RANDOM QUASI-LINEAR UTILITY

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ABSTRACT. I refine the random utility model (RUM) of Block and Marschak (1959) and represent stochastic choice data with quasi-linear types. In my framework, choices are observed across pairs of consumption goods and money. Such primitives provide unique identification for the probability measure over the underlying quasi-linear types. Moreover, the uniqueness in identification implies a unique social welfare aggregator that is consistent with Pareto efficiency criteria.

*** This paper is work in progress. Link to most recent version.

1. Introduction

Stochastic choice data is common in empirical settings and can naturally arise from heterogeneous preferences. Such heterogeneity can be revealed by any generic group of agents. Even a single agent who faces repeated choices can often vary her responses (Agranov & Ortoleva, 2017). Such variations can result from factors like information, mood, social situations, and so on (e.g., Harsanyi, 1973).

Block and Marschak (1959) model stochastic choice data via the *random utility model (RUM)* where the probabilities of the choices reflect some endogenous distributions over the underlying types. In general, Block and Marshack assume that each possible type has a utility representation but impose no other constraints. Falmagne (1978) showed that the sufficient representation condition for the general RUM is the non-negativity of Block-Marschak polynomials. The proof in Falmagne (1978) is constructive. In the construction, although the marginal distributions of the types are recovered uniquely, the joint distribution is not. Hence, the general RUM typically cannot be uniquely identified. (see discussion in McClellon et al., 2015; Turansick, 2021).[1] Therefore, the RUM constructed from the stochastic data may be different from the true distribution of types.

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[1]Any general RUM with full support is non-unique (McClellon et al., 2015; Turansick, 2021). In particular, Block-Marschack polynomials, $q(x, A)$, characterizes the probability weight put on orders for which the strict upper contour set of x is exactly $X \setminus A$. One can construct a probability flow that respects observed probabilities from Block-Marschak polynomials. However, such construction is non-unique since the flow at a two-in-two-out

Uniqueness can be restored in refinements of RUM where additional structure is imposed on the types, such as the expected utility case of Gul and Pesendorfer (2006) (henceforth, GP) and the single-crossing property of Apesteguia, Ballester, and Lu (2017) (henceforth, ABL)[2]. Uniqueness can also be restored when the space of observations is enriched, for example, when both ex-ante and ex-post choices are observed (Ahn & Sarver, 2013), when choices under uncertainty (Lu, 2016, 2021) are observed, or when dynamic choices (Duraj, 2018; Frick, Iijima, & Strzalecki, 2019; Lu & Saito, 2018) are observed. Parametric models such as in Apesteguia and Ballester (2018) and Fudenberg and Strzalecki (2015) also identify the random utility model uniquely.

In this paper, I study another natural refinement of RUM that makes its identification unique and comes along with several other benefits. All types are restricted to be quasi-linear concerning prices. More precisely, each type has a representation $U : Z \times \mathbb{R} \to \mathbb{R}$:

$$U(i, c_i) = v_i - c_i, \tag{1.1}$$

where $w_i \in \mathbb{R}$ is the price of physical good i, and can be interpreted as a reduction in wealth. v_i is the agent's reserve value for good i. I call this model *random quasi-linear utility model (RQUM)*.

Quasi-linearity is natural in many settings in mechanism design, auction theory, bargaining theory, public welfare analysis, etc. When choice alternatives are augmented with the prices (or equivalently, wealth), the space of available observations is greatly expanded. This added richness of primitives makes the identification of my model unique.

To illustrate RQUM, consider a dataset of the ordinal RUM without monetary variations. Let $Z = \{a, b, c, d\}$, and $\rho(x, A)$ be the probability of choosing x from A for $A \subseteq Z$. Consider the following choices:

$$\begin{aligned} &\rho(a, \{a, b, c, d\}) = \rho(b, \{a, b, c, d\}) = 0.5, \quad \rho(a, \{a, c, d\}) = \rho(b, \{b, c, d\}) = 1, \\ &\rho(a, \{a, b, c\}) = \rho(b, \{a, b, c\}) = \rho(a, \{a, b, d\}) = \rho(b, \{a, b, d\}) = 0.5, \\ &\rho(a, \{a, b\}) = \rho(b, \{a, b\}) = 0.5, \quad \rho(c, \{c, d\}) = \rho(d, \{c, d\}) = 0.5, \\ &\rho(a, \{a, c\}) = \rho(a, \{a, d\}) = \rho(b, \{b, c\}) = \rho(b, \{b, d\}) = 1. \end{aligned} \tag{1.2}$$

Observations (1.2) are consistent with two distinct random utility functions (Fishburn, 1998; Turansick, 2021):

$$\pi_1 = \begin{cases} a \succ b \succ c \succ d & w.p. \quad 0.5 \\ b \succ a \succ d \succ c & w.p. \quad 0.5 \end{cases} \quad \text{and} \quad \pi_2 = \begin{cases} a \succ b \succ d \succ c & w.p. \quad 0.5 \\ b \succ a \succ c \succ d & w.p. \quad 0.5 \end{cases}$$

branching can go two different ways and thus can be generated by different random utility functions. See detailed discussion in Turansick (2021).

[2]However, RUM with some classes of risk preferences still have the non-uniqueness problem, see Lin (2020).

In contrast, the money dimension in the RQUM produces many more observations. Associate observations (1.2) with the price vector $(0,0,0,0)$, and suppose the wealth can vary contingently on (a,b,c,d), respectively. Unlike the general case, some representations that are consistent with (1.2) can be refuted by varying the wealth vector. For instance,

$$\pi_3 = \begin{cases} v(a,b,c,d) = (5,3,2,1) & w.p. \quad 0.5 \\ v(a,b,c,d) = (3,4,1,2) & w.p. \quad 0.5 \end{cases} \tag{1.3}$$

is consistent with observation (1.2) with the price vector is $(0,0,0,0)$. Take the price vector $(0,2,0,0)$. Then the utility associated with π_3 is

$$u(a,b,c,d) = \begin{cases} (5,1,2,1) & w.p. \quad 0.5 \\ (3,2,1,2) & w.p. \quad 0.5. \end{cases} \tag{1.4}$$

In this case, $\rho(a,\{a,b,c,d\}) = 1$. Any conflicting observations will rule out π_3.

The uniqueness of RQUM is very convenient for aggregating social welfare. Note that ordinal types that appear in the general RUM will always make Pareto comparisons incomplete and sometimes extremely so. For example, if opposite types like $a \succ b \succ c \succ d$ and $d \succ c \succ b \succ a$ are possible, then we cannot make Pareto comparisons on any two distinct alternatives. Pareto aggregation is still problematic in the expected utility case of GP, where Pareto aggregation has many free parameters (Harsanyi, 1955). By contrast, the Pareto criterion in my RQUM delivers a *unique* aggregate social welfare for any consumption alternative paired with costs:

$$W(i,c_i) = \sum_{v\in V} \pi(v)(v_i - c_i) = \left(\sum_{v\in V} \pi(v) v_i\right) - c_i, \tag{1.5}$$

where V is the set of types, and the probability π represents the proportion of agents in each type.

My main result (Theorem 1) characterizes RQUM in terms of an observable stochastic choice function. To construct RQUM, I employ an inclusion-exclusion technique that is reminiscent of Falmagne (1978) and is distinctive from the identification approaches in GP or ABL. However, instead of constructing the probabilities from cylinders and upper contour sets as in the ordinal RUM, I directly calculate the probability that there is a reserve value vector equal to a given cost vector from the observations by perturbing the cost vector. The probability is positive only when there is a full tie. Theorem 1 does not characterize tie-breaking rule, except that it is wealth independent.

This paper contains several extensions and applications of Theorem 1. Conjecture 1 characterizes the uniform tie-breaking rule for RQUM. Conjecture 2 demonstrates that the number of types can be counted from the observations. Conjecture 3 establishes the representation result on a discretized domain. Conjecture 4 characterizes the case when the types are a continuum.

There is a close relation between RQUM and path-independent quasi-linear choice functions. In particular, Proposition 2 shows that the choice function that includes all choice alternatives with positive probability is path-independent. Finally, I propose an alternative construction of the probability distributions over the types based on the likelihood of choosing the status quo over all other alternatives.

1.1. **Related Literature.** Falmagne (1978) established that the Block-Marschalk polynomials are sufficient for ordinal RUM, while McFadden and Richter (1991) showed that the axiom of revealed stochastic preference is also a sufficient condition for the ordinal. Both conditions, in the absence of additional structures, characterize the general RUM. The two characterizations, however, imply distinct refinements and applications. The latter leads to the econometric study of discrete choice models, which is interpreted as agents with a single utility but an error term that results in distinct choice behavior. While the former is studied in the decision theory literature.

There are many refinements to the RUM that address the issue of uniqueness. For example, McClellon et al. (2015) studies the random choice over Savage acts. Lu (2016) studies random choices in the Anscombe-Aumann setting, in which the agents have deterministic utility functions but stochastic beliefs. Apesteguia et al. (2017) imposes an exogenous order on the choice domain, whereas Filiz-Ozbay and Masatlioglu (2020) considers a more general setting with exogenous sorted choice functions. These refinements are useful in their respective settings. My RQUM model contributes to this literature by imposing a quasi-linear structure on the utilities in the RUM and restores uniqueness. This refinement is natural in the market settings where where utility on consumption goods can be compensated by money.

One important special case of the RUM which is commonly used in the discrete choice application is Luce's Model (Luce, 1960), which characterizes a random choice rule which respects the Independence of Irrelevant Relatives (IIA) condition. A random choice rule that satisfies Luce's Model is rationalizable by heterogeneous preferences.[3] Let $P_A(x)$ denote the probability that the choice alternative x is chosen in menu A. Let $\{x, y\} \subset A$. Then IIA requires

$$\frac{P_{\{x,y\}}(x)}{P_{\{x,y\}}(y)} = \frac{P_A(x)}{P_A(y)}. \tag{1.6}$$

The IIA condition is problematic empirically. Suppose the agent faces the choice set {bus, car}, and have $\frac{1}{2}$ probability of choosing either transportation. When the choice set is enlarged to be {blue bus, red bus, car}, IIA requires the agent to choose each transportation with probability

[3]Luce's model is heavily used in applied work. However, such popularity is not due to the multi-utility interpretation but instead the interpretation that an agent's utility on a choice alternative is the average utility plus an error term. With such an interpretation, it is has a close connection with discrete choice models in econometrics (see e.g. McFadden et al. (1973)).

