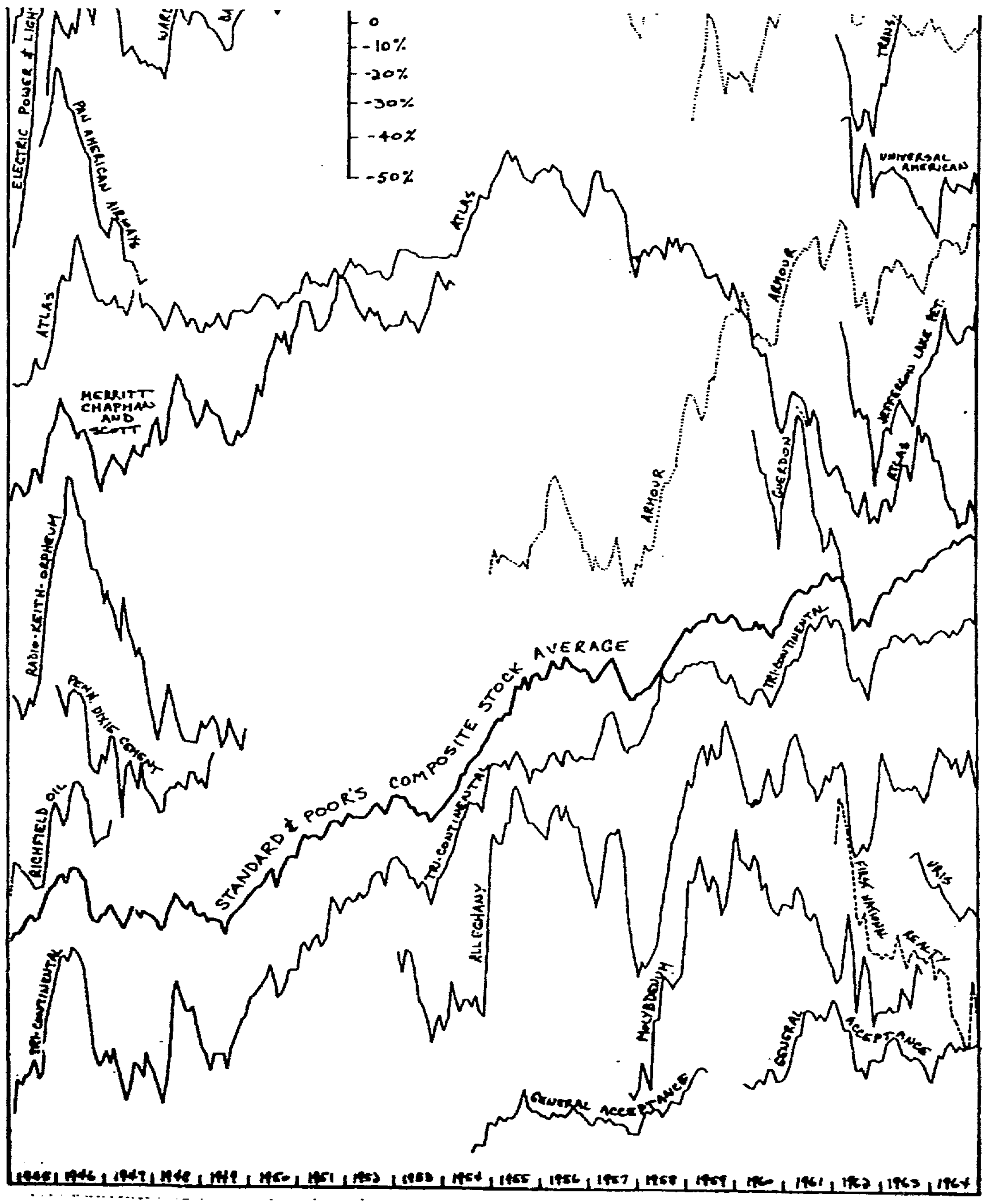
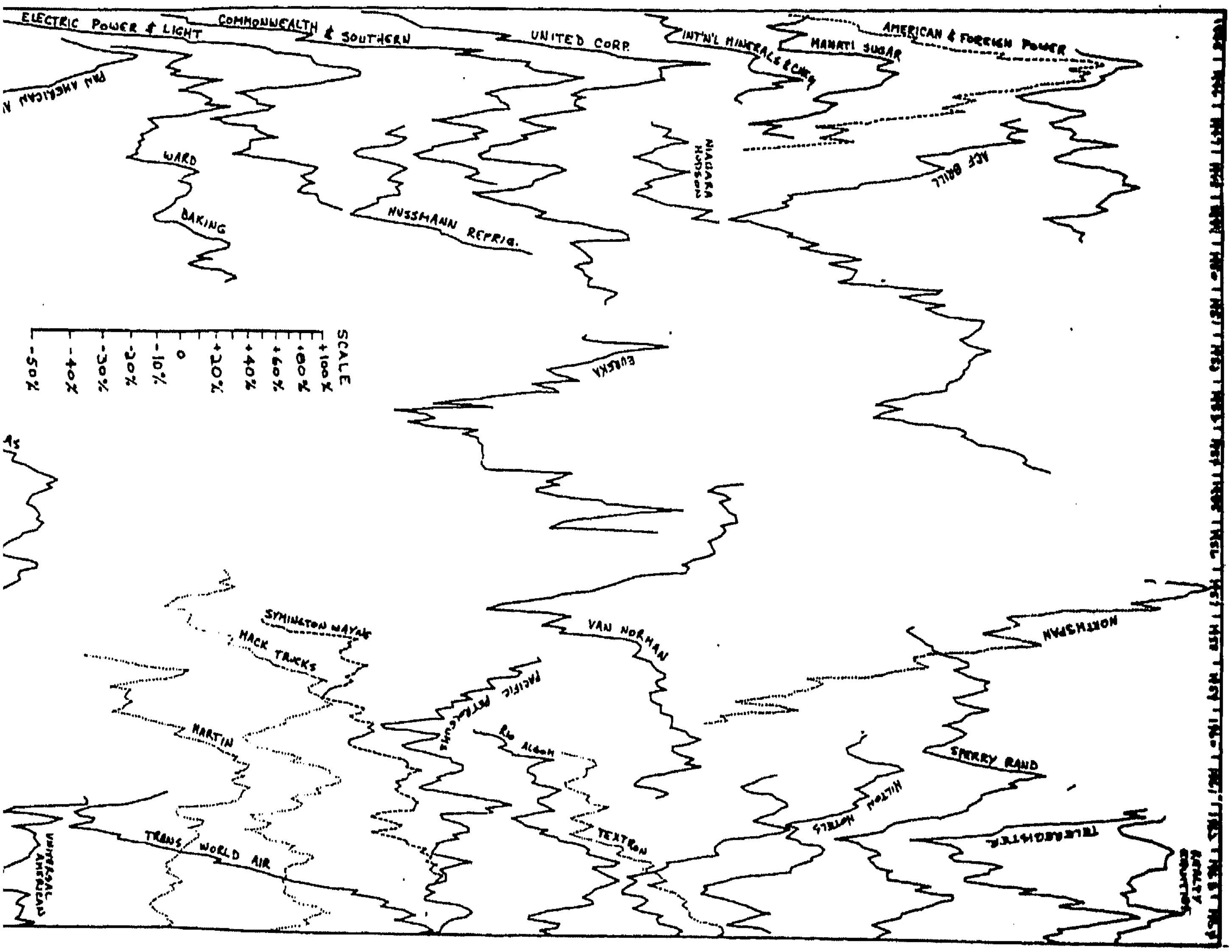


