

Gradual Bargaining in Decentralized Asset Markets*

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Abstract

We introduce a new approach to bargaining, with strategic and axiomatic foundations, into models of decentralized asset markets. Bargaining with an agenda assumes that asset portfolios are partitioned into bundles sold sequentially, and encompasses both the Nash and Kalai solutions as special cases. Gradual negotiations are optimal for portfolio holders and, in general equilibrium, they raise social welfare by reducing asset misallocation. In the presence of multiple assets, our theory generates a pecking order, and differences in returns and turnover. We apply our model to the study of open-market operations and the determination of the exchange rate between (crypto-)currencies.

JEL Classification: D83

Keywords: decentralized asset markets, bargaining with an agenda, Nash program, rate-of-return dominance.

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1 Introduction

Modern monetary theory and financial economics formalize asset trades in the context of decentralized markets with explicit game-theoretic foundations (e.g., Duffie et al., 2005; Lagos and Wright, 2005). These models replace the elusive Walrasian auctioneer by a market structure with two core components: a technology to form pairwise meetings and a strategic or axiomatic mechanism to determine prices and trade sizes. This paper focuses on the latter: the negotiation of asset prices and trade sizes in pairwise meetings.

The need to revisit the bargaining foundations of models of decentralized asset markets follows from recent methodological advances allowing agents to hold unrestricted asset portfolios. Going back to Diamond (1982), the search-theoretic literature has placed stark restrictions on individual asset inventories, typically $a \in \{0, 1\}$. As a result, in versions of the model with bargaining (e.g., Shi, 1995; Trejos and Wright, 1995; Duffie et al., 2005), the only item to negotiate in pairwise meetings — the *agenda* of the negotiation — is the price of an indivisible asset in terms of a divisible commodity.¹ Because models with $a \in \{0, 1\}$ cannot tackle many important questions in monetary economics and finance, recent incarnations (surveyed in Lagos et al., 2017) allow for unrestricted portfolios of divisible assets, $\mathbf{a} \in \mathbb{R}_+^J$ with $J \in \mathbb{N}$.

A key conceptual difference between the bargaining problems when $a \in \{0, 1\}$ and $\mathbf{a} \in \mathbb{R}_+^J$ is that the agenda of the negotiation can no longer be left implicit in the latter case, i.e., there is a large set of possible agendas for a given portfolio when $\mathbf{a} \in \mathbb{R}_+^J$. Any ordered partition of $\mathbf{a} \in \mathbb{R}_+^J$ constitutes an agenda, where the elements of this partition correspond to items to be negotiated sequentially. For instance, agents can sell their whole portfolio at once, as a large block, or they can partition their portfolio into smaller bundles of same or different sizes to be added to the negotiation table one after another. When the portfolio is composed of multiple assets, $J \geq 2$, those assets can be traded in different orders. The possibility of negotiating asset sales according to different agendas raises several questions regarding trading strategies and price formation in decentralized asset markets. Do agendas matter when agents have perfect foresight and information is complete? What is the optimal strategy to partition the portfolio, e.g., should the portfolio be divided into smaller parts or negotiated as a whole? Does the outcome depend on the side (buyer or seller) choosing the agenda of the negotiation? What are the (implicit) agendas of standard bargaining solutions, such as the Nash or Kalai solutions? Does the order according to which distinct assets are sold matter for the distribution of rates of return?

¹A thorough treatment of the axiomatic and strategic solutions for such bargaining problems is provided by Osborne and Rubinstein (1990). In Osborne and Rubinstein (1990) agents trade an indivisible consumption good and pay with transferable utility. The interpretation is reversed in Shi (1995) and Trejos and Wright (1995) where the indivisible good is fiat money and agents negotiate over a divisible consumption good. In Duffie et al. (2005) the indivisible good is a consol and agents pay with transferable utility.

Our contribution is to introduce a new and generalized approach to bargaining over portfolios of assets in models of decentralized asset market with the notion of agenda at the forefront, under both strategic and axiomatic foundations. This approach, which encompasses existing bargaining solutions such as Nash (1950) and Kalai (1977), assumes that agents sell their assets sequentially according to a pre-specified agenda. We start with a simple agenda by partitioning a portfolio of homogeneous assets into N bundles of equal size. The extensive-form bargaining game is composed of N rounds. In each round, one asset bundle is up for negotiation. For simplicity, one player makes an ultimatum offer, and the identity of the proposer alternates across rounds. This alternating-ultimatum-offer bargaining game is nonstationary since the amount of assets left to negotiate decreases over time. We show the existence and uniqueness of a subgame-perfect equilibrium (SPE) and characterize equilibrium payoffs through a system of difference equations. We consider the limit as N goes to infinity, which we call the *gradual solution*. We show it gives a simple and intuitive relationship between asset prices and trade sizes, and it has properties that make it tractable for general equilibrium analysis, including monotonicity and concavity of payoffs in trade size. We generalize the game by studying an agenda composed of an uneven partition of the asset holdings, thereby generating asymmetric gradual solutions.