$\frac{1}{3}$ (See, Debreu, 1960). This requirement is not reasonable since the color of the bus need not affect the choice of means of transportation.[4]

RQUM does not overlap with the Luce model and does not suffer from IIA. Intuitively, when a choice alternative associated with a very low price level is added to the choice set, all agents will change their choices in favor of this choice alternative. For example, suppose $Z = \{a, b, c\}$, and there are two quasi-linear agents, with types $v_1(a, b, c) = (1, 3, 2)$, $v_2(a, b, c) = (1, 2, 3)$. When the price vector associated with the three consumption goods is $w = (0, 0, 0)$, IIA is satisfied for positive choice probabilities. But if the price level is $w = (0, -3, -3)$, IIA is violated in sets $\{b, c\}$ and $\{a, b, c\}$.

The Luce model violates Axiom 3 in RQUM. This Axiom imposes an important monotonicity feature on the set function derived from the observations. In particular, I construct a *capacity function* by perturbing the observation on each cost vector and breaking the ties. Hence, the probability of types that have at least one maximizer in the set will be accounted for in the value of the set function. Axiom 3 will guarantee the existence of a probability measure on the set of types (Barthélemy, 2000; Grabisch, 2015; Zhou, 2013). Detailed treatment on the theory of capacity can be found in Grabisch et al. (2016),

In RQUM, the probability of the types that have ties on any subset can be identified. Hence one can impose any tie-breaking rules. In general, the tie-breaking rule in a random utility model is difficult to characterize. The ordinal RUM in Block and Marschak (1959) and the random expected utility model in Gul and Pesendorfer (2006) are both restricted to the case where there are no ties. Gul and Pesendorfer (2013) studies the ordinal RUM with weak orders and characterizes ties with the "double total monotonicity" condition. Lu (2016) treats ties as non-measurable sets in the Anscombe-Aumann framework.

The remainder of the paper is organized as follows. Section 2 introduces the primitives and the axioms and develops Theorem 1. Additionally, it provides the construction of the random quasi-linear utility function (RQUF) and illustrates the role of Axiom 3 with two examples. Finally, it sketches out a proof for Theorem 1. Section 3 is currently a work in progress. It discusses several extensions and applications of Theorem 1. The application of RQUM to utility aggregation is examined. Conjecture 1 imposes the uniform tie-breaking rule. Conjecture 2 discusses counting the number of types from observations. Conjecture 3 studies the case where money unit variation is discrete. Conjecture 4 studies the case where the types are a continuum. Proposition 2 discusses the relation between RQUM and the path-independent choice functions. Finally, an alternative construction of the RQUF based on the likelihood of choosing the status quo is provided. The proof for Theorem 1 is in Section 5.

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[4]Specifications of the Luce model, which does not suffer from the blue-bus red-bus fallacy, has been studied in (e.g., Fudenberg, Iijima, & Strzalecki, 2015; Gul, Natenzon, & Pesendorfer, 2014).

2. Main Model

Let $Z = \{0, 1, ..., n\}$ be a finite domain of consumption goods. Let $\Delta(Z)$ be the simplex of probability distributions on Z. As money is desirable, any consumption good should be selected with its lowest available cost. Therefore, it is without loss of generality to associated each consumption good with one cost.

A function $\rho : \mathbb{R}^{n+1} \to \Delta(Z)$ is called a *random choice rule* (RCR) on Z. Interpret the probability $\rho_i(c)$ as the observed likelihood of consumption good i when the cost vector is c. For any $A \subseteq Z$,

$$\rho_A(c) = \sum_{i \in A} \rho_i(c)$$

is the combined likelihood of all $i \in A$ at cost vector c.

A utility function $U : Z \times \mathbb{R}$ is called *quasi-linear* if

$$U(i, \alpha) = v_i - \alpha \tag{2.1}$$

for some vector $v \in \mathbb{R}^{n+1}$. Let $\mathbb{R}_0^{n+1}$ be the set of all vectors $v \in \mathbb{R}^{n+1}$ such that $v_0 = 0$. Obviously, the value of good 0 can be restricted to zero without loss of generality. Let Π be the set of all probability distributions that have finite support in $\mathbb{R}_0^{n+1}$.

Say that $t : \mathbb{R}_0^{n+1} \times \mathbb{R}^{n+1} \to \Delta(Z)$ is a tie-breaking rule if

$$t_i(v, c) > 0 \implies v_i - c_i = \max_{j \in Z} v_j - c_j.$$

In other words, t assigns positive probability only to the optimal choices in Z according to the quasi-linear utilities. If the maximizer i is unique, then $t_i(v, c) = 1$. Let T be the set of all quasi-linear tie-breaking rules such that

$$t(v, c) = t(v, c + \alpha\mathbf{1}), \forall \alpha \in \mathbb{R}. \tag{2.2}$$

where $\mathbf{1} = \{1, .., 1\} \in \mathbb{R}^{n+1}$, and $\alpha\mathbf{1}$ is the constant vector $(\alpha, ..., \alpha)$. This property is analogous to the quasi-linear structure of utility functions.

Say $(\pi, t) \in \Pi \times T$ is a *random quasi-linear representation (RQR)* for ρ if for any $c \in \mathbb{R}^{n+1}$ and $v \in \mathbb{R}_0^{n+1}$,

$$\rho(c) = \sum_{v \in \mathbb{R}_0^{n+1}} \pi(v) t(v, c). \tag{2.3}$$

π is the probability distribution on the finite types. $t_i(v, c) > 0$ only if i is a maximizer for v. So the likelihood of i at cost vector c, $\rho_i(c)$, is the weighted sum of the probability of types for which i is a maximizer.

To characterize representation (2.3), consider several conditions for the RCR, ρ. First, adapt a standard invariance property for the quasi-linear utility model.

Axiom 1 (Wealth Invriance (WI)). *For all* $c \in \mathbb{R}^{n+1}$, $\alpha \in \mathbb{R}$,

$$\rho(c) = \rho(c + \alpha\mathbf{1}).$$

It is assumed here that the optimal consumption choice as well as the tie-breaking rule is invariant of the constant wealth variations $\alpha\mathbf{1}$.

For any $A \subset Z$, $c, c' \in \mathbb{R}^{n+1}$, write $c \gg_A c'$ if $c_i > c'_i$ for $i \in A$ and $c_j \leq c'_j$ for $j \notin A$.

Axiom 2 (Monotonic Demand (MD)). $c \gg_A c' \implies \rho_A(c) \leq \rho_A(c')$.

This condition assumes that the aggregate demand for goods in the set A should not decrease if all goods in A become cheaper without reducing the costs of any other goods. In the standard theory of demand, this condition corresponds to positive price effects.

For any subset $A \subseteq Z$, let the characteristic vector $\mathbf{1}_A \in \mathbb{R}^{n+1}$ be equal to 1 if $i \in A$ and equal to 0 if $i \notin A$.

Axiom 3 (Alternation). *For all* $\beta > \alpha > 0$,

$$\sum_{A \subseteq Z, |A| \text{ is odd}} \rho_A(c - \beta\mathbf{1}_A) \geq \sum_{A \subseteq Z, |A| \text{ is even}} \rho_A(c - \alpha\mathbf{1}_A). \tag{2.4}$$

Axiom 3 is analogous to the *alternating* property of capacity (see, e.g., Grabisch et al., 2016). To get some intuition for Axiom 3, rewrite (2.4) as

$$\sum_{A \subset Z, |A| \text{ is odd}} \sum_{i \in A} [\rho_i(c - \beta\mathbf{1}_A) - \rho_i(c)] \geq \sum_{A \subset Z, |A| \text{ is even}} \sum_{i \in A} [\rho_i(c - \alpha\mathbf{1}_A) - \rho_i(c)]$$

The above inequality is equivalent to (2.4) becasuse ρ is a additive set function. The quantity $\rho_i(c - \alpha\mathbf{1}_A) - \rho_i(c), i \in A$ is the demand change for consumption good i when it is in the set of goods that have price discounts. Axiom 3 imposes restriction on such changes. When $Z = \{0, 1, 2\}$. Axiom 3 requires

$$\sum_{i=0}^{2} (\rho_i(c_i - \beta, c_i) - \rho_i(c)) - (\rho_i(c) - \rho_i(c_i + \alpha, c_{-i})) \geq 0. \tag{2.5}$$

This is very similar to the convexity condition. It requires that the average demand increase when price fall is greater than the average demand decrease when price rise.

The last axiom is technical and restricts the number of types to be finite.

Axiom 4 (Finite Range (FR)). *For any* $i \in Z$, $c \in \mathbb{R}^{n+1}$, *the function* $\rho_i(\alpha, c_{-i})$ *has a finite range in* $[0, 1]$ *that includes both* 0 *and* 1.

