Econometrica, Vol. 37, No. 4 (October, 1969)

AN ECONOMETRIC MODEL FOR OPTION PRICE WITH IMPLICATIONS FOR INVESTORS' EXPECTATIONS AND AUDACITY

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Stock option prices result from the interaction of many investors of many persuasions. Previous theories of option price have been micronormative, thus having tenuous connections with observed option prices. This paper makes no assumptions about individual expectations or utilities; instead a model is specified for actual prices and tested against twenty years of data. Inferences are then made concerning the aggregate change in investors' expectations and risk attitudes through time.

1. INTRODUCTION

PRESENT THEORIES OF option price [**1**, **2**, **3**, **4**, **5**, and **7**] assume that an individual investor's expectations and risk attitude can be expressed by certain parameters. If these subjective phenomena differ among investors, it is not simple to test these theories against observed option prices which result from the interaction of many investors.

This paper will specify an econometric model for *observed* prices (Sections 2 and 3), and give the results of an empirical test of the model using twenty years of price data on common stock purchase warrants with some inferences concerning expectations and risk attitudes (Section 4).

2. SOME A PRIORI CONSIDERATIONS

It will be convenient to consider in detail one specific form of security option: common stock purchase warrants. The model to be constructed can be extended *mutatis mutandis* to any convertible security; Section 4 explicitly relates the model of warrant price to convertible bond price.

A common stock purchase warrant is a security which the holder may exchange, at his option, for equity capital. The exchange may be effected by surrendering the warrant and a prespecified sum of money before a prespecified date to the corporation issuing the common stock. The act of conversion is usually called the *exercise* of the warrant, and the accompanying money the *exercise price.*

Let X be the price of a unit of common, let Y be the price of a warrant, and let A be the exercise price of Y. For the remainder of this study, we shall measure X and Y in units of A so that $Y/A = y$ is the price of $1/A$ warrants and $X/A = x$ is the price of $1/A$ common shares. In this way, y plus \$1 can be converted into shares

[1] This paper is based in part on the author's doctoral dissertation completed at Columbia University in 1965. The facilities of the Western Data Processing Center were used. The author is indebted to Professor Arthur F. Burns, whose insatiable appetite for facts led to a model that could be tested empirically. The author is also grateful to two referees who have pointed out a number of inconsistencies in an earlier version of this paper. The author is solely responsible for all remaining errors.

with a value of x dollars, i.e., the exercise price of y will always equal 1. Of course if we specify a model for $y = f(x, \ldots)$, we have also specified a model for Y.[2]

The world we live in, alas, is not always linear. For a model of warrant price this is immediately evident: y cannot be linear in x. A warrant cannot sell for less than the difference between the price of its associated common stock and its exercise price, except momentarily, nor for more than the price of the common stock; i.e., $x \geqslant y \geqslant x - 1$, and since warrants never take on negative values, this is equivalent to $x \geqslant y \geqslant x - 1$, when $x > 1$, and $x \geqslant y \geqslant 0$ otherwise.

In the x-y plane, these conditions carve out a rhomboidal region in the positive quadrant—a region to which the functional relation is restricted. In Figure 1, this region is the unshaded portion of the quadrant.

The lower bound of this region is defended by arbitrageurs: if the warrant could be purchased for less, a simultaneous short sale of the common would yield an instantaneous and riskless profit. Thus arbitrage activity tends to raise the price of the warrant and depress the price of the common until the price of the warrant is above or at the lower bound of the region defined above.

The upper bound is dictated by reason: the warrant is inferior to the common with respect to dividends, voting power, and length of life. Since the warrant has no offsetting characteristics, its value should not exceed the value of the common.

In addition, it seems clear that the graph of this function in the x-y plane should pass through the origin: if the common is worthless then so is the warrant. (If the warrant takes the value zero for positive values of the common, then the implication is that there is no possibility for the common to advance beyond the exercise price before the date of expiration.) Now y cannot be linear in x *and* have a graph passing through the origin, except the case where $y = x$.

The standard way to estimate $f(x, \ldots)$ is to assume the function "sufficiently smooth" in all its arguments so that it can be represented by a Taylor polynomial, i.e., to introduce higher powers of the explanatory variables and cross products between these new terms as "interaction" variables. (Interaction terms will be necessary. As one of the explanatory variables other than x changes, the graph of f will move up and down in the admissible region of Figure 1, and $\partial f/\partial x|_x$ will not remain constant for changes in this other variable.) If there are four or five explanatory variables for y, this polynomial might have as many as twenty terms and as many coefficients to estimate.