In order to relate the gradual solution to the Nash solution, we extend the game so that agents play an alternating-offer game with risk of breakdown, as in Rubinstein (1982), in each of the N rounds. In contrast to our original game, agents alternate making offers within each round, and not simply across rounds. For all $N \leq +\infty$, the path of agreements is a sequence of Nash solutions with endogenous disagreement points. The outcome of the negotiation varies with N , even though agents have perfect foresight, and it coincides with the Nash solution and the gradual solution in the limiting cases $N = 1$ and $N = +\infty$, respectively. Hence, the gradual solution is remarkably robust to the protocol in each round provided that the symmetry between players is preserved. If we let asset owners choose N in order to maximize their utility, then $N = +\infty$, i.e., they bargain gradually, one infinitesimal unit of asset at a time.²

Our description of the negotiation encompasses over-the-counter (OTC) bargaining games in the finance literature. We reinterpret our model as one where agents trade an illiquid asset, for which they have idiosyncratic valuations, in exchange for a liquid asset that is commonly valued (e.g., fiat money). So far, we considered agendas where agents trade their liquid assets gradually over time. Alternatively, if agents negotiate the illiquid asset one infinitesimal unit at a time, then the outcome coincides with the proportional solution of Kalai (1977). Remarkably, while the Kalai solution is not scale invariant, the gradual bargaining

²Interestingly, the gradual aspect of asset trades is a key characteristic of many trading practices observed on financial markets. For example, broker-dealers are known to break large orders ("block orders") into smaller ones and execute them over the span of several days (see, e.g., Chan and Lakonishok, 1995).

solution is ordinal (O’Neill et al., 2004).

We check the robustness of our solution by adopting the axiomatic approach of O’Neill et al. (2004) that abstracts from the details of the extensive-form game. The agenda, which is part of the definition of the bargaining problem, is formalized as a collection of expanding bargaining sets. We choose the agenda to be consistent with the assumption that agents add assets on the negotiating table gradually over time. The solution of O’Neill et al. (2004) is a path that shares three axioms with the Nash (1950) solution, Pareto optimality, scale invariance, and symmetry, and satisfies two new axioms, directional continuity and time consistency. The unique solution satisfying these five axioms coincides with the SPE of the alternating-ultimatum-offer bargaining game when $N = +\infty$.

The second part of the paper incorporates bargaining solutions with an agenda into a general equilibrium model of decentralized asset markets with endogenous portfolios. We use the fact that an agenda has an explicit time dimension (the items of the agenda are negotiated sequentially) to introduce a new asset characteristic – negotiability.³ Asset negotiability is defined as the amount of time required for the sale of each unit of the asset to be finalized, e.g., each asset added to the negotiation table needs to be authenticated and ownership rights take time to transfer.⁴ Hence, negotiability depends on the characteristics of an asset, e.g., tangibility and authenticability, as well as the technology to transfer asset ownership, e.g., by physical transfer of the asset if it is tangible, through a physical or electronic ledger, or through a blockchain technology.

We make this negotiability relevant by assuming that the time agents have to complete their negotiation is stochastic and exponentially distributed. On the positive side, the general equilibrium spread between the rate of return of the asset and the rate of time preference is the product of four components: the search friction, the bargaining power, the negotiability friction, and marginal gains from trade. Thus, bargaining affects asset prices through both bargaining powers and delays to reach and confirm an agreement.

On the normative side, if the asset is scarce, then the decentralized choice of asset negotiability (when it is endogenous) is too low relative to the planner’s choice, even if asset owners have all the bargaining power, because of a pecuniary externality. Assuming the negotiability constraint never binds (because the time horizon of the negotiation is sufficiently large), the equilibrium under Nash bargaining ($N = 1$) features asset misallocation: a fraction of the asset supply ends up being held by agents with no liquidity needs. In

³The concept of negotiability dates back to the 17th century and referred to institutional arrangements aiming at enhancing liquidity by “centralizing all rights to the underlying asset in a single physical document, [...] reducing the costs a prospective purchaser incurs in acquiring [...] information about the asset” (Mann, 1996). The concept of blockchains - immutable, decentralized ledgers that can record ownership and transfer of intangible assets - can be seen as a digital incarnation of the original idea of negotiability.

⁴According to Duffie (2012) search and matching frictions encompass not only “delays associated with reaching an awareness of trading opportunities” but also delays due to “arranging financing and meeting suitable legal restrictions, negotiating trades, executing trades, and so on.” For evidence on these delays, see, e.g., Saunders et al. (2002) and Pagnotta and Philippon (2018).

contrast, under gradual bargaining ($N = +\infty$), the first best is implemented as long as the asset supply is sufficiently abundant. This finding is especially stark in an OTC version of the model with fiat money: under Nash bargaining the OTC market shuts down and the equilibrium achieves its worst allocation whereas under gradual bargaining the OTC market is active and the equilibrium achieves first best.

Finally, we extend our environment to allow for any arbitrary number of assets. All assets, except fiat money, generate the same stream of dividends but differ by their negotiability. For instance, more complex assets take more time to be negotiated than simpler ones. If we let asset owners choose the order according to which assets are negotiated, then our model generates an endogenous pecking order: assets that are more negotiable are put on the negotiating table before the less negotiable ones. In equilibrium, the most negotiable assets have lower rates of return and higher velocities. Hence, our model explains rate-of-return differences of seemingly identical assets.