Axiom 2 and 4 together implies that $\rho_i(\alpha, c_{-i})$ is a monotonic step functions such that $\rho_i(\alpha, c_{-i})$ is decreasing. Due to tie-breaking, $\rho_j(\alpha, c_{-i})$ is not necessarily increasing. To see this, consider a type that has i, j, k as maximizers at c , and the tie breaking rule is choosing j. At $(c_i + \varepsilon, c_{-i}), \varepsilon \to 0$, the type has only two maximizers, j, k. The tie-breaking rule is to choose k. Then, $\rho_j(c_i + \varepsilon, c_{-i}) < \rho_j(c), \varepsilon \to 0$. However, $\rho_j(c)$ is increasing in all but finitely many points, since Axiom 4 restricts changes in $\rho_i(\alpha, c_i)$ to occur at finitely many points. Moreover, they satisfy the following asymptotic properties:

$$\lim_{\alpha\to-\infty} \rho_i(\alpha, c_{-i}) = \lim_{\alpha\to\infty} \rho_j(\alpha, c_{-i}) = 0, \quad \text{and} \quad \lim_{\alpha\to\infty} \rho_i(\alpha, c_{-i}) = 1.$$

My main result is as follows:

Theorem 1. *An RCR ρ satisfies Axioms 1 – 4 if and only if ρ is represented by $(\pi, t) \in \Pi \times T$. Moreover, such π is unique.*

Theorem 1 characterizes my main model in terms of observable properties of the random choice rule. The representation have two components: the distribution π over quasi-linear types $v \in \mathbb{R}_0^{n+1}$, and the tie-breaking rule $t \in T$. π in the RQUM has the desirable property of being unique and thus overcomes the identification issues in the RUM model . The uniqueness of π allows to define a Pareto efficient utility aggregator. In contrast to Block and Marschak (1959) and Gul and Pesendorfer (2006), ties are allowed in the RQUM model, and one can impose more structures on ties. I further discuss uniform tie-breaking rule in section 3.2.

To understand why the distribution π is unique in my model, I demonstrate the construction of π. Denote $f(\varepsilon, c) = \pi(v \in \mathbb{R}_0^{n+1} : v_i - c_i \text{ is within } \varepsilon \text{ from } \max_j v_j - c_j \ \forall i \in Z)$. By inclusion-exclusion principle,

$$f(\varepsilon, c) = \sum_{A\subseteq Z} (-1)^{|A|+1} \pi\{v \in \mathbb{R}_0^{n+1} : \text{there is } i \in A \text{ s.t. } v_i - c_i \text{ is within } \varepsilon \text{ from } \max_{j\in Z} v_j - c_j\}.$$

In the model, for $\alpha = \varepsilon, \beta = 2\varepsilon$,

$$\rho_A(c - \beta\mathbf{1}_A) \geq \pi\{v \in \mathbb{R}_0^{n+1} : \text{there is } i \in A \text{ s.t. } v_i - c_i \text{ is within } \alpha \text{ from } \max_{j\in Z} v_j - c_j\},$$

since all types with $(v_i - c_i) + \alpha \geq max_{j\in Z} v_j - c_j$ would chose i; while

$$\rho_A(c - \alpha\mathbf{1}_A) \leq \pi\{v \in \mathbb{R}_0^{n+1} : \text{there is } i \in A \text{ s.t. } v_i - c_i \text{ is within } \alpha \text{ from } \max_{j\in Z} v_j - c_j\}.$$

since types with $(v_i - c_i) + \alpha = max_{j\in Z} v_j - c_j$ would divide between i and j with a tie-breaking rule. Therefore

$$f(\varepsilon, c) \leq \sum_{A\subseteq Z, |A| \text{ is odd}} \rho_A(c - \beta\mathbf{1}_A) - \sum_{A\subseteq Z, |A| \text{ is even}} \rho_A(c - \alpha\mathbf{1}_A).$$

and π on any $v = c$ can be constructed by taking $\varepsilon \to 0$. Write $\pi(c) = \pi(v \in \mathbb{R}_0^{n+1} : v = c)$. Then

$$\pi(c) = \lim_{\varepsilon\to 0} f(\varepsilon, c) = \lim_{\varepsilon\to 0} \sum_{A\subseteq Z, |A| \text{ is odd}} \rho_A(c - 2\varepsilon\mathbf{1}_A) - \sum_{A\subseteq Z, |A| \text{ is even}} \rho_A(c - \varepsilon\mathbf{1}_A). \tag{2.6}$$

Axiom 3 ensures that the construction (2.6) is non-negative at any $c \in \mathbb{R}^{n+1}$. I provide an examples to show that Axiom 3 is not implies by Axioms 1, 2, 4.

Example 1. *$Z = \{0, 1, 2\}$, and assume a uniform tie-breaking rule. Consider the set of types $\{v^i, i = 1, 2, 3, 4\}$ associated with a charge measure as follows:*

	0	1	2	charge
v^1	0	0	0	-0.5
v^2	0	-1	0	0.5
v^3	0	0	-1	0.5
v^4	0	1	1	0.5

By assuming quasi-linearity, ρ satisfies Axiom 1. Axioms 2, 4 are satisfied by the stochastic function generated by $v^i, i = 1, 2, 3, 4$ according to (2.3). Figure 1 illustrates this.

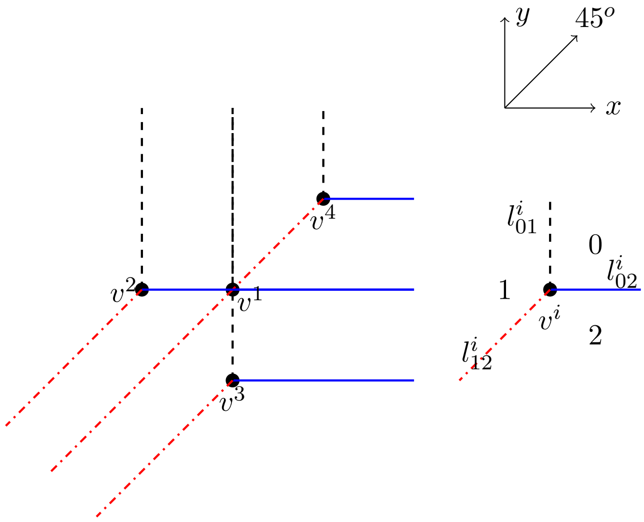

FIGURE 1. Negative Charge

Each type $v^i, i = 1, 2, 3, 4$ is associated with three half-lines starting from it: l^i_{01}, l^i_{02}, and l^i_{12}. These three-half lines divides $\mathbb{R}^2$ into three regions: $0, 1, 2$. Type v^i chooses k when c falls in the region k associate with it, $k = 0, 1, 2$, and has a tie between j, k on the half-line $l^i_{j,k}$, $j, k = 0, 1, 2$.

ρ is always non-negative. Indeed, notice that c falls in region k for v^1, it always falls in the same region for another two types. Hence, whenever type v^1 chooses k, two other types also choose k. ρ has a discontinuity when the line of chose change intersects with one of the twelve half-lines $l^i_{j,k}, i = 1, 2, 3, 4, j, k = 0, 1, 2$. So there is a finite number of changes in ρ in any direction of cost change. The cost (∞, y), $y \in \mathbb{R}$ is not in the 1 region for any type, and $(-\infty, y), y \in \mathbb{R}$ is in the 1 region for all agents. So $\rho_1(0, -\infty, y) = 1, \rho_1(0, \infty, y) = 0$. So Axiom 4 is satisfied.

For any $c \in \mathbb{R}^3$, c_1, c_2 increases along the x, y axes, respectively, while c_0 decreases in the direction of the 45 degree line. Axiom 2 can be verified by observing that increases in cost directions $A \subseteq \{0, 1, 2\}$ make the cost vector less likely to fall in regions in A.

In the following example, I show that Axiom 3 separates the RQUM from the logit function.

Example 2. *Consider* $Z = \{0, 1, 2\}$. *Let*

$$\rho_i(c) = \frac{e^{-c_i}}{\sum_{i\in Z} e^{-c_i}}.$$

It is easy to verify that ρ *satisfies Axioms 1 and 2. Let* $c = (0, 0, 0)$. *Then* $\rho_i(x, c_{-i}) = \frac{e^{-x}}{2+e^{-x}}$. *It is concave for* $x < -\log 2$, *and is convex for* $x > -\log 2$.

However, Axiom 3 requires that

$$\sum_{i=0}^{2} \rho_i(c_i - \beta, c_i) - 2\rho_i(c) + \rho_i(c_i + \alpha, c_{-i}) \geq 0. \tag{2.7}$$

Condition (2.7) *does not hold for say,* $\beta = 2, \alpha = 1$.

2.1. **Sketch of Proof for Theorem 1.** By Axiom 1, we can fix the cost on good 0 to be 0. When $Z = \{0, 1\}$, ρ can be thought of as an increasing one-variable step function. I define a distribution function F from ρ by changing the values of discontinuity points of ρ to make it right-continuous. The realizations of the random variable are the types. I obtain the types on consumption good 1 using the Skorokhod construction of random variables. Axiom 4 requires that this random variable takes finitely many realizations. I show that they types are quasi-liner as in (2.1). The proof is relegated to Section 4.

A type is identified when there is tie between goods 0 and 1. Hence, when there is a tie, only one type is indifferent between the two goods. A wealth invariant tie-breaking rule always exists. Hence, Axioms 1, 2, 4 ensure that there is a distribution of finite quasi-linear types when $|Z| = 2$, see Lemma 1:

Lemma 1. *When* $Z = \{0, 1\}$, *an RCR* ρ *satisfies Axioms 1, 2, and 4 if and only if* ρ *maximizes RQUF* $(\pi, t) \in \Pi \times T$. *Moreover, such* π *is unique.*

Next we consider the case $|Z| > 2$. For any cost vector $c \in \mathbb{R}^{n+1}$, define the following:

$$\rho_A^+(c) = \lim_{\varepsilon\to 0} \sum_{i\in A} \rho_i(c - \varepsilon \mathbf{1}_A), \quad \rho_A^-(c) = \lim_{\varepsilon\to 0} \sum_{i\in A} \rho_i(c + \varepsilon \mathbf{1}_A). \tag{2.8}$$

$\rho_A^+(c)$ is the revealed probability that at least one of the consumption good in the set A is optimal; and $\rho_A^-(c)$the revealed probability that at lease one of the consumption good in the set S is strictly optimal. (2.6) can be written in terms of ρ^+:

$$\pi(c) = \sum_{A\subset Z} (-1)^{|A|+1} \rho_A^+(c) \geq 0. \tag{2.9}$$

Hence, the set of types is

$$\text{supp}(\pi) = \{c \in \mathbb{R}_0^{n+1} : \pi(c) > 0\}. \tag{2.10}$$

Take $c \in \mathbb{R}_0^{n+1}$. When $|\{i \in Z : \rho_i(c) > 0\}| = 2$, it WLOG to consider $Z = \{0, 1\}$. In this case, take $c = (0, c_1)$. (2.9) becomes

$$\begin{aligned} \pi(c) &= \rho_0^+(c) + \rho_1^+(c) - 1 \\ &= \rho_0(-\varepsilon, c_1) + \rho_1(0, c_1 - \varepsilon) - 1 \\ &= \rho_1(0, c_1 - \varepsilon) - (1 - \rho_0(0, c_1 + \varepsilon)) \quad \text{By Axiom 1} \\ &= \rho_1^+(c) - \rho_1^-(c). \end{aligned} \tag{2.11}$$

This is the same as the construction in the proof of Lemma 1. I show in Lemma 3 in section 4 that that π constructed as in (2.9) satisfies the consistency condition necessary for a probability measure. Hence, the case $|\{i \in Z : \rho_i(c) > 0\}| = 2$ recovers the marginal distribution of the quasi-linear types, while for general $c \in \mathbb{R}_0^{n+1}$, (2.9) identifies the unique joint distribution of types. When $|\{i \in Z : \rho_i(c) > 0\}| = n + 1$, π is the unique discrete probability measure on $\mathbb{R}_0^{n+1}$.