Fig. 17--Price chart of common stocks in sample


ELECTRIC POWER & LIGHT
COMMONWEALTH & SOUTHERN
UNITED CORP.
INT'N'L MINERALS & CHEM.
MANATI SUGAR
AMERICAN & FOREIGN POWER
WARD
BAKING
NUSSMANN REFRIG.
NIAGARA HUDSON
SCALE
+100%
+80%
+60%
+40%
+20%
0
-10%
-20%
-30%
-40%
-50%
EUREKA
SYMINGTON WAYNE
MACK TRUCKS
VAN NORMAN
MARTIN
TRANS WORLD AIR
TEXTRON
SPERRY RAND
UNIVERSAL AMERICAN

APPENDIX B

LISTING OF THE DELETED OBSERVATIONS

Warrants listed on the American Stock Exchange during any November in the years 1945 through 1964 represent observations from the universe under study. Some of these observations were deleted from the test sample. These deletions will be enumerated here with the reason for their exclusion.

In December 1961, the Van Norman Corporation was merged with the Universal American Corporation. Each share of Van Norman common was replaced with one share of Universal American common plus one Universal American (1962) warrant. Thus, each Van Norman warrant became an option on one share of Universal American common plus one Universal American (1962) warrant. The Van Norman warrant, born in 1955, was then renamed the Universal American (1955) warrant. Because of this unique conversion privilege, the Universal American (1955) warrant violates the a priori conditions supporting this study. For instance, it is not true that this warrant's price must be less than the price of the common--it must be less than the price of the common plus the price of the Universal American (1962) warrant. Furthermore, this warrant will have an intrinsic value long before the common reaches the exercise price of $16. 50. The point at which the "package" of one Universal American common share plus one (1962) warrant equals $16. 50 is the point at which the (1955) warrant begins to have a minimum value greater than zero. This point is reached when the Universal American common is approximately $12, for then the (1962) warrant will be about $4. 50. Since this point will vary with the market valuation of the (1962) warrant, it is difficult to express the value of the (1955) warrant as a function of the price of the common. Therefore, the following observations were deleted:

D1	Universal American (1955)	Nov. 1962
D2	Universal American (1955)	Nov. 1963
D3	Universal American (1955)	Nov. 1964

In November 1962 the prevailing exercise price for the Symington Wayne warrant was $10 and the price was to jump to $15 on May 1, 1963. Holders of the warrant apparently treated this enormous jump as almost equivalent to an expiration of their privilege, for during this $5\frac{1}{2}$ month period, 200, 240 of the 261, 281 outstanding warrants, or 77%, were exercised. Shortly after, the warrant was delisted from the American Stock Exchange and was traded over-the-counter.