We conclude the paper by considering two applications that will showcase the relevance of our generalized approach to bargaining to address standard puzzles in monetary and financial economics, e.g., the rate-of-return-dominance puzzle and the nominal exchange rate indeterminacy. The first application has money and interest-bearing government bonds and studies the effects of open-market operations (OMOs). Our model predicts that an open market sale of bonds raises the nominal interest rate and reduces output because fiat money is replaced by less-negotiable bonds. Our second application is a dual-currency economy where the supplies of the two currencies grow at different rates and currencies differ in their negotiabilities. While it has been argued that the exchange rate is indeterminate in a world with multiple fiat currencies (e.g., Kareken and Wallace, 1981), we show that the exchange rate is determinate once one takes into account differences in negotiability: the currency with higher negotiability appreciates vis-a-vis the high-return currency if the frequency of trades increases, if the consumers' bargaining power increases, or if the time horizon of the negotiation shortens.

Related literature

Models of decentralized markets adopting a strategic approach to the bargaining problem in pairwise meetings were pioneered by Rubinstein and Wolinsky (1985). Bargaining with an agenda composed of multiple issues was first studied by Fershtman (1990). The axiomatic formulation with a continuous agenda used in this paper was developed by O'Neill et al. (2004). To the best of our knowledge, we provide its first application.⁵

We show how to identify the agenda of the negotiation in the context of decentralized asset market models,

⁵An early application can be found in the working paper of Rocheteau and Waller (2005) in the context of a pure currency economy. The bargaining solutions of Zhu and Wallace (2007) and Rocheteau and Nosal (2017) can also be interpreted as negotiations with an agenda, where the agenda bundles together assets of the same type (e.g., money holdings as one item and bond holdings as a separate item). These solutions, however, lack axiomatic or strategic foundations.

we propose different ways to endogenize it, and we provide strategic foundations.

While O'Neill et al. (2004) are silent about the strategic foundations of the solution, an earlier working paper by Wiener and Winter (1998) conjectures that a bargaining game with alternating offers should generate the same outcome. We formalize this conjecture in our context with two extensive-form games: an alternating-ultimatum-offer bargaining game and a "repeated" Rubinstein game. Our second game is related to the Stole and Zwiebel (1996) game in the literature on intra-firm wage bargaining. See Brugemann et al. (2018) for a recent re-examination of this game. In the Stole-Zwiebel game a firm with a strictly concave production function bargains sequentially with N workers. In each negotiation the wage is determined according to a Rubinstein game with alternating offers and exogenous risk of breakdown. While we describe sequential negotiations within a bilateral match, we could reinterpret our game as one where the asset owner bargains sequentially with multiple buyers provided that the disutility associated with the payment is linear. There are differences. In the intra-firm bargaining literature workers sell an indivisible unit of labor, whereas in models of asset markets agents sell divisible assets. Moreover, we let agents choose both the quantity of assets to sell and the number of rounds of the negotiation. The structure of the game is also different. In our game, if agents fail to reach an agreement in one round, they move to the next round, but the agreements of earlier rounds are preserved. In the Stole-Zwiebel game, all previous agreements are erased.

The extensive-form bargaining games we study are not stationary. Coles and Wright (1998) describe the strategic negotiation of indivisible units of money in continuous time in the non-stationary monetary equilibria of the model of Shi (1995) and Trejos and Wright (1995).

The concept of agenda has a natural time dimension since different parts of the portfolio are sold sequentially (see, e.g., O'Neill et al., 2004). Tsoy (2018) formalizes bargaining delays in the absence of common knowledge. He studies an alternating-offer bargaining game in OTC markets with $a \in \{0, 1\}$ where agents have private values that are affiliated. At the limit, when values become perfectly correlated, there is a class of equilibria converging to the Nash division of the surplus but agreements are reached with delays.

One of our results shows that agents prefer to bargain gradually, one infinitesimal unit of asset at a time. Relatedly, Gerardi and Maestri (2017) formalize the bargaining of a divisible asset under private information and show that gradual trading emerges endogenously. There is also a literature on the optimal execution of large asset orders, e.g. Bertsimas and Lo (1998).

The general equilibrium framework into which we incorporate bargaining games with an agenda corresponds to a version of the Lagos and Wright (2005) model with divisible Lucas trees, as in Geromichalos et al. (2007) and Lagos (2010).⁶ We also consider a variant where agents trade assets because of idiosyncratic

⁶In those models, the asset owner has all the bargaining power. Rocheteau and Wright (2013) adopt the proportional

valuations, as in Duffie et al. (2005). See also Lagos and Rocheteau (2009) and Uslu (2018) with unrestricted portfolios; Geromichalos and Herrenbrueck (2016a), Lagos and Zhang (2018), and Wright et al. (2018), with asset trades financed with money.⁷ Our paper clarifies the role of different assumptions regarding the bargaining protocol in those models, e.g., Lagos and Zhang (2018) use Nash while Wright et al. (2018) use Kalai.

Our extension with multiple assets contributes to the literature on asset price puzzles in markets with search frictions, e.g., Vayanos and Weill (2008) based on increasing-returns-to-scale matching technologies; Rocheteau (2011), Li et al. (2012) and Hu (2013) based on informational asymmetries; Lagos (2013) based on self-fulfilling beliefs in the presence of assets' extrinsic characteristics; and Geromichalos and Herrenbrueck (2016b) based on matching and bargaining frictions differentials across the secondary markets where each asset is traded. The application to open-market operations is related to Rocheteau et al. (2018) and references therein. The application to the determination of the exchange rate in a two-currency economy is related to Zhang (2014) and Gomis-Porqueras et al. (2017). Related to our notion of negotiability, Chiu and Koepl (2018) study the optimal design of the transfer of asset ownership using blockchain technologies.