But there is a more elegant approach that permits nonlinearity between x and y without introducing so many interaction and higher power variables. Consider

[2] If we assume $Y(X, A)$ is homogeneous of degree 1 with respect to X and A then $Y(X/A) = Y(X, A)/A$. Samuelson [**5**, p. 18] explicitly makes this assumption, but he does not recognize it as an assumption. His justification is a "property of competitive arbitrage ... a property that says no more than that two shares always cost just twice one share." Unfortunately, this is a property that does not always hold in the stock market. It is notorious that stock splits occasionally change nothing except the number of shares outstanding and yet it has been observed that the price of the split shares is not that fraction of the unsplit share that the number of original shares is of the total number of new shares. There is a distinct possibility of the existence of "share price illusion" in the market—whereas it would be hard to imagine two \$5 bills to be worth more than one \$10 bill, it is quite possible investors feel better after a two for one stock split, ceteris paribus. The model to be developed here allows a test of this "property" and the evidence suggests that "share price illusion" exists.

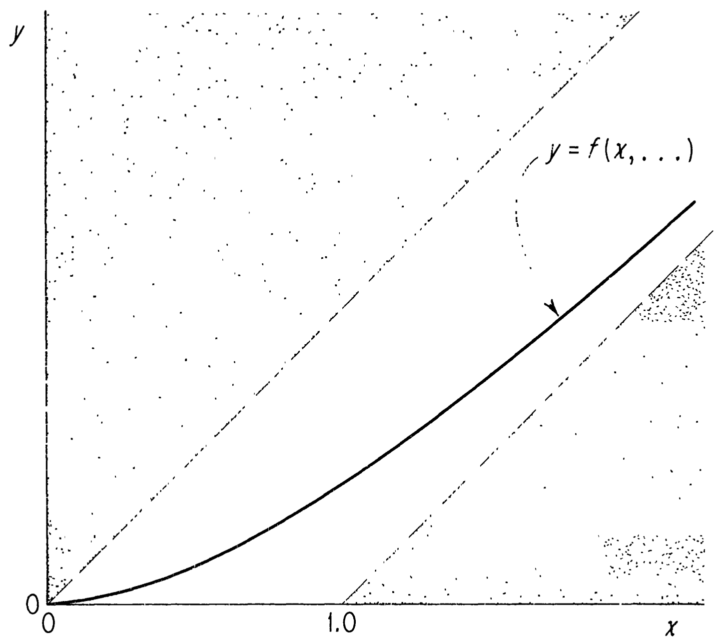

FIGURE 1.—Possible relationship between price of common and price of warrant.

the family of functions:

$$y(z) = (x^z + 1)^{1/z} - 1$$

where $1 \leqslant z < \infty$. Then $y(1) = x$, and $y(\infty) = 0$ when $x < 1$, $y(\infty) = x - 1$ otherwise. That is, $y(1)$ is the upper boundary of the rhomboidal region to which the model is restricted, and $y(\infty)$ is the lower boundary. Every point in the region lies on a unique z curve. (Figure 2 shows the graphs of some selected z curves.) If the functional relation between x and y were of this form, then a model for y could be approached indirectly by building a linear model for z.

3. FORM OF THE MODEL

It is well known that the length of time remaining before expiration affects the price of the warrant. Let t be the number of months before expiration, then $z = g(t, \ldots)$. If it is assumed that z is a decreasing function of t, with $z = \infty$ when $t = 0$, it seems reasonable as a first approximation that $z = \alpha + \beta/t$. Figure 3 is a scatter of $1/t$ on z for a selected warrant during a period when most of the other explanatory variables to be introduced were almost constant. This indicates that the assumption of linearity between z and $1/t$ is not too wild. Other variables that might affect z (i.e., y) are: R, the dividend yield on common (per cent); D, the number of outstanding warrants per number of outstanding shares (i.e., potential dilution ratio); E_1, the slope of the least squares line fitted to logarithms