Next, I show that π represents ρ and thus the construction make sense. First, define the following:

Definition 2.1 (Gap function)**.** *Define the gap function for any $c \in \mathbb{R}^{n+1}, i \in Z$ as follows:*

$$gap_i(c) = \rho_i^+(c) - \rho_i^-(c). \tag{2.12}$$

The function gap_i is the probability that i is a weak maximizer. See Lemma 4.

Definition 2.2 (Set of maximizers and generic points)**.** *Let*

$$M(v, c) = \arg\max_{i \in Z}(v_i - c_i) \tag{2.13}$$

be the set of all consumption goods that maximizes the quasi-linear utility at the cost vector $c \in \mathbb{R}_0^{n+1}$. c is called generic if

$$\pi(c \in \mathbb{R}_0^{n+1} : |M(v, c)| > 1) = 0. \tag{2.14}$$

Condition (2.14) includes all $c \in \mathbb{R}^{n+1}$ that induces a unique maximizer for each type. As π has finite support, the set of cost vectors c that satisfy (2.14) is dense in $\mathbb{R}^{n+1}$.

Lemma 4 in section 4 shows that $gap_i(c)$ can be expressed as local perturbations of $\pi(c)$ for any $c \in \mathbb{R}^{n+1}$. Take c_0 such that $c_i \to \infty$ for all $i \in Z \setminus \{0\}$. Then $\rho_0(c_0) = 1$ by Axioms 2 and 4. Take any generic c. One can always find a path on $\mathbb{R}^{n+1}$ between c_0 and c such that $|M(v, c)| \leq 2$. I illustrate such a path when $|Z| = 3$ in Figure 2.

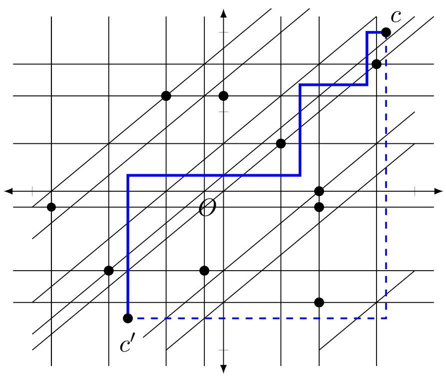

FIGURE 2. Path with Continuous cost

I show in Lemma 5 in section 4 that one can recover $\rho'(c)$ from π constructed on the path, and that the constructed $\rho'(c)$ is the same as the observation $\rho(c)$. In particular, at any generic point, $gap_i(c) = 0$. When $M(v, c) = \{i, j\}$, $gap_i(c)$ is the amount of increase in ρ_i and decrease in ρ_j (or the reverse). $\rho'(c)$ is obtained by modifying $\rho(c_0)$ on the path by the gap functions where there are two-way ties. Rewrite the gap functions in π using Lemma 5, and rewrite π in ρ with (2.9), one obtain $\rho'(c)$ that agrees with the observation for any generic $c \in \mathbb{R}_{n+1}$.

To show the existence of $t \in T$ on non-generic points, I transform the problem into a matching problem that can be solved using Hall's marriage theorem on rationals, and then use continuity to approximate any real numbers. The nodes in the matching problems are units of agents and probabilities, where a unit is defined as follows:

Definition 2.3. *A unit for $\pi, \rho \in \mathbb{Q}$ at $c \in \mathbb{R}^{n+1}$ is $\frac{1}{k}$, $k \in \mathbb{N}$ such that*

$$k\pi(v) \in \mathbb{N} \ \forall \ v \in supp(\pi), \quad and \quad k\rho_A(c) \in \ N \ \forall A \subset Z.$$

I consider $\pi(v : v \in \text{supp}(\pi), M(v, c) \subset A)$ to be a collection of units of agents (a-units) demanding the collection of unit of probabilities in $\rho_A(c)$ (p-units). I show in Lemma 6 that there is a perfect matching between a-units and p-units. The tie-breaking rule $t_i(c)$ is the number of matchings between a-units demanding i and p-units of i divided by the total number of all a-units demanding i at c.

3. EXTENSION AND APPLICATIONS

3.1. **Utility Aggregation.** There are many situations that the planner needs to know the private values. Directly eliciting the valuations on all options in Z from the underlying population is difficult, since truthful reporting is not always a dominant strategy. If the planner has aggregate stochastic choice data on the set Z associated with various costs, and the observation conform to the RQUM model, then the private values can be constructed with (2.9). I give two example where private values are needed.

Example 1. Suppose a planner wants to implement a costly policy $i \in Z$ in a Pareto efficient way, for example, a location choice. Theorem 1 provides a unique way to identify the Pareto optimal option under any cost c:

$$i = \arg\max \left(\sum_{v \in \mathbb{R}^Z} : \pi(v) v_i \right) - c_i. \tag{3.1}$$

Example 2. In an assignment problem, the planner needs to know the cost matrix to make cost minimizing matching between the workers and the jobs. Suppose

$$|\text{supp}\pi| = |Z \setminus \{0\}|, \quad \text{and} \quad \pi(v_j) = \frac{1}{|\text{supp}\pi|}, \ j = 1, ..., n.$$

Then from the aggregate choice data, the planner can construct the cost matrix. An assignment matrix x assigns distinctive agents to distinct jobs. The objective is to find the assignment matrix x that minimizes the total cost:

$$\begin{aligned} &\min_x \sum_{i=1}^{n} \sum_{j=1}^{n} v_{i,j} x_{i,j}, \quad \text{where} \\ &x_{i,j} = \begin{cases} 1 & \text{if type } v_j \text{ is assigned to job } i \\ 0 & \text{if type } v_j \text{ is not assigned to job } i \end{cases} \\ &s.t. \sum_{i=1}^{n} x_{i,j} = 1, j = 1, ..., n, \\ &\sum_{j=1}^{n} x_{i,j} = 1, i = 1, ..., n. \end{aligned} \tag{3.2}$$

This problem can be solved with the standard Hungarian algorithm.

3.2. **Uniform Tie-breaking.** Take any $A \subseteq Z$. The probability that at least one good in $A' \subset A$ is a maximizer at cost vector $c \in \mathbb{R}_0^{n+1}$ is $\rho^+_{A'}(c)$. By the inclusion-exclusion principle, the probability that all consumption goods in A are maximizers is:

$$\pi_A(c) = \sum_{A' : A' \subset A} (-1)^{|A'|+1} \rho^+_{A'}(c) \tag{3.3}$$

Furthermore, by inclusion-exclusion principle, the probability that a consumption good is an maximizer at cost vector $c \in \mathbb{R}^{n+1}$ if and only if it is in A, $\tilde{\pi}_A(c)$, can be constructed as follows:

$$\tilde{\pi}_A(c) = \sum_{A' \subseteq Z : A \subset A'} (-1)^{|A' \setminus A|} \pi_{A'}(c)) \tag{3.4}$$

One can impose any tie-breaking rule. I impose a uniform tie-breaking rule here. The *Uniform Tie-breaking RQUM Representation* (U-RQUM representation) is

$$\rho_i(c) = \sum_{A \subseteq Z, i \in A} \frac{1}{|A|} \tilde{\pi}_A(c) \quad \forall i \in Z, c \in \mathbb{R}^{n+1}. \tag{3.5}$$

Substitute in (3.3) and (3.4), the following axiom imposes uniform tie-breaking.

Axiom 5 (Uniform tie-breaking)**.**

$$\rho_i(c) = \sum_{A\subseteq Z, i\in A} \frac{1}{|A|} \sum_{A'\subseteq Z: A\subset A'} (-1)^{|A'\setminus A|} \sum_{A'': A''\subset A'} (-1)^{|A''|+1} \rho^{+}_{A''}(c) \quad \forall i \in Z, c\in \mathbb{R}^{n+1}.$$

Conjecture 1. *An RCR ρ satisfies Axioms 1– 5 if and only if ρ maximizes the U–RQUM representation. Moreover, the representation is unique.*

3.3. **Counting Number of Types.** The number of types can be revealed by the number of discontinuities in ρ in certain directions.

Axiom 6. *For $k \geq 1$, take any $d \in \mathbb{R}^{n+1}$ such that $d_i > 0, d_j \leq 0$ for all $j \neq i$, the number of gaps in $\rho_i(c + \alpha d), \alpha \in \mathbb{R}$ is capped by k.*

Conjecture 2. *An RCR ρ satisfies Axioms 1-4 and 6 if and only if ρ maximizes $(\pi, t) \in \Pi\times T$. Moreover, π is unique, and $k - 1 \leq |supp(\pi)| \leq k$.*

Each type defines $n+1$ rays and divides the space into $n+1$ subspace, where each consumption goods in Z is chosen by this type in one of the subspace. For a type to have discontinuity in $\rho_i(c + \alpha d)$, $d_i > 0, d_j \ \leq 0 \ \forall j \neq i$ that coincide with the discontinuity of another type for all $i \in Z$, this type must locate at the intersection of a ray from all other types. There is at most 1 such intersection. So at most 1 type will not be counted in the discontinuity.