2 Environment

There is an infinite (countable) number of periods, where each period is divided into two stages. In the first stage, there is a decentralized market, labeled DM, where agents trade goods and assets in pairwise meetings over some time interval $[0, \bar{\tau}] \subseteq \mathbb{R}_+$. The second stage, labeled CM (for centralized market), features a centralized Walrasian market. There is one good in each stage and we take the CM good as numeraire. There is a continuum of agents with measure two evenly divided between two types, called consumers and producers. An agent's type corresponds to his role in the first stage, where only consumers wish to consume the DM good while only producers have the technology to produce it. Throughout most of the paper we think of consumers as natural asset holders who receive liquidity shocks while producers are potential buyers of those assets. During the DM, a fraction α of consumers and producers are matched bilaterally.

Consumers' preferences are represented by the period utility function, $u(y) - h$, where y is total DM consumption and h is the disutility of producing h units of numeraire. Producers' preferences are represented by $-v(y) + c$, where y corresponds to total production of the DM good and c is the consumption of the numeraire. We assume $u'(y) > 0$, $u''(y) < 0$, $u'(0) = +\infty$, $u(0) = v(0) = v'(0) = 0$, $v'(y) > 0$, $v''(y) > 0$, and $v(\bar{y}) = u(\bar{y})$ for some $\bar{y} > 0$. Let y^* denote the solution to $u'(y^*) = v'(y^*)$. All agents share the same

bargaining solution, endogenize participation, and consider non-stationary equilibria. Lester et al. (2012) introduce a costly acceptability problem. Rocheteau (2011) and Li et al. (2012) add informational asymmetries.

⁷See Trejos and Wright (2016) for a model that nests Shi (1995), Trejos and Wright (1995) and Duffie et al. (2005).

discount factor across periods, $\beta \equiv (1 + \rho)^{-1} \in (0, 1)$.

Example 1 (*Interpretation of the DM as an OTC market.*) *At the end of each period, a unit measure of agents, called sellers, receive an equal endowment of one-period-lived real assets, Ω , that pay off at the end of the following period. The payoff to a seller from holding ω units of assets is $f_\ell(\omega)$. There is a distinct unit measure of agents, called buyers, who do not receive any asset in the CM but who value it according to the payoff function $f_h(\omega)$ where $f'_h(\omega) > f'_\ell(\omega) > 0$.⁸ Assets can only be reallocated in the DM, interpreted as an OTC market. For this formulation, $u(y) \equiv f_h(y)$ and $v(y) \equiv f_\ell(\Omega) - f_\ell(\Omega - y)$.*

Agents, who are anonymous, cannot issue private IOUs. This assumption creates a need for liquid assets. There is an exogenous measure of long-lived Lucas trees indexed on $[0, A]$ that are perfectly durable, storable at no cost, and non-counterfeitable. For now all trees are identical and a unit measure of trees pays off $d > 0$ units of numeraire in the CM. We will consider later the case of fiat money, $d = 0$. We denote ϕ_t the price of Lucas trees in terms of the numeraire.

In pairwise meetings the negotiation and transfer of assets take place during a time interval $[0, \bar{\tau}]$. The technology to authenticate and transfer ownership of assets is such that δ units of assets can be negotiated per unit of time. In the first part of the paper, δ plays no role because the time available for the negotiation, $\bar{\tau}$, is assumed to be large. However, in Sections 5, 6, and 7 the constraint imposed by $\bar{\tau}$ becomes relevant.

Before turning to the analysis of the pairwise negotiation we first derive some preliminary results that will be useful to set up the problem. We restrict our attention to stationary equilibria where the price of Lucas trees is constant at ϕ and hence their gross rate of return is also constant and equal to $R = 1 + r = (\phi + d)/\phi$. We measure a consumer's asset holdings in the DM in terms of their value in the upcoming CM. More precisely, a units of asset in the DM are worth $z = (\phi + d)a$. The lifetime expected utility of a consumer (i.e., buyer of DM goods) with wealth z in the CM is

$$W^b(z) = \max_{z', h} \{-h + \beta V^b(z')\} \quad \text{s.t.} \quad z' = R(z + h), \quad (1)$$

where z' are next-period asset holdings, and $V^b(z')$ is the value function at the start of the DM. From (1) the consumer chooses his production of numeraire and future asset holdings in order to maximize his discounted continuation value net of the disutility of production. According to the budget constraint, next-period asset holdings are equal to current asset holdings plus output from production, everything multiplied by the gross

⁸In Appendix D we study a version of this model with ex ante identical agents. One can interpret $f(\omega)$ as a production function and ω as physical capital, as in Rocheteau and Nosal (2017) or Wright et al. (2018). One can also think of $f(\omega)$ as reduced form utility for different services provided by the asset, e.g., liquidity and hedging services, as in Duffie et al. (2005) and Lagos and Rocheteau (2009).

rate of return of assets. Substituting h by its expression coming from the budget identity into the objective, we obtain

$$W^b(z) = z + \max_{z' \geq 0} \left\{ -\frac{z'}{R} + \beta V^b(z') \right\}. \quad (2)$$

As is standard, W^b is linear in wealth. There is a similar equation defining the value function of a producer (seller of the DM goods), $W^s(z)$.