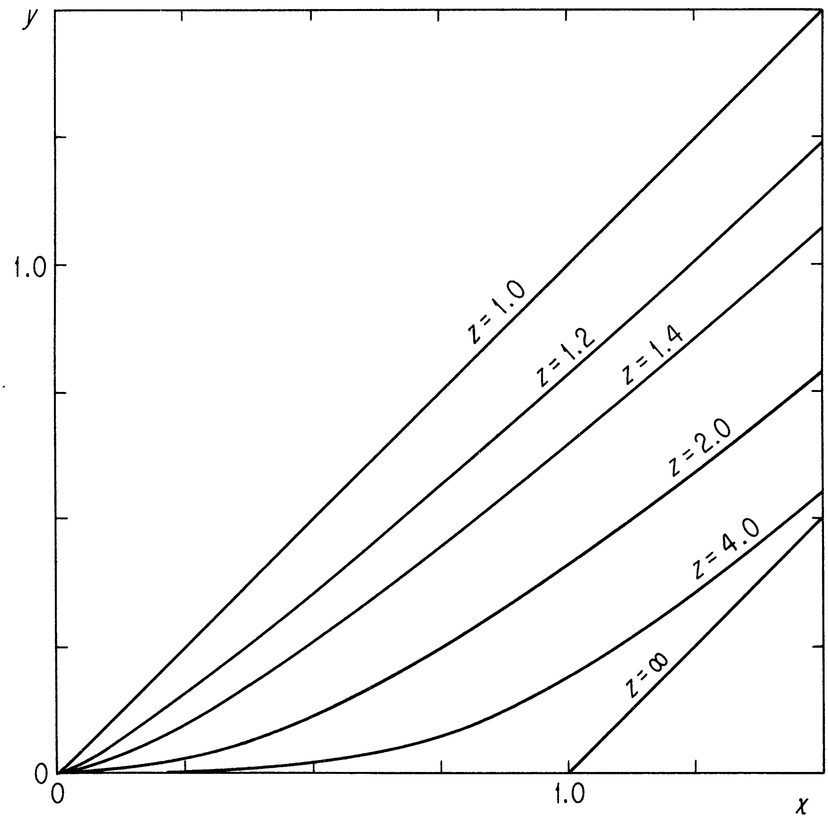

FIGURE 2.—Selected members of the family of functions $y(z) = (x^z + 1)^{1/z} - 1$.

of the monthly mean price for common for the previous eleven months; and E_2, the standard deviation of (natural) logarithms of the monthly mean price for the common for the previous eleven months.

The dividend yield of the common will add to the return promised by the stock, so that the higher this yield, all else equal, the higher the value of the stock *relative* to the value of the warrant. In other words, the higher the dividend yield, the less valuable the warrant as an alternative investment, so it is reasonable to expect z to be monotonically increasing with R, the dividend yield on the common.

It has also been hypothesized that the higher D, the less valuable the warrant, or the higher z.[3]

It has also been presumed that the greater the "speculative possibilities" of the common stock, the greater the price of the warrant.[4] E_1 and E_2 are possible proxies for investors' expectations for the common, if recent price performance of the stock is believed to yield a clue to future performance.

[3] Benjamin Graham and David L. Dodd, *Security Analysis*, 4th ed., McGraw-Hill (New York, 1965) p. 658.

[4] Ibid, p. 658.