3.4. **Discrete Domain.** In practice, money unit variations are often discrete. In this section, I consider the case when the variations in money units are discrete, but the changes in valuations are greater than the smallest money unit. Denote the smallest money unit variation to be δ. Then possible money valuations is $G = (-N\delta, ..., -\delta, 0, \delta, ..., N\delta)$, the set of possible money variation be G^{n+1}. Let the types be points on $G^{n+1}_0 = \{0\} \times G^n$ such that $v_i = 2k\delta$, $-N \leq 2k \leq N$, $i = 1, ..., n$.

Conjecture 3. *An RCR satisfies Axioms 1 to 4 restricted to the domain G^{n+1} if and only if it is represented by $(\pi, t) \in \Pi \times T$. Moreover, such π is unique.*

To show Theorem 3, I augment the observation to the entire space $\mathbb{R}^{n+1}$. In this case, at any non-type point $c \in G^{n+1}_0$, the open hypercube $c_\delta = \{0\} \times^n_{i=1} (c_i - \delta, c_i + \delta)$ do not contain any type. So let $\rho(c') = \rho(c)$ for all $c' \in c_\delta$. This way, the observation is extended to the entire space $\{0\}\times\big(-(N+1)\delta, (N+1)\delta\big)^n$. For the space any point c'' in $\{0\}\times\big(\mathbb{R}^n \setminus \big(-(N+1)\delta, (N+1)\delta\big)^n\big)$, let $\rho_0(c'') = 1$. Finally, use Axiom 1 to extend ρ to the space $\mathbb{R}^{n+1}$.

I show that Axioms 1 to 4 restricted to the discrete points can be extended to the entire space, and thus Theorem 1 can be applies. Furthermore, (π, t) obtained on the augmented domain can still satisfy Axioms 1 to 4 restricted to the discrete points.

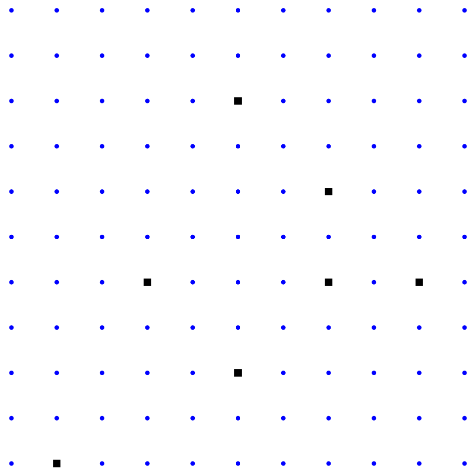

FIGURE 3. Path on grid

3.5. **Continuous types.** I impose Axiom 7 so that the types are in a continuum. In this case, ties are everywhere, but they have probability 0.

Axiom 7 (Infinite Range)**.** *ρ is continuous with range* $[0,1]$.

Let

$$A = (a_0, b_0] \times ... \times (a_n, b_n], \quad V = \{a_0, b_0\} \times ... \times \{a_n, b_n\}$$

where $-\infty < a_i < b_1 M \infty$. A is a finite rectangle, and V are the vertices of the rectangle A. If $v \in V$, let

$$\Delta_A F = \sum_{v \in V} (-1)^{\#\text{of } a's \text{ in } v} F(v).$$

In this case, $F(c) = \rho(-c)$ become a distribution function of the types, since it satisfies the following:

(1) $F(c)$ is non-decreasing by Axiom 2
(2) $F(c)$ is right continuous.
(3) $\Delta_A F \geq 0$ for all finite rectangles A by Axioms 3.

The theorem of the joint distributions are as follows:

Theorem (Joint Distribution)**.** *Suppose $F : \mathbb{R}^d \to [0,1]$ satisfies (1)–(3) given above. Then there is a unique probability measure on μ on $(\mathbb{R}^d, \mathcal{R}^d)$ so that $\mu(A) = \Delta_A F$ for all finite rectangles.*

Conjecture 4. *An RCR ρ satisfies Axioms 1, 2, 4, and 7 if and only if ρ is represented by a continuous probability measure $\mu \in \Delta(\mathbb{R}_0^{n+1})$. Moreover, such μ is unique.*

Since all ties have probability 0, I am agnostic of the ties.

3.6. **Alternative Construction.** Suppose the observations satisfy Axioms 1, 2, 3, 4. Then the observations conform to a RQUM by Theorem 1. I provide an alternative construction for (2.9). This alternative construction requires less observations.

One can find the discontinuity points on each coordinate, $V_i, i = 1, ..., n+1$, by setting $c_j \to \infty$ for all $j \in Z_{-i}$ and vary α in the cost vector (α, c_{-i}). Then the types are in the grid $V = \times_{i=1}^n V_i$. I construct $P \in \Delta(V)$, such $\pi(c_{-0}) = P(c_{-0})$ for any $c \in \mathbb{R}_0^{n+1}$. $V = \times_{i=1}^n V_i$ is a product space. Notice that

$$\rho_0^+(c) = \lim_{\varepsilon \to 0} P(v \in V, 0 + \varepsilon \geq v_i - c_i, i = 1, ..., n) = P(v \in V, v \leq c) = \sum_{v \leq c, v \in V} P(v). \tag{3.6}$$

The following lemma suggests that $P(v)$ can be recovered as in 3.7:

Lemma 2 (Mobius Inversion Theorem)**.** *Let $(L, \leq)$ be a poset, let μ be its Mobius function, let $P : L \to \mathbb{R}$ and let $\rho : L \to \mathbb{R}$ be given by $\rho(x) = \sum_{y \leq x} P(y)$. Then $P(x) = \sum_{y \leq x} \mu(y, x)\rho(y)$.*

P defined on any poset can be constructed by Mobius inversion, as in Definition 3.1:

Definition 3.1. *Take any poset $(L, \leq)$. Define $P_L : L \to \mathbb{R}$ as follows:*

$$P_L(w) = \sum_{v \in L, v \leq w} \mu(v, w)\rho_0^+(v) \tag{3.7}$$

$$\mu(v, w) = \begin{cases} 1 & v = w \\ -\sum_{v \leq w'' < w} \mu(v, w'') & v < w. \end{cases} \tag{3.8}$$

and $P_L(\emptyset) = 0$.

For any $A \subseteq \{1, ..., n\}$, $V_A = \times_{i \in A} V_i$, $P : V_A \to \mathbb{R}$ as in Definition 3.1.

3.7. **Path Independent Choice Functions.** Choice functions are defined on menus. In my framework, a menu is a collection of pairs of consumption goods and finite wealth levels. Define $V^* = \times_{i=1}^n V_i \cup \{\infty\}$. Let $w \in V^*$. The menu A associated with w is such that $A = \{i \in Z : w_i \in V_i\}$. The choice function is defined as $\varphi(A) = \{\rho_i(w_\varepsilon) > 0, i \in A\}$.

Proposition 2. *$\varphi(A)$ satisfies path independence. That is,*

1. *Sen's α: $A \subset B$, then $\varphi(B) \cap A \subset \varphi(A)$.*
2. *Aizerman and Malishevski (AM): $\varphi(B) \subset A \subset B \implies \varphi(A) \subset \varphi(B)$.*

Proof. Take $A \subset B \subset Z$. Therefore, B is associated with $w \in V^*$, and A is associated with $w' \in V^*$ such that $w'_i = w_i$ for $i \in A$, and $w'_j = \infty$ for $j \in Z \setminus A$.

Show Sen's α. Take $i \in A$. Then Marginal Consistency, $\rho_i(w'_\varepsilon) = \rho_i(w''_\varepsilon)$, such that $w''_i = w_i$ for $i \in A$,and $w''_j = \infty$ for $j \in B \setminus A$. Then by Monotonic Finite Jumps and $\rho_i(w_\varepsilon) < \rho_i(w''_\varepsilon) = \rho_i(w'_\varepsilon)$. Thus $\rho_i(w_\varepsilon) > 0 \implies \rho_i(w'_\varepsilon) > 0$. Hence if $i \in \varphi(B) \cap A$, $i \in \varphi(A)$.

Show AM. $\pi \in \Delta(V)$. Since $\pi(v \in V : v_j + w_j > v_k + w_k, k \in B) = 0$ for all $j \in B \setminus A$, $v_j + w_j < v_k + w_k$ for some $k \in A$ for all v with $\pi(v) > 0$. Suppose $i \in \varphi(A)$. There is v such that $\pi(v) > 0$ where $v_i + w_i > v_l + w_l$ for all $l \in A$. Since $k \in A$, $v_i + w_i > v_k + w_k > v_j + w_j$ for all $j \in B \setminus A$. Therefore, $v_i + w_i \geq v_h + w_h$ for all $h \in B$. Hence $i \in \varphi(B)$. □

Path independent choice function can be decomposed into the union of choices of a collection of types (Aizerman & Malishevski, 1981). By Chambers and Yenmez (2017, Theorem 1), when the agents' choice functions are path independent, the worker-proposing deferred acceptance algorithm produces the worker-optimal stable matching. This matching is also the firm-pessimal stable matching.

4. Proof of Theorem 1

Several steps are requires to prove the sufficiency of the Axioms. First, I show that the types are quasi-linear for each consumption good. Following that, I show that the types are in the grid of join valuations of each consumption good, construct the RQUM $\pi \in \Pi$, and show that it is a discrete probability measure. Then I show that π represents ρ on generic points. Finally, I prove the existence of tie-breaking rules in T for non-generic points.

4.1. Types are quasi-linear on $\{0, i\}, i \in Z$.

In this subsection, I prove Lemma 1.

In the first step, I construct the set of types on each coordinate i. Define $c^i \in \mathbb{R}_0^{n+1}$ to be such that $c^i_j \to \infty$ for all $j \in Z \setminus \{0\}, j \neq i$. By Axioms NC and FR, $\rho_j(c^i) = 0$ for all $j \neq i, 0$. $\rho_i(x, c^i_{-i})$ is piece-wise constant in x and is discontinuous at finitely many points of $x \in \mathbb{R}$. Denote the set of discontinuity points of x to be $V_i = \{v_{i,1}, ..., v_{i,m_i}\}$, where $v_{i,1} < v_{i,2} < ... < v_{i,m_i}$.