3 Gradual bargaining

We describe a negotiation between a consumer (i.e., buyer of the DM good) holding $z > 0$ units of assets, expressed in terms of the numeraire, and a producer (i.e., seller of the DM good) with no asset. For now all Lucas trees are indistinguishable so that the agenda of the negotiation is simply a partition of the interval $[0, z]$. We start with a simple partition of N subsets of equal size, z/N , but we will also consider asymmetric agendas later on. We first propose an extensive-form game and then adopt an axiomatic approach to show the robustness of the solution.

3.1 The alternating-ultimatum-offer bargaining game

The game has N rounds. In each round, the consumer can negotiate at most z/N units of assets for some DM output. The round-game corresponds to a two-stage ultimatum game: in the first stage an offer is made; in the second stage the offer is accepted or rejected.⁹ In order to maintain some symmetry between the two players (when N is large), the identity of the proposer alternates across rounds. We assume N is even and the producer is the one making the first offer. These assumptions will be inconsequential when we consider the limit as N becomes large. The game tree is represented in Figure 1.

We define $\tau \equiv nz/(\delta N)$ as the time at the end of the n^{th} round (in each of the n rounds, z/N assets are up for negotiation, and each asset takes $1/\delta$ units of time to be negotiated.) The utility accumulated by the consumer up to τ is

$$u^b(\tau) = u[y(\tau)] + W^b[z - p(\tau)] = u[y(\tau)] - p(\tau) + u_0^b, \quad (3)$$

where $y(\tau)$ is the consumer's cumulative consumption at time τ , $p(\tau)$ is his cumulative payment with Lucas trees (expressed in terms of the numeraire), and $u_0^b = W^b(z)$. The utility accumulated by the producer up to τ is

$$u^s(\tau) = -v[y(\tau)] + p(\tau) + u_0^s, \quad (4)$$

⁹A feature of our game is that if an offer is rejected, the z/N units of assets that are unsold cannot be renegotiated later in the game. The solution to our game, however, is robust to this feature. See Appendix B.

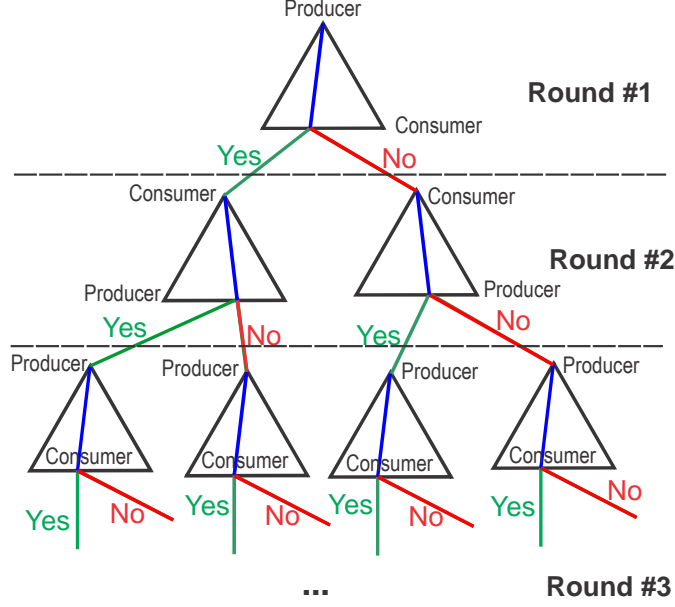


Figure 1: Game tree of the alternating-ultimatum-offer game

where $u_0^s = W^s(0)$. Given the feasibility constraint $p(\tau) \leq \delta\tau$, we can define a Pareto frontier for each τ , i.e.,

$$u^b = \max_{y, p \leq \delta\tau} \{u(y) - p + u_0^b\} \quad \text{s.t.} \quad -v(y) + p + u_0^s \geq u^s.$$

These Pareto frontiers play a key role to solve for the SPE of the game by backward induction.

Lemma 1 (Pareto frontiers) *The Pareto frontier at time τ satisfies $H(u^b, u^s, \tau) = 0$ where*

$$H(u^b, u^s, \tau) = \begin{cases} u(y^*) - v(y^*) - (u^b - u_0^b) - (u^s - u_0^s) & \text{if } u^s - u_0^s \leq \delta\tau - v(y^*) \\ \delta\tau - v[u^{-1}(\delta\tau + u^b - u_0^b)] - (u^s - u_0^s) & \text{otherwise} \end{cases}. \quad (5)$$

The function H is continuously differentiable, increasing in τ (strictly so if $y < y^*$), decreasing in u^b and u^s . Consequently, each Pareto frontier has a negative slope:

$$\left. \frac{\partial u^s}{\partial u^b} \right|_{H(u^b, u^s, \tau)=0} = \begin{cases} -1 & \text{if } u^s - u_0^s \leq \delta\tau - v(y^*) \\ -\frac{v'(y)}{u'(y)} & \text{otherwise} \end{cases}$$

The Pareto frontier is linear when $y = y^*$. When $y < y^*$, it is strictly concave.

We call a bargaining round an *active round* if there is trade. We say that a SPE is *simple* if in each active round the consumer offers z/N units of assets, except possibly for the last active round, and active rounds are followed by inactive rounds (if any).