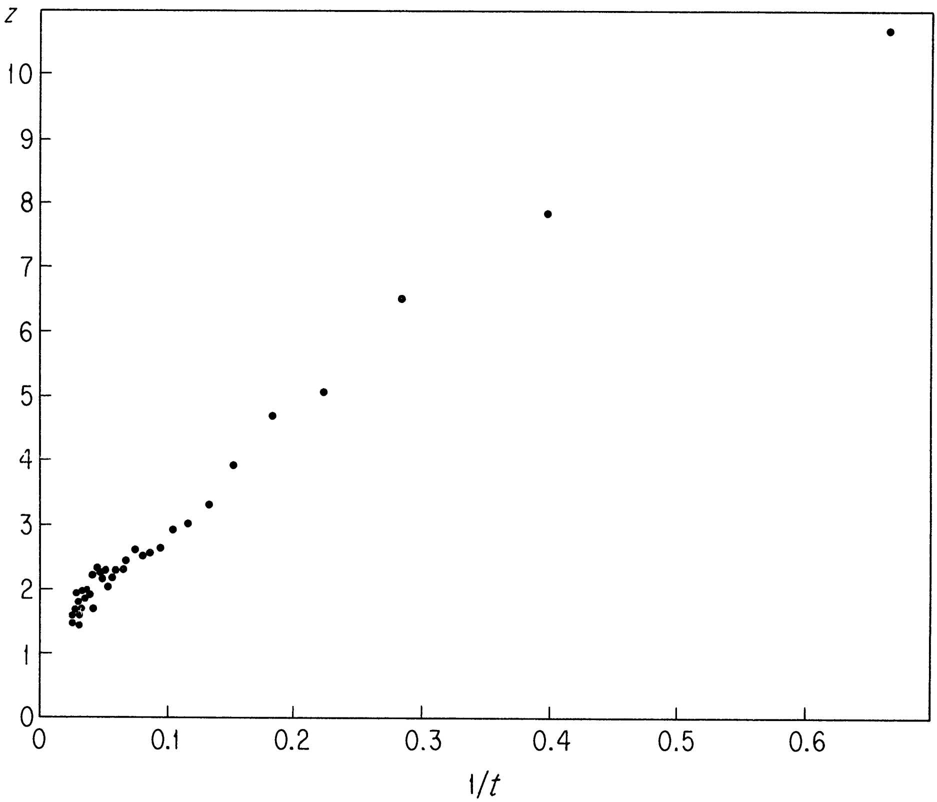

FIGURE 3.—Relation between z and $1/t$, Pennsylvania Dixie Cement Warrant, January, 1946–April, 1949.

It would be too much to hope that the price of a warrant slides up and down z curves with varying x, changing curves only when one of the other explanatory variables changed. To allow departure from these curves, x is introduced as an explanatory variable for z. Also to test whether investors have a preference for warrants with different exercise prices, A is introduced into the model.

The model proposed for z is

$$z = k_1 + k_2/t + k_3R + k_4D + k_5E_1 + k_6E_2 + k_7x + k_8A + \varepsilon,$$

where ε is a stochastic variable with zero mean. Even if the least squares prediction of z is unbiased, the resulting estimate of y is biased downward:

$$y(z) = (x^z + 1)^{1/z} - 1,$$

and it can be shown that $y'' > 0$. Therefore, by Jensen's inequality,[5] $E(\hat{y}) > y(E(\hat{z})) = \hat{y}$, where $\hat{y} = y(\hat{z})$. It should be noted, however, that from the Slutsky-Cramer theorem and the assumption that the estimate for z is consistent, that the estimate for y is consistent.

[5] William Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, John Wiley (New York, 1966) p. 151.

4. EMPIRICAL FINDINGS

To estimate the seven parameters of the model specified for z, observations were chosen in the following manner: For the years 1945 through 1964 every warrant listed on the American Stock Exchange for the month of November was tabulated. (Warrants were not listed on the New York Stock Exchange during this period; the month of November was chosen arbitrarily.) The means of November's high-low prices were used for x and y.[6] Then, using Newton's method and iterating for z, each observation gave rise to a unique "observed" value of z. (In 12 cases out of 229 observations $y \leqslant x - 1$ so that z is undefined or equal to ∞. Of necessity, these observations could not be used. Their exclusion reduced the efficiency of the data to some extent, but perhaps dropping them from the sample could be justified by noting that these prices were clearly unstable prices—arbitrage in most cases would have yielded finite z's. Of course, this kind of argument would make *every* observation suspect and attempting to introduce criteria so that only "equilibrium" positions remained in the sample would be a herculean task.) The values of the independent variables were taken from Standard & Poor's Monthly Stock Guides and from Moody's Handbooks. The results of a least squares regression utilizing these 217 observations are:

$$\hat{z} = \underset{}{1.220} + \underset{(0.754)}{5.748/t} + \underset{(2.043)}{13.042R} + \underset{(0.136)}{0.360D} + \underset{(0.116)}{0.258E_1} - \underset{(0.986)}{2.814E_2}$$

$$+ \underset{(0.122)}{0.480x} + \underset{(0.007)}{0.016A}, \qquad R^2 = .429, \hat{s} = .942.$$

Correlation Matrix

	$1/t$	R	D	E_1	E_2	x	A
z	.338	.452	.093	.044	−.267	.265	.015
$1/t$		.001	−.068	−.054	.076	−.118	−.118
R			−.003	−.088	−.328	.183	.045
D				.060	.082	.020	−.147
E_1					.351	.038	.115
E_2						−.250	−.025
x							−.316

To see how well Y is explained, the computed z's were converted into computed Y's and the new residuals (the differences between the computed and observed Y's) were calculated. The variance of these residuals is .248, giving rise to a "new" R^2 of .832.