$\rho_i(x, c^i_{-i})$ is not right continuous nor increasing as a discrete probability cumulative distribution function. Instead, the behavior at a jump is determined by a tie-breaking rule. However, we can construction $F^i : \mathbb{R} \to [0, 1]$ from $\rho_i(x, c^i_{-i})$ as follows.

$$F^i(x) = \begin{cases} 1 - \rho_i^-(x, c^i_{-i}) & \text{for } x \in V_i, \\ 1 - \rho_i(x, c^i_{-i}) & \text{otherwise} \end{cases}$$

By Axiom 2 and 4,

$$\lim_{x \to -\infty} F(x) = \lim_{x \to \infty} \rho_i(x, c^i_{-i}) = 0 \quad \text{and} \quad \lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \rho_i(x, c^i_{-i}) = 1.$$

$F^i(x)$ is piecewise constant, non-decreasing, and right continuous. Hence, F^i is a distribution function of random variable V_i, where $F^i(x) = Prob(V_i \le x)$, and

$$Prob(V_i = x) = F^i(x) - \lim_{\varepsilon \to 0} F^i(x - \varepsilon) = \begin{cases} \rho_i(x, c^i_{-i}) - \rho^+_i(x, c^i_{-i}) = 0 & x \notin V_i \\ \rho^-_i(x, c^i_{-i}) - \rho^+_i(x, c^i_{-i}) = gap_i(x) & x \in V_i. \end{cases}$$

Define $\{\succeq^i\}$ to be a set of weak orders for consumption good i such that

$$Prob((i, x) \succeq^i (0, 0)) = Prob(V_i \le x) = F^i(x).$$

Hence, $F^i((i, x) \succeq^i (0, 0)) = F^i(x)$. $Prob((i, x) \sim^i (0, 0)) = Prob(V_i = x)$ is positive only for $x \in V_i$, and there is a finite set of rankings $\{\succeq^i_k\}_{k=1}^{m_i}$, such that

$$Prob((i, v_{i,k}) \sim^i_k (0, 0)) = Prob(V_i = v_{i,k}), k = 1, ..., m_i.$$

Take $v_{i,k} \in V_i$, $(i, x) \succ^i_k (0, 0)$ for $x < v_{i,k}$, $(i, x) \prec^i_k (0, 0)$ for $x > v_{i,k}$, and $(i, x) \sim^i_k (0, 0)$ for $x = v_{i,k}$.

Let $\mathbf{1} \in \mathbb{R}^{n+1}$ be such that $\mathbf{1} = (1, 1, ..., 1)$. By Axiom 1, for any constant vector $\alpha\mathbf{1}$ for $\alpha \in \mathbb{R}$,

$$F^i(x + \alpha) = \begin{cases} 1 - \rho^-_i(x + \alpha\mathbf{1}) & \text{for } x \in V_i, \\ 1 - \rho_i(x + \alpha\mathbf{1}) & \text{otherwise} \end{cases} = \begin{cases} 1 - \rho^-_i(x) & \text{for } x \in V_i, \\ 1 - \rho_i(x) & \text{otherwise} \end{cases} = F^i(x).$$

Define $Prob((i, x) \succeq^i (0, \alpha)) = Prob(V_i \le \alpha + x) = F^i(x + \alpha)$. So

$$Prob((i, x + \alpha) \succeq^i (0, \alpha)) = F^i(x + \alpha) = F^i(x) = Prob((i, x) \succeq^i (0, 0))$$

and

$$Prob((i, x + \alpha) \sim^i (0, \alpha)) = F^i(x) - \lim_{\varepsilon \to 0} F^i(x - \varepsilon) = Prob((i, x) \sim^i (0, 0)).$$

Hence, $Prob((i, x + \alpha) \sim^i (0, 0)) = Prob((i, x) \sim^i (0, 0)) > 0$ only when $x \in V_i$, and

$$\begin{aligned} (i, v_{i,k}) \sim^i_k (0, 0) &\iff (i, v_{i,k} + \alpha) \sim^i_k (0, \alpha) \\ &\iff (i, \alpha) \sim^i_k (0, -v_{i,k} + \alpha) \iff (i, 0) \sim^i_k (0, -v_{i,k}), k = 1, ...m_i. \end{aligned} \tag{4.1}$$

Furthermore, $(i, x + \alpha) \succ^i_k (0, \alpha)$ when $x < v_{i,k}$, $(i, x + \alpha) \sim^i_k (0, \alpha)$ when $x = v_{i,k}$, and $(i, x + \alpha) \prec^i_k (0, \alpha)$ when $x > v_{i,k}$, where $\alpha \in \mathbb{R}$ is arbitrary. Therefore,

$$y < y' \iff (i, y) \sim^i_k (0, -v_{i,k} + y) \succ^i_k (i, y') \sim (0, -v_{i,k} + y'). \tag{4.2}$$

Let $U_k(i, x), x \in \mathbb{R}$ represents $\succ^i_k$. (4.2) suggests

$$U_k(0, -v_{i,k} + y) >^i_k U_k(0, -v_{i,k} + y') \iff y < y'.$$

this preference can be represented by $U_k(0, x) = -x$ for $x \in \mathbb{R}$. (4.1) suggest that

$$U_k(i, \alpha) = U_k(0, -v_{i,k} + \alpha) = v_{i,k} - \alpha, k = 1, ..., m_i.$$

Hence, for $V_i = \{v_{i,1}, ..., v_{i,m_i}\}$, and $Z = \{0, i\}$, U_k is quasi-linear. The same argument goes for any $j \in Z \setminus \{0\}$. The probability distribution on the types $\pi_{\{0,i\}} \in \Delta(V_i)$ is such that $\pi_{\{0,i\}}(v_{i,k}) = Prob(v_{i,k})$. It is obvious that $\pi_{\{0,i\}} \in \Delta(V_i)$ is a probability measure for any $i \in Z \setminus \{0\}$.

4.2. **Show π is a Discrete Probability Measure.** When $|Z| = 2$, the probabiliby on the types constructed with (2.9) is the same as the the PDF for the random variables in step 1. I have shown that types are quasi-linear on each consumption goods. In this subsection, I show that when $|Z| > 2$, the RQUF $\pi \in \Delta(\mathbb{R}_0^{n+1})$ on the types can be constructed from ρ as in (2.9), and is a probability measure on $\{0\} \times_{i=1}^n V_i$.

Denote the set of types for which $i \in Z$ is a maximizer for $c \in \mathbb{R}^{n+1}$ as follows:

$$M_i(c) = \{v \in \text{supp}(\pi) : i \in M(v, c)\}. \tag{4.3}$$

The set function $\rho_A^+(c)$ is the probability on $\cup_{i\in A} M_i$. $\pi(\cap_{i\in A} M_i(c))$ is the probability on the type that ranks all elements in A as the maximizers. So

$$\pi(\cap_{i\in A} M_i(c)) = \pi(v \in \text{supp}(\pi) : v_i = c_i \ \forall \ i \in A) = \sum_{A' \subset A} (-1)^{|A'|+1} \rho_{A'}^+(c) \tag{4.4}$$

Notice that the collection of sets that only restricts the types on $A \subset Z$, i.e., $\mathcal{S} = \{\{c \in \{0\} \times_{i=1}^n : c_i = c_i^*, c_i^* \in \mathbb{R}, \forall i \in A, A \subset Z\}\}$ forms a finite semialgebra. Further, $\pi(\emptyset) = 0$ by construction. So to show $\pi \in \Delta(\{0\} \times_{i=1}^n$ one just need to show that (1) π is additive on $\mathcal{S}$, and that (2) $\pi(\{0\} \times_{i=1}^n V_i) = 1$. Denote

$$\pi_A(c) = \pi(\cap_{i\in A} M_i(c)).$$

Lemma 3 implies that π is additive on $\mathcal{S}$.

Lemma 3. *Take $c \in \mathbb{R}^{n+1}$ such that $\sum_{i\in S} \rho_i(c) = 1$ for some $S \subseteq Z$. For any $j \notin S$,*

$$\pi_S(c) = \sum_{\alpha \in \mathbb{R}} \pi_{S\cup\{j\}}(\alpha, c_{-j}) \tag{4.5}$$

Proof. By Axiom 4, there is a set of finite discontinuity point $V_j = \{\alpha_1, ..., \alpha_m\}$, $\alpha_1 < \alpha_2 < ... < \alpha_m$ for $\rho(\alpha, c_{-j})$ such that $\rho_j^+(\alpha_1, c_{-j}) = 1, \rho_j^-(\alpha_m, c_{-j}) = 0$. Furthermore,

$$\rho_j^-(\alpha_k, c_{-j}) = \rho_j^+(\alpha_k, c_{-j})$$

for $k = 1, ..., m-1$. When c_{-j} is fixed, we can obtain the conditional probability from $\rho_j(\alpha, c_{-j})$:

$$\frac{\pi_{S\cup\{j\}}(\alpha_k, c_{-j})}{\pi_S(c)} = \rho_j^+(\alpha_k, c_{-j}) - \rho_j^-(\alpha_k, c_{-j}) = gap_j(\alpha_k, c_{-j}).$$

Therefore,

$$\sum_{\alpha\in\mathbb{R}} \frac{\pi_{S\cup\{j\}}(\alpha_k, c_{-k})}{\pi_S(c)} = \rho_j(\alpha_1 - \varepsilon, c_{-j}) - \rho_j(\alpha_m + \varepsilon, c_{-j}) = 1.$$

Hence (4.5) holds. □

Hence, if $\pi(c) > 0$, $\pi_i(c_i) = \sum_{c_{-i}\in\mathbb{R}^n} \pi(c_{-i}, c_i) > 0$. Therefore $\text{supp}(\pi) \subseteq \{0\} \times_{i=1}^n V_i$. Next I show $\pi(\{0\} \times_{i=1}^n V_i) = 1$. Let $V_1 = \{\alpha_1, ..., \alpha_{m_1}\}$.

$$\pi(\{0\} \times_{i=1}^n V_i) = \sum_{c_1\in V_1} ... \sum_{c_n\in V_n} \pi(0, c_1, ..., c_n) = \sum_{c_1\in V_1} .. \sum_{c_{n-1}\in V_{n-1}} \pi_{Z\setminus\{n\}}(0, c_1, ..., c_{n-1}) =$$
$$= ... = \sum_{c_1\in V_1} \pi_{\{0,1\}}(0, c_1) = \sum_{c_1\in V_1} \rho_1^+(0, c_1) + \rho_0^+(0, c_1) - 1$$
$$= \sum_{c_1\in V_1} \rho_1^+(0, c_1) - \rho_1^-(0, c_1) = \rho_1^+(0, \alpha_1) - \rho_1^-(0, \alpha_{m_1}) = 1.$$

So far, I have shown that agents are quasi-linear types on the consumption goods, and $\pi \in \{0\} \times_{i=1}^n V_i$ is a probability distribution on the finite types. Next, I show that π represents the random choice rule ρ.