Proposition 1 (SPE of the alternating-ultimatum-offer game.) *All SPE of the alternating-ultimatum-offer game share the same final payoffs. If final y is less than y^* , then the SPE is unique and simple; otherwise, there is a unique simple SPE. Moreover, in any simple SPE, the intermediate payoffs,*

$\{(u_n^b, u_n^s)\}_{n=1,2,\dots,N}$, converge to the solution, $\langle u^b(\tau), u^s(\tau) \rangle$, to the following differential equations as N approaches $+\infty$:

$$u^{b'}(\tau) = -\frac{1}{2} \frac{\partial H(u^b, u^s, \tau) / \partial \tau}{\partial H(u^b, u^s, \tau) / \partial u^b} \quad (6)$$

$$u^{s'}(\tau) = -\frac{1}{2} \frac{\partial H(u^b, u^s, \tau) / \partial \tau}{\partial H(u^b, u^s, \tau) / \partial u^s}. \quad (7)$$

Proposition 1 (proved in Appendix B) establishes that the SPE of the alternating-ultimatum-offer game is essentially unique — any multiplicity when $y = y^*$ is due to differences in the timing of asset sales that are payoff-irrelevant. When N approaches $+\infty$, i.e., bargaining becomes gradual, equilibrium payoffs are characterized by the system of differential equations, (6)-(7). The interpretation of this solution is as follows. An increase in τ by one unit expands the bargaining set by $\partial H / \partial \tau$. The maximum utility gain that the consumer could enjoy from this expansion is $-(\partial H / \partial \tau) / (\partial H / \partial u^b)$, as illustrated by the horizontal arrow in Figure 2. According to (6), the consumer enjoys half of this gain. The same holds true for the producer. By combining (6) and (7), the slope of the gradual agreement path is:

$$\frac{\partial u^s}{\partial u^b} = \frac{\partial H(u^b, u^s, \tau) / \partial u^b}{\partial H(u^b, u^s, \tau) / \partial u^s}. \quad (8)$$

According to (8), the slope of the gradual bargaining path is equal to the opposite of the slope of the Pareto frontier.

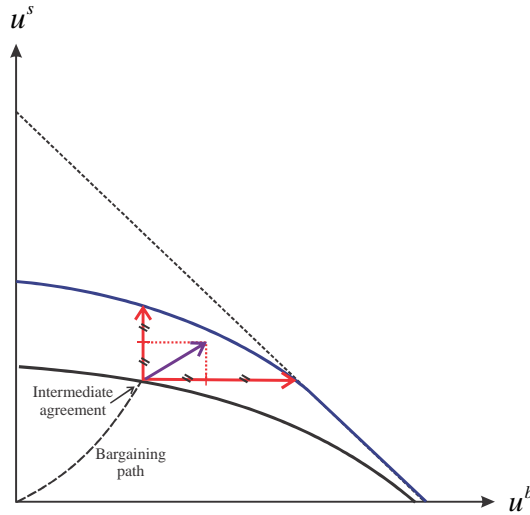


Figure 2: Solution to a gradual bargaining problem

The proof of Proposition 1 consists of two steps: first, we characterize the SPE for any (sub)game with an arbitrary number of rounds, N . In the second part, we establish that the sequence of intermediate payoffs of the SPE converges to the solution to the system of differential equations, (6) and (7), as N approaches

$+\infty$. The intuition goes as follows. Suppose the negotiation enters its last round, N , and the two agents have agreed upon some intermediate payoffs (u_{N-1}^b, u_{N-1}^s) . The consumer makes the last take-it-or-leave offer, which maximizes his payoff by keeping the producer's payoff unchanged at u_{N-1}^s . Graphically, the final payoffs are constructed from the intermediate payoffs by moving horizontally from the lower Pareto frontier, to which (u_{N-1}^b, u_{N-1}^s) belongs, to the upper Pareto frontier corresponding to an increase in assets of z/N , as shown in the left panel of Figure 3.

We now move backward in the game by one round. Suppose that the negotiation enters round $N - 1$ with some intermediate payoffs, (u_{N-2}^b, u_{N-2}^s) , with the producer making the offer. Now, if the consumer rejects the producer's offer, the negotiation enters its last round and the consumer's payoff is obtained as before, i.e., by moving horizontally from the lower frontier to the upper frontier. Given the consumer's payoff, the producer's payoff is obtained such that the pair of payoffs is located on the last Pareto frontier. Graphically, there is first a horizontal move from the initial payoff, (u_{N-2}^b, u_{N-2}^s) , to the next Pareto frontier that determines the consumer's terminal payoff, $u_{N-1}^b = u_N^b$, and then a vertical move to the following frontier that determines the producer's payoff, u_N^s , as shown in the right panel of Figure 3. We iterate this procedure backward until we reach the start of the game with initial payoffs (u_0^b, u_0^s) .

