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AN OPTIMAL ALLOCATION BETWEEN TREASURY BILLS AND COMMON STOCKS

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The characteristics of a diversified common stock portfolio have been well documented. For example, for the twenty five year period ending December 1974, the quarterly total returns (dividends plus capital gains or losses) for the Standard & Poor's 500 stock index averaged 2.71% with a standard deviation of 7.22%. The geometric mean was 2.44%, or equivalently, this index grew at an annual rate of 9.75% compounded quarterly.

We examine here the characteristics of a portfolio composed of varying proportions of the S&P 500 and ninety day U.S. Treasury bills. In particular, we address the question of finding the optimal allocation between bills and stocks for an investor who wishes to maximize the long term growth rate of his portfolio. We assume that at the start of each quarter assets are costlessly reallocated between the S&P 500 and ninety day bills. Let

μ = expected return on stocks

σ = standard deviation of stock returns

T = certain return on Treasury bills

p = proportion of portfolio in Treasury bills.

Then the expected return on the portfolio is $pT + (1-p)\mu$ and the standard deviation of portfolio return is $(1-p)\sigma$. The quarterly return on this portfolio, r, is a random variable. Its growth rate $G \doteq E(\log(1+r))$ can be approximated from the Taylor expansion about μ neglecting terms of order greater than two:

$$G = \log(1+pT+(1-p)\mu) - \frac{(1-p)^2\sigma^2}{2(1+pT+(1-p)\mu)^2}$$

To optimize G by selecting the appropriate p at the start of each quarter we set

$$\frac{\partial G}{\partial p} = \frac{(T-\mu)(1+\mu+p(T-\mu))^2+\sigma^2(1-p)(1+T)}{(1+\mu+p(T-\mu))^3} \qquad \text{(A)}$$

equal to zero. Considering only $0 \leq p \leq 1$ (effectively prohibiting margin and short sales), with $\mu = .027$ and $\sigma = .072$, bills do not play a role in the optimal portfolio unless they yield more than 8.8% annually. Between 8.8% and 10.8%, the proportion in bills ranges almost linearly between zero and 100%. This indicates that a growth maximizer during this period would have been 100% invested in common stocks the entire time. If available, he would have utilized margin. (If borrowing costs were 4% he would have borrowed more than three times his assets; if costs were 6%, he would have borrowed two and one half times his assets.)

We have assumed thus far that the investor "knows" μ and σ and that they are constant. We extend this by now assuming that he also knows the relationship between μ and T. During this period, bill yields and stock returns were significantly negatively related with $\rho = -.39$. A regression of bill rates on stock returns yields the estimating equation:

$$\mu = .08 - 5.98\ T.$$

Assuming μ to be this function of T, setting equation A equal to zero yields $0 \leq p \leq 1$ only when $.043 \leq T \leq .046$ (annual rates). Thus when the negative correlation between bill rates and stock returns is assumed known, the optimal portfolio (constrained not to sell short or borrow) will be fully invested in stocks if bill rates are less than 4.3% and fully invested in bills if the bill rate exceeds 4.6%. The proportion of the portfolio in bills rises almost linearly from zero to 100% when bill rates rise from 4.3% to 4.6%.

This can be extended further by assuming that σ is a function of T using the standard error of estimate of the regression of T on μ. Then

$$\sigma^2 = 1.95\ T^2 - .0348\ T + 1.989 \times 10^{-4}.$$

This makes the interval for positive p even smaller: $0 \leq p \leq 1$ when $4.58 \leq T \leq 4.60$. Maximizing G throughout this period involved a highly unstable portfolio consisting of all stocks when the bill rate was less than 4.58% and all bills when the rate exceeded 4.60%. Such a portfolio had an annual growth rate, compounded quarterly, of 12.61% compared to the all stock portfolio growth rate of 9.75%. The average return per quarter was increased to 3.29% from 2.71% and the standard deviation was reduced to 5.19% from 7.22%.

This dramatic improvement in growth rate and average return per period along with a reduction in variability of return was not practically achievable. Liquidating an entire portfolio at the start of a quarter and possibly completely reinvesting it at the end of the quarter imposes very high transaction, liquidity and psychological costs. We turn now to a more feasible strategy that does not entail the instability inherent in maximizing G.

Seeking a portfolio that "smoothly" reallocates assets between bills and stocks, we constrain p in the following manner: p is a continuously differentiable function of T on the interval $0 \leq T \leq 9$, with p = 0 when T = 0 and p = 1 when T = 9. A family of functions satisfying these constraints is

$$p = 1 - (1 - (1 - (T/9)^{\alpha})^{1/\alpha}.$$

Using the actual sequence of returns for bills and stocks we find by iteration that growth is maximized when $\alpha = .69$, yielding the following table of "optimal" p for different annual bill rates:

Annual bill rate	Proportion in bills	Proportion in stocks
0	.00	1.00
1	.02	.98
2	.07	.93
3	.13	.87
4	.20	.80
5	.29	.71
6	.40	.60
7	.53	.47
8	.70	.30
9	1.00	.00

A portfolio that allocated assets quarterly in this way over the twenty five year period ending December 1974 had the following characteristics relative to the S&P 500 index:

	S&P 500	"Optimal" portfolio
Annual growth, %	9.75	10.99
Avg quarterly, %	2.71	2.89
Stand. dev., %	7.22	5.43