4.3. **Recover ρ from π.** Lemma 4 shows that j is not a maximizer or j is a unique maximizer associated with the vector $c \in \mathbb{R}^{n+1}$ if and only if ρ is constant under small variations on the cost of j, i.e., $\rho_j^+(c) = \rho_j^-(c)$. This condition relates the changes of π under perturbation in several coordinates to the change in a single coordinate of ρ.

$\pi_A(c)$ is the probability on all types such that $A \subset M(v, c)$. Denote

$$\tilde{\pi}_A(c) = \pi(v : M(v, c) = A).$$

By inclusion-exclusion principle,

$$\tilde{\pi}_A(c) = \sum_{A'\subseteq Z: A\subset A'} (-1)^{|A'\setminus A|}\pi_{A'}(c)) \tag{4.6}$$

Lemma 4. *For any $i \in Z$, $c \in \mathbb{R}^{n+1}$,*

$$gap_i(c) = \sum_{S\subset Z_{-i}, |S|\geq 1} \tilde{\pi}_{S\cup\{i\}}(c). \tag{4.7}$$

Proof of Lemma 4. I show Lemma 4 by showing the following:

$$\sum_{S\subset Z_{-j}, |S|\geq 1} \tilde{\pi}_{S\cup\{j\}}(c) \overset{(1)}{=} \sum_{S\subset Z_{-j}, |S|\geq 1} (-1)^{|S|-1}\pi_{S\cup\{j\}}(c) \overset{(2)}{=} \rho_j^+(c) + \rho_{Z_{-j}}^+(c) - \rho_Z^+(c) \tag{4.8}$$

$$= \rho_j^+(c) - (1 - \rho_{Z_{-j}}^+(c)) = gap_j(c) \tag{4.9}$$

First show (1). From (4.6), for any $S \subset Z$, the coefficient on $\pi_{S\cup\{j\}}$ will include contributions from $\tilde{\pi}_{S'}(c)$ for all $S' \subset S$. The coeffiicent on $\pi_{S\cup j}$ is:

$$\sum_{|S'|=1}^{|S|} (-1)^{|S\setminus S'|} \frac{\binom{n}{|S'|}\binom{n-|S'|}{|S|-|S'|}}{\binom{n}{|S|}} = (-1)^{|S|} \sum_{|S'|=1}^{|S|} (-1)^{|S'|} \frac{\binom{n}{|S|}\binom{|S|}{|S'|}}{\binom{n}{|S|}}$$
$$=(-1)^{|S|} \sum_{|S'|=1}^{|S|} (-1)^{|S'|} \binom{|S|}{|S'|} = (-1)^{|S|} \left(\sum_{|S'|=0}^{|S|} (-1)^{|S'|} \binom{|S|}{|S'|} - (-1)^0 \right)$$

$$=(-1)^{|S|}\left((1-1)^{|S|}-1\right)=(-1)^{|S|-1}$$

The term $\binom{n}{|S'|}$ is the number of sets with cardinaitliy $|S'|$. The term $\binom{n-|S'|}{|S|-|S'|}$ is the number of sets with cardiality $|S|$ containing a set with cardinality $|S'|$. The multiplication of the two term gives the total nubmer of sets with cardinality $|S|$ associated with all sets with cardiaility $|S'|$. The term $\binom{n}{|S|}$ is the number of sets with cardinailty $|S|$ formed from the set Z_{-j}. So the fraction $\frac{\binom{n}{|S'|}\binom{n-|S'|}{|S|-|S'|}}{\binom{n}{|S|}}$ is the absolute value of the coefficient on each $\pi_{S\cup\{j\}}$ from all $\tilde{\pi}_{S'\cup\{j\}}$ with $S'\subseteq S$, $S'\neq\emptyset$ fixed. The signs are obtained by observing (4.6).

Next show (2). Notice that by (4.4), for ρ_j^+, the coefficients are:

$$\binom{n}{1}-\binom{n}{2}+...+(-1)^{n+1}\binom{n}{n}=-(1-1)^n+\binom{n}{0}=1.$$

(4.4) also requires that $\rho^+_{S\cup\{j\}}$ and ρ^+_S for $1\leq|S|\leq n-1$ come from $\pi_{S'\cup\{j\}}$ for all $S\subset S'$. The coefficient on ρ^+_S and $\rho^+_{S\cup\{j\}}$ are the same:

$$\sum_{|S'|=|S|}^{n}(-1)^{|S|-1}\frac{\binom{n}{|S'|}\binom{|S'|}{|S|}}{\binom{n}{|S|}}=\sum_{|S'|=|S|}^{n}(-1)^{|S|-1}\frac{\binom{n}{|S|}\binom{n-|S|}{|S'|-|S|}}{\binom{n}{|S|}}=\sum_{|S'|=|S|}^{n}(-1)^{|S|-1}\binom{n-|S|}{|S'|-|S|}$$
$$=\sum_{i=0}^{n-|S|}(-1)^{|S|-1}\binom{n-|S|}{i}=(-1)^{|S|-1}(1-1)^{n-|S|}=0.$$

The term $\binom{n}{|S'|}$ is the number of all $\pi_{S'\cup\{j\}}$ with fixed cardinality $|S'|$. $\binom{|S'|}{|S|}$, $S\subset S'$ is the number of sets with cardinaility $|S|$ that can be obtained from sets with cardinaility $|S'|$. The multiplication gives the total number of $\rho^+_{S\cup\{j\}}$ for a fixed cardinality $|S|$. $\binom{n}{|S|}$ gives the number of distinct $\rho^+_{S\cup\{j\}}$ for fixed cardinaility $|S|$. So the term $\frac{\binom{n}{|S'|}\binom{|S'|}{|S|}}{\binom{n}{|S|}}$ is the absolute value of coefficients on the term $\rho^+_{S\cup\{j\}}$ for each S.

Note that $\rho^+_{Z_{-j}\cup\{j\}}=\rho^+_Z$ and $\rho^+_{Z_{-j}}$ only come from the term $(-1)^{n+1}\pi_Z(c)$. The coefficient on ρ^+_Z is $(-1)^{n-1}(-1)^n=-1$, and the coefficient on $\rho^+_{Z_{-j}}$ is $(-1)^{n-1}(-1)^{n-1}=1$. Hence (2) holds.

The rest of (4.8) comes from the obervation that $\rho^+_Z=1$ and $\rho_j(c_j+\varepsilon,c_{-j})=1-\rho^+_{Z_{-j}}$ by Axiom 1 and that ρ is a probability measure. □

Lemma 5. *One can construct $\rho'(c)$ with π such that $\rho'(c)=\rho(c)$ for any generic $c\in\mathbb{R}^{n+1}$.*

Proof. Take a point $c_0\in\mathbb{R}^{n+1}$ such that $\rho_0(c_0)=1$. Find a gird-like path from c_0 to any $c_1\in R^{n+1}$ such that there are at most two-way ties on the path. This path exists because the types are finite and the generic points are dense in $\mathbb{R}^{n+1}$.

Take any point c on the path. Denote

$$c^-=\lim_{\varepsilon\to 0}(c_i+\varepsilon,c_{-i}),\quad c^+=\lim_{\varepsilon\to 0}(c_i-\varepsilon,c_{-i}).$$

Let $\rho'(c)$ be the constructed choice function on c. If c is a generic point, then $\pi_{S\cup\{i\}} = 0$ for all $S \subseteq\in Z_{-i}$. Suppose c^- is also on the path. Then let $\rho'(c^-) = \rho'(c)$.

If c has two-way tie where i and j are the maximizers, then suppose both c^+ and c^- are on the path. This requires the path to not take a turn at points with two-way ties. At c, $\pi_{\{i,j\}} > 0$ and $\pi_{S\cup\{i\}} = 0$ for all $S \subseteq Z_{-i}, S \neq \{j\}$. Construct $\rho'(c^-)$ as follows.

$$\rho'_i(c^-) = \rho'_i(c^+) - \pi_{i,j}(c), \quad \rho'_j(c^-) = \rho'_j(c^+) + \pi_{i,j}(c), \quad \rho'_k(c^-) = \rho'_k(c^+)\ \forall\ k \neq i, j.$$

I show that if c is a generic point, then if $\rho(c) = \rho'(c)$, then $\rho(c^-) = \rho'(c^-)$. If i, j are the only two maximizers for c, if $\rho(c^+) = \rho'(c^+)$, then $\rho'(c^-) = \rho'(c^-)$.

Case 1: points with no ties. By Lemma 4, $gap_i(c) = 0$ for all $i \in Z$. So $\rho(c) = \rho(c^+) = \rho(c^-)$. So if $\rho(c) = \rho'(c)$, then $\rho(c^-) = \rho'(c^-)$.