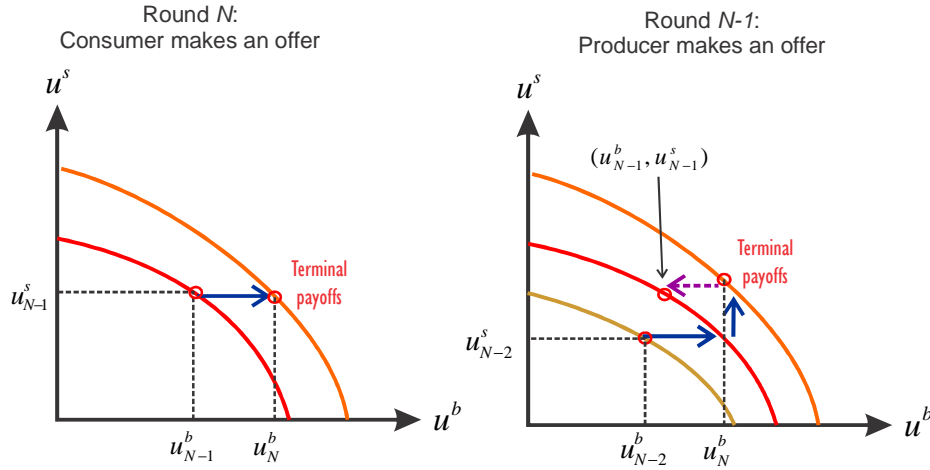


Figure 3: Left panel: offer in last round; Right panel: offer in $(N - 1)^{\text{th}}$ round

Once we have the terminal payoffs, we use another backward induction to determine the sequence of intermediate payoffs. The intermediate payoffs at the end of the $(N - 1)^{\text{th}}$ round lie on the $(N - 1)^{\text{th}}$ frontier and are obtained by moving horizontally from the N^{th} frontier to the $(N - 1)^{\text{th}}$ frontier since the consumer is making the last offer. The intermediate payoffs on the $(N - 2)^{\text{th}}$ frontier are obtained by moving first vertically, from the N^{th} frontier to the $(N - 1)^{\text{th}}$ frontier, and then horizontally from the $(N - 1)^{\text{th}}$ frontier to

the $(N - 2)^{\text{th}}$ frontier by using the same reasoning as above. It turns out that the two sequences constructed above get closer to one another as N becomes large, and, both converge to the gradual bargaining path according to (8).

3.2 Negotiated price and trade size

We now turn to the implications of the gradual bargaining solution for asset prices and trade sizes. From the definition of H in (5), the solution to the bargaining game, (6)-(7), can be reexpressed as

$$u^{b'}(\tau) = \delta \frac{u'(y) - v'(y)}{2v'(y)} \quad (9)$$

$$u^{s'}(\tau) = \delta \frac{u'(y) - v'(y)}{2u'(y)}, \quad (10)$$

if $\delta\tau < u^s - u_0^s + v(y^*)$ and $u^{b'}(\tau) = u^{s'}(\tau) = 0$ otherwise. From (9) and (10) the slope of the gradual bargaining path is $\partial u^s / \partial u^b = v'(y)/u'(y)$, which is increasing in y , i.e., it becomes steeper as the negotiation progresses. The producer's share in the match surplus increases throughout the negotiation as the gap between $u'(y)$ and $v'(y)$ shrinks over time.

Proposition 2 (Prices and trade sizes) *Along the gradual bargaining path, the price of the asset in terms of DM goods is*

$$\frac{y'(\tau)}{\delta} = \frac{1}{2} \left(\overbrace{\frac{1}{v'(y)}}^{\text{bid price}} + \overbrace{\frac{1}{u'(y)}}^{\text{ask price}} \right) \quad \text{for all } y < y^*. \quad (11)$$

The overall payment for y units of consumption is

$$p(y) = \int_0^y \frac{2v'(x)u'(x)}{u'(x) + v'(x)} dx. \quad (12)$$

If $z \geq p(y^*)$ then $y = y^*$ and $y = p^{-1}(z)$ otherwise.

According to (11), the negotiated price is the arithmetic average of the bid and ask prices. The bid price of one unit of asset at time τ , i.e., the maximum price in terms of DM goods that the producer is willing to pay to acquire it, is equal to $1/v'(y)$. The ask price at time τ , i.e., the minimum price in terms of DM goods that the consumer is willing to accept to give up the asset, is $1/u'(y)$. The bid price decreases with y because the producer incurs a convex cost to finance an additional unit of asset. The ask price increases with y because the consumer enjoys a decreasing marginal utility in exchange of an additional unit of asset. So the negotiated price can be non-monotone with the size of the trade. From (12) we can compute the consumer's surplus from a trade:

$$u(y) - p(y) = \int_0^y \frac{u'(x) [u'(x) - v'(x)]}{u'(x) + v'(x)} dx, \quad \text{for all } y \leq y^*.$$

The surplus increases with y , is strictly concave for all $y < y^*$, and is maximum when $y = y^*$. We will emphasize the importance of the monotonicity of the surplus later when we turn to the general equilibrium.

3.3 Asymmetric agenda

So far the agenda of the negotiation corresponds to a uniform partition of the portfolio, $[0, z]$, where each asset bundle has the same size, z/N . In the following we modify the agenda to provide a noncooperative foundation for asymmetric bargaining powers. We still assume that N is even. In each round where the consumer is making the offer, the amount of assets that can be negotiated is $2\theta z/N$ where $\theta \in [0, 1]$. In rounds where the producer is making the offer, the amount of assets up for negotiation is $2(1 - \theta)z/N$. Note that $\theta = 1/2$ corresponds to the bargaining game studied earlier. We show in Appendix B that the solution to this bargaining game generalizes (6)-(7) as follows:

$$u^b(\tau) = -\theta \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^b} \quad (13)$$

$$u^s(\tau) = -(1 - \theta) \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^s}, \quad (14)$$

where $\theta \in [0, 1]$ is interpreted as the consumer's bargaining power.¹⁰ By the same reasoning as above, the DM price of assets evolves according to

$$\frac{y'(\tau)}{\delta} = \left(\theta \overbrace{\frac{1}{v'(y)}}^{\text{bid price}} + (1 - \theta) \overbrace{\frac{1}{u'(y)}}^{\text{ask price}} \right). \quad (15)$$