A company's motive in graduating an exercise price is clear: less of the potential increase in the value of the corporation is relinquished. The effect of a graduated price, however, is not always patent. In the case of the Symington Wayne warrant, the effect was to cause conversion of a substantial number of warrants. Had the jump in the exercise price been less severe, it is probable that fewer warrants would have been exercised. The crucial variable determining the number of warrants that will be exercised is the value of the common stock immediately prior to the increase in the exercise price--if the stock is below the prevailing exercise price, obviously, no warrants will be converted; if it is substantially above and the jump in the exercise price is relatively large, almost all of the warrants will be converted. In setting the terms of conversion, management must realize that if the graduation is too great, it is possible that most of the warrants will be exercised long before their expiration. If this is inimical to the interests of management because the need for sudden and massive doses of cash is not anticipated or because it is wished to delay the dilution of the common as long as possible, it would be wiser to relinquish less of the potential increase in the value of the corporation by other means, viz., setting a higher but constant exercise price.

Because the adjustment described on page did not seem appropriate for such a large increase in exercise price, the following observation was deleted:

D4 Symington Wayne Nov. 1962

In 1946, although the New York City Omnibus warrant was listed on the American Stock Exchange, there was no real market--it rarely traded and no transactions took place in November, so this observation was deleted:

D5 New York City Omnibus Nov. 1946

In 1960, the United Industrial warrant was traded on the American Stock Exchange but because the Corporation was beset with internal difficulties it was delisted at the end of December 1960. The move was well known in advance so that in November it could not be considered a member of the population under study, so it was deleted:

D6 United Industrial Nov. 1960

BIBLIOGRAPHY

Subjective Probability and the Expected Utility Hypothesis

Carnap, Rudolph. Logical Foundations of Probability. Chicago: University of Chicago Press, 1950.

Friedman, Milton, and Savage, L. J. "The Utility Analysis of Choices Involving Risk," The Journal of Political Economy, LVI (1948), 279-304. Reprinted in Stigler, George J., and Boulding, Kenneth E. (eds.) Readings in Price Theory, 57-96. Chicago: Richard D. Irwin, 1952.

Keynes, John Maynard. A Treatise On Probability. London: Macmillan and Company, 1921.

Luce, Duncan R., and Raiffa, Howard. Games and Decisions: Introduction and Critical Survey. New York: John Wiley and Sons, 1957.

Markowitz, Harry M. Portfolio Selection: Efficient Diversification of Investments. New York: John Wiley and Sons, 1959.

Ramsey, Frank Plumpton. The Foundations of Mathematics and Other Logical Essays. New York: Harcourt Brace and Company, 1931.

Savage, Leonard J. The Foundations of Statistics. New York: John Wiley and Sons, 1954.

Shackle, G. L. S. Expectations in Economics. Cambridge, England: University Press, 1949.

__________. Uncertainty in Economics and Other Reflections. Cambridge, England: University Press, 1955.

Von Neumann, John, and Morgenstern, Oskar. The Theory of Games and Economic Behavior. 3rd ed.; Princeton: Princeton University Press, 1954.

Evaluation of Options

Boness, A. James. "Elements of a Theory of Stock Option Value," Journal of Political Economy, LXXII (1964), 163-75.

Filer, Herbert. Understanding Put and Call Options. New York: Crown Publishers, Inc., 1959.

Fried, Sidney. The Speculative Merits of Common Stock Warrants. New York: R. H. M. Associates, 1961.

Hallingby, Paul, Jr. "Speculative Opportunities in Stock Purchase Warrants," The Analysts Journal (now the Financial Analysts Journal), III (1947), 41-49.

Kassouf, S. T. Evaluation of Convertible Securities. Middle Village, New York: Analytic Investors, Inc., 1962.

Skelly, William S. Convertible Bonds: A Study of Their Suitability for Commercial Bank Bond Portfolios. New York: Salomon Bros. and Hutzler, 1959.

Whiting, Richard. Profitable Trading With Puts and Calls. New York: R. H. M. Associates, 1960.

__________. Increase Your Investment Profits With Warrants and Convertibles. Garden City, New York: Investing and Management Compass, Inc., 1961.