Case 2: Points with a two-way-tie Suppose at c there is a two-way tie between i and j. So $\pi_{i,j}(c) > 0$, $\pi_{S\cup\{i\}}(c) = 0$ for all other $S \subseteq Z_{-i}$, $S \neq \{j\}$. By (4.6), $\tilde{\pi}_{\{i,j\}} = \pi_{\{i,j\}}$. By Lemma 4,

$$gap_i(c) = gap_j(c) = \pi_{\{i,j\}}(c), \quad gap_k(c) = 0\ \forall\ k \in Z, k \neq i, j.$$

So $\rho_i^-(c) = \rho_i^+(c) - gap_i(c)$, equivalently, $\rho_i(c^-) = \rho_i(c^+) - gap_i(c)$. I show $\rho_j(c^+) = \rho_j(c^-) - gap_j(c)$.

By Axiom 2,

$$\rho_k^-(c) \le \rho_k(c) \le \rho_k(c^-) \le \rho_k(c_i + \varepsilon, c_j + \varepsilon, c_{-i,j}),$$
$$\rho_k(c_i - \varepsilon, c_j - \varepsilon, c_{-i,j}) \le \rho_k(c^-) \le \rho_k(c) \le \rho_k^+(c).$$

for all $k \in Z_{-i,j}$ If $\rho_{ij}^-(c) \neq \rho_{ij}^+(c)$, then there exists $k \in Z_{-ij}$ such that

$$\rho_k(c_i - \varepsilon, c_j - \varepsilon, c_{-i,j}) < \rho_k(c_i + \varepsilon, c_j + \varepsilon, c_{-i,j}).$$

Hence, $\rho_k^-(c) < \rho_k^+(c)$, and therefore $gap_k(c) > 0$, this is a contradiction. Hence, $\rho_{ij}^-(c) = \rho_{ij}^+(c)$. $\rho_{ij}^+(c) = \rho_{ij}^-(c)$ implies that $\rho_{ij}^+(c) = \rho_{ij}^-(c) = \rho_{ij}(c)$, and thus $\rho_k(c_i - \varepsilon, c_j - \varepsilon) = \rho_k(c_i + \varepsilon, c_j + \varepsilon)$. So $\rho_k(c^+) = \rho_k(c^-) = \rho_k(c)$. Hence $\rho_i(c^+) + \rho_j(c^+) = \rho_i(c^-) + \rho_j(c^-)$. Therefore

$$\rho_j(c^-) = \rho_i(c^+) + \rho_j(c^+) - \rho_i(c^-) = \rho_j(c^+) + gap_j(c)$$

Hence, in the observation,

$$\rho_i(c^-) = \rho_i(c^+) - gap_i(c), \quad \rho_j(c^-) = \rho_j(c^+) + gap_j(c), \quad \rho_k(c^+) = \rho_k(c^-)\ \forall\ k \neq i, j$$

So if $\rho(c) = \rho'(c)$, then $\rho(c^-) = \rho'(c^-)$.

Therefore, on a the grid-like path from c_0 to $c_1 \in \mathbb{R}^{n+1}$ such that the there are at most two-way ties on the path, $\rho'(c) = \rho(c)$ for any point c on the path, and thus $\rho'(c_1) = \rho(c_1)$. $\square$

4.4. **Existence of a Tie-breaking Rule.** At non-generic point c, for some $v \in \text{supp}(\pi)$, $M(v,c) > 1$. I show that there is $t : \mathbb{R}_0^{n+1} \times \mathbb{R}^{n+1} \to \Delta(Z)$ for all $v \in \text{supp}(\pi)$ such that

$$\rho_A(c) = \sum_{v \in \text{supp}(v)} \pi(v)t(v,c).$$

Consider the case that $\rho_A(c) \in \mathbb{Q}$ and $\pi(v) \in \mathbb{Q}$ for all $A \subseteq Z$ and and $v \in \text{supp}(\pi)$. Let $\frac{1}{k} \in \mathbb{N}$ be a *unit* as defined in Definition 2.3. So $\pi(v)k \in \mathbb{N}$ for all $v \in \text{supp}(\pi)$ and $k\rho_A(c) \in \mathbb{N}$ for all $A \subseteq Z$. Each type $v \in \text{supp}(\pi)$ is treated as $k\pi(v)$ copies agent units (a-units) v' where $\pi(v') = \frac{1}{k}$; and each $\rho_i(c)$ is treated as $k\rho_A(c)$ copies of probability units (p-units) i'. So $\pi(v') = \frac{\pi(v)}{k}$, and $\rho_{i'}(c) = \frac{\rho_i(c)}{k}$.

For any $A \subset Z$, each a-units v' has edges to all p-units i' such that $i \in M(v,c)$, $i \in A$. Call the edge a demand. The goal is to show that for any $\rho_A(c)$, there is enough p-units $\frac{1}{k}$ to satisfy the demands. This problem can be formulated in to a bipartite matching of assigning the p-units to the a-units in a one-to-one matching.

By Hall's marriage theorem, the necessary and sufficient condition for the existence of a one-to-one matching is:

$$|i' : i \in \rho_A(c)| \geq |\{v' : v \in \text{supp}(\pi), M(v,c) \subset A\}|$$

Since $|i' \in A| = k\rho_A(c)$, and $|\{v' : v \in \text{supp}(\pi), M(v,c) \subset A\}| = k\pi\{v \in \text{supp}(\pi) : M(v,c) \subset A\}$, Lemma refhall shows that hall's condition can be satisfied.

Lemma 6. *For any non-generic c.*

$$\rho_A(c) \geq \pi\{v \in supp(\pi) : M(v,c) \subset A\}.$$

Proof. Take a generic point c' such that $\|c - c'\| < \varepsilon$ for any $\varepsilon > 0$, $c' \gg_A c$ for any $A \subset Z$. Then

$$\rho_A(c') = \pi(v \in \text{supp}(\pi) : M(v,c') \subset A).$$

Since π represents ρ on c'. By Axiom 2,

$$\rho_A(c) \geq \rho_A(c').$$

Furthermore, $\{v \in \text{supp}(\pi) : M(v,c) \subset A\} \subseteq \{v \in \text{supp}(\pi) : M(v,c') \subset A\}$ since $c' \gg_A c$. So

$$\pi\{v \in \text{supp}(\pi) : M(v,c) \subset A\} \leq \pi\{v \in \text{supp}(\pi) : M(v,c') \subset A\}.$$

Therefore,$\rho_A(c) \geq \pi\{v \in \text{supp}(\pi) : M(v,c) \subset A\}$ holds. □

Hence there exists a matching between the a-units and p-units. Denote (v', i') to be a matching. For any $v \in \text{supp}(\pi)$,

$$t_i(v,c) = \frac{|(v',i')|}{|v'|}.$$

The construction (2.9) suggests that if $\rho_A(c) \in \mathbb{Q}$ for all $A \subset Z, c \in \mathbb{R}^{n+1}$, then $\pi(v) \in \mathbb{Q}$ for all $v \in \text{supp}(\pi)$. Suppose $\rho_A(c) \notin \mathbb{Q}$ for some $A \subseteq Z$, $c \in \mathbb{R}^{n+1}$. Construct a sequence $\rho^j, j = 1, 2, ...$ such that $\rho^j_A(c) \in \mathbb{Q}$ for all $A \subset Z, c \in \mathbb{R}^{n+1}$, $\rho^j \to \rho$. Hence, (2.9) construct π^j from ρ^j such that $\pi^j(v) \in \mathbb{Q}$ for all $v \in \text{supp}(\pi^j)$, where $\pi^j \to \pi$.

Each (ρ^j, π^j) is a matching problem in the rationals, and by the discussion before, there exists t^j that solves this problem. Since Z is finite, $\Delta(Z)$ is compact. $t(v, c)$ is in $\Delta(Z)$ for any $v \in \mathbb{R}^{n+1}_0, c \in \mathbb{R}^{n+1}$. Hence, $t^j(v^j, c)$ is a bounded sequence. By Bolzano-Wierstrauss theorme, it has a covergent subsequence, $t^{j_k}(v^{j_k}, c) \to t(v, c)$, where $t(v, c)$ solves the problem for ρ, π with $v \in \text{supp}(\pi)$.

4.5. **Proof of Necessity.** Assume there exists $(\pi, t) \in \Pi \times T$ such that (2.3) holds.

Show Axiom 1. The agents have quasi-linear types. S Take $\alpha \in \mathbb{R}$. Then by (4.3)

$$M_i(c) = \{v \in \text{supp}(\pi) : v_i - c_i > v_j - c_j\} = \{v \in \text{supp}(\pi) : v_i - (c_i + \alpha) > v_j - (c_j + \alpha)\} = M_i(c + \alpha\mathbf{1}).$$

By (2.2) and the fact that $M_i(c) = M_i(c + \alpha\mathbf{1})$, Axiom 1 holds.

Axiom 2 follows from (2.2) and the representation (2.3) because $M_i(c)$ is a non-increasing set function with c, and therefore $t_i(v, c)$ is non-increasing with c_i by definition.

Axiom 3 holds because π is a probability measure and for $0 < \alpha < \beta$,

$$\rho_A(c - \beta\mathbf{1}_A) \geq \pi\{v \in \mathbb{R}^{n+1}_0 : \text{there is } i \in A \text{ s.t. } v_i - c_i \text{ is within } \alpha \text{ from } \max_{j \in Z} v_j - c_j\},$$

since all types with $(v_i - c_i) + \alpha \geq max_{j \in Z} v_j - c_j$ would chose i; while

$$\rho_A(c - \alpha\mathbf{1}_A) \leq \pi\{v \in \mathbb{R}^{n+1}_0 : \text{there is } i \in A \text{ s.t. } v_i - c_i \text{ is within } \alpha \text{ from } \max_{j \in Z} v_j - c_j\}.$$

since types with $(v_i - c_i) + \alpha = max_{j \in Z} v_j - c_j$ would choose i with a tie-breaking rule.

Axiom 4 holds because π is a discrete probability measure. Since $\text{supp}(\pi)$ is finite, and each $v_k \in \text{supp}(\pi)$ is associated with one point α_k such that $v_k \in M(\alpha, c_{-i}) \iff \alpha \leq \alpha_k$, $t_i(v, c)$ changes only finitely many times for $\alpha \in \mathbb{R}$. Hence $\rho_i(\alpha, c_{-i})$ changes only finitely many times for $\alpha \in \mathbb{R}$.

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