While the coefficient of A suggests that the exercise price *does* influence z, its effect is probably very small. Therefore, to assume homogeneity in the warrant-common relationship is probably not a great distortion of reality.

[6] A referee has correctly observed that simultaneous prices would have been more appropriate. For example, if the high and low observation both lie on the same z-curve, their mean will not. For many of the warrants in this study, simultaneous prices were not available for many days when either the warrant or the common did not trade. Strictly speaking, this is not a model for warrant price, but for *monthly mean* price. Since the relation is apparently convex, the mean of high and low observations is greater than a simultaneous price. Therefore, the estimate obtained by this model is probably greater than the estimate that would be obtained with observations on simultaneous prices.

The coefficient of x is also significantly positive implying that the price of warrants does not slide up and down z curves, i.e., z varies with x.

The negative coefficient of E_2 suggests that the more "volatile" the price of the common has been in the recent past, the more valuable investors consider the warrant. If past volatility is a guide to future volatility, this seems reasonable behavior, for greater volatility increases the expected value of the warrant under many circumstances. (See [7, p. 190].)

The positive coefficient for E_1 suggests that the "better" the performance of the common stock in the preceding eleven months, the *less* investors were willing to pay for the warrant. The correlation between E_1 and E_2, however, casts doubt upon this inference. In any case, the total effect of E_1 is very small; the average value of E_1 in the sample was .057.

The data used covered a period of 20 years during which risk attitudes could easily have fluctuated widely. In an attempt to see if the model could yield clues to investor audacity, the observations were partitioned into two subsets, the first containing 110 observations for the years 1945 through 1957 and the second containing 107 observations for the years 1958 through 1964. Regressions on these subsets yielded the following results:

1945–1957

$$\hat{z} = -1.061 + \underset{(1.000)}{6.922}(1/t) + \underset{(2.967)}{8.768}R + \underset{(0.286)}{1.876}x + \underset{(0.186)}{0.357}D + \underset{(0.015)}{0.074}A,$$

$$R^2 = .505, \hat{s} = 1.101.$$

1958–1964

$$\hat{z} = 1.526 + \underset{(.960)}{2.717}(1/t) + \underset{(1.898)}{13.421}R + \underset{(0.072)}{0.301}x - \underset{(0.801)}{1.340}E_2,$$

$$R^2 = .491, \hat{s} = .488.$$

In the earlier period, neither E_1 nor E_2 were significant in explaining warrant price. In the later period, D, E_1, and A were not significant in explaining price. Since E_1 was not significant in either period, it may not be too great a distortion to say that expectations in both periods were about the same. Proceeding with this assumption, and substituting the mean value of the variables for the combined period, yields the following estimates for z:

1945–1957: $\hat{z} = 2.817$, $\hat{Y} = 4.87$;

1958–1964: $\hat{z} = 2.201$, $\hat{Y} = 6.50$;

1945–1964: $\hat{z} = 2.302$, $\hat{Y} = 6.20$.

The means of the explanatory variables for the different periods are:

	1945–64	1945–57	1958–64
x	.9767	.8581	1.0987
$1/t$	.0321	.0430	.0208
R	.0336	.0411	.0258
D	.4472	.5428	.3488
A	18.4651	18.2970	18.6380
E_1	.0571	.1208	−0.0083
E_2	.1206	.1290	0.1121

The estimates assert that a warrant with 31.2 months before expiration, an exercise price of \$18.47, with the common stock selling slightly below exercise price, and with all the properties indicated by the mean values of the variables indicated above would have sold for \$4.87 in the earlier period, \$6.50 in the later period, and \$6.20 in the combined period, all "on average." If these differences are significant, one might conclude that investors in the later period were more audacious—were willing to pay more for a warrant in this period, ceteris paribus. Since the periods involved span twenty years, and since within each subperiod investor audacity could have fluctuated widely, these tentative conclusions do not provide much information about changing risk attitudes through time. Furthermore, the difference in these estimates may only be due to the effect of shifts in the values of the explanatory variables. If cross sectional estimates could be made, then some indication of investors' subjective schedules might be revealed. Unfortunately, the number of warrants listed in any one year never exceeded twenty-one, and frequently was less than ten, so that this model could hardly yield reliable estimates. But there is a richer universe of options that may be mined for this purpose.[7] It is hoped that further research dealing with larger contemporaneous samples will help quantify shifts in investors' audacity and optimism.

5. CONCLUDING REMARKS

In studying option price, other researchers have often imposed strong assumptions concerning investor behavior, e.g., Samuelson [**5**, pp. 19–20] assumes that investors consider a warrant to be in a different risk class than its associated common stock and would thus price it so that its expected yield compensates for the difference. This is very probably true, but to assume that this difference is *constant* for any price of the common stock is questionable. If leverage, or the elasticity of the warrant determines its riskiness, then investors may wish to *first* determine the probable leverage for different prices of the common and *then* determine the differential they require over the common. It may be that investors

[7] Convertible bonds clothe an intrinsic warrant. Consider the Philips Petroleum $4\frac{1}{4}\%$ debenture, due 1987. On December 31, 1964 this bond sold for \$1185. It was convertible into twenty shares of common which on that date was selling at \$53.50 per share. It has been estimated that without the conversion feature the bond would probably have sold for \$990. Thus it might be argued that this bond consisted of an ordinary debenture, worth \$990, plus twenty warrants, worth \$195, or \$9.75 each. The exercise price was \$49.50 (\$990 divided by 20).

consider a warrant whose associated common stock is low relative to its exercise price more risky than one for which the common is greater than the exercise price. In one sense, this seems reasonable: when the a priori region of the warrant-common relationship is plotted on a log-log grid, as in Figure 4, the elasticity of the warrant is equal to the slope of the graph. Since the region becomes choked off as X increases, the elasticity of the warrant tends to 1, that is, it of necessity becomes less risky, whereas in the lower values of X, the slope of the graph can approach infinity.

This study has attempted to make as few assumptions as possible and has further attempted to actually estimate the relation that exists between warrant

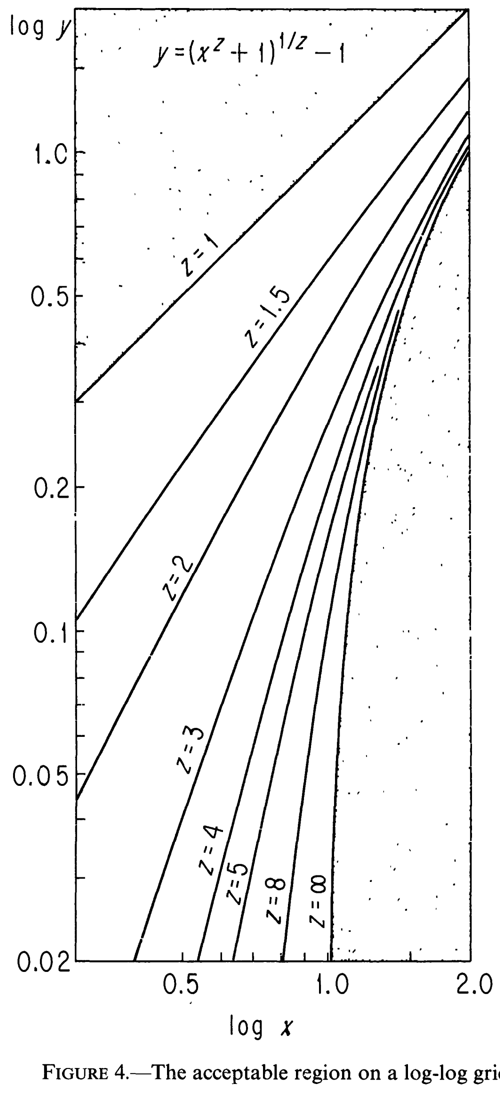

FIGURE 4.—The acceptable region on a log-log grid.

and common price so that the property of leverage and riskiness could *then* be attributed to the warrant.

Assumptions concerning risk attitudes have been purposely avoided; in particular, the usual first approximation of risk neutrality leading to the criterion of maximizing expected value has been rejected. Furthermore, no assumptions were made concerning the distribution of future stock prices and therefore no deductions could be made concerning the distribution of warrant prices. No pretense is made that the foregoing model "explains" the warrant-common price relationship—but it is hoped that it is a good description that may eventually lead to theoretical substantiation.

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