It is now a weighted average of the bid and ask prices where the weights are given by the relative bargaining powers of the consumer and the producer. From (15) the DM price of the asset is increasing in θ . The payment for y units of DM consumption is

$$p(y) = \int_0^y \frac{u'(x)v'(x)}{\theta u'(x) + (1 - \theta)v'(x)} dx \quad \text{for all } y \leq y^*. \quad (16)$$

3.4 An axiomatic approach

An axiomatic approach, by abstracting from the details of the bargaining game, provides a sense of the robustness of our solution.¹¹ The Nash definition of a bargaining problem, which does not include the notion

¹⁰This solution coincides with the axiomatic solution of Wiener and Winter (1998). One could make the bargaining power a function of time, τ , or output traded, y , without affecting the results significantly.

¹¹As written by Serrano (2008) in his description of the Nash program:

The non-cooperative approach to game theory provides a rich language and develops useful tools to analyze strategic situations. One clear advantage of the approach is that it is able to model how specific details of the interaction may impact the final outcome. One limitation, however, is that its predictions may be highly sensitive to those details. For this reason it is worth also analyzing more abstract approaches that attempt to obtain conclusions that are independent of such details. The cooperative approach is one such attempt.

of agenda, was extended by O’Neill et al. (2004). The agenda takes the form of a family of feasible sets indexed by time. The difficulty is to identify the relevant agenda for the problem at hand. In the context of our model where agents negotiate gradually the sale of assets, the bargaining problem is defined as follows.

Definition 1 *A gradual bargaining problem between a consumer holding z units of asset and a producer is a collection of Pareto frontiers, $\langle H(u^b, u^s, \tau) = 0, \tau \in [0, z/\delta] \rangle$ and a pair of disagreement points, (u_0^b, u_0^s) .*

A gradual agreement path is a function, $o : [0, z] \rightarrow \mathbb{R}_+ \times [0, z]$, that specifies an allocation (y, p) for all $\tau \in [0, z/\delta]$ and associated utility levels, $\langle u^b(\tau), u^s(\tau) \rangle$. The gradual solution of O’Neill et al. (2004) is the unique solution to satisfy five axioms: Pareto optimality, scale invariance, symmetry, directional continuity, and time-consistency. The first three axioms are axioms imposed by Nash (1950). The last two axioms are specific to the new definition of the bargaining problem. Directional continuity imposes a notion of continuity for the bargaining path with respect to changes in the agenda. The requirement of time-consistency specifies that if the negotiation were to start with the agreement reached at time τ as the new disagreement point, then the bargaining path onwards would be unchanged. The theorem of O’Neill et al. applied to the bargaining problem above leads to the following result.

Theorem 1 (Ordinal solution of O’Neill et al., 2004) *There is a unique solution to the gradual bargaining problem given by $\langle H(u^b, u^s, \tau) = 0, \tau \in [0, z/\delta] \rangle$ and it satisfies (6)-(7).*

The equilibrium payoffs of the alternating-ultimatum-offer bargaining game coincide with the axiomatic solution from O’Neill et al. (2004). While scale invariance was imposed as an axiom, the solution exhibits ordinality endogenously: the solution is covariant with respect to any order-preserving transformation. This result is noteworthy because Shapley (1969) shows that for standard Nash problems with two players, no single-valued solution can satisfy Pareto efficiency, symmetry, and ordinality. Finally, if the axiom of symmetry is dropped, then the generalized ordinal solutions corresponding to the bargaining problem in Definition 1 solve (13)-(14).

4 Relation to Nash and Kalai bargaining

We now describe bargaining games with an agenda that admit the two most commonly used bargaining solutions, namely, the Nash and Kalai solutions, as particular cases. First, we generalize the game of Section 3.1 by adopting the Rubinstein (1982) alternating-offer game in each round. The Nash solution corresponds to the particular case where $N = 1$ while the gradual solution corresponds to $N = +\infty$. In a second part,

we set up an alternative agenda under which agents negotiate gradually over the DM good (instead of the liquid assets). We show that this agenda generates the Kalai (1977) solution as $N \rightarrow +\infty$.

4.1 The repeated Rubinstein game

We generalize the game studied in Section 3.1 so that each round, $n \in \{1, \dots, N\}$, is composed of an infinite number of stages during which the two players bargain over z/N units of assets following an alternating-offer protocol as in Rubinstein (1982). The consumer is the first proposer if n is odd, and the producer is the first proposer otherwise. The round-game, illustrated in Figure 4, is as follows. In the initial stage, the first proposer makes an offer and the other agent either accepts it or rejects it. If the offer is accepted, round n ends and agents move to round $n + 1$. If the offer is rejected then there are two cases. With probability $(1 - \xi_n)$ round n is terminated and the players move to round $n + 1$ without having reached an agreement. With probability ξ_n the negotiation continues and the responder becomes the proposer in the following stage. We focus on the limit case where ξ_n converges to one, and the order of convergence is from ξ_N to ξ_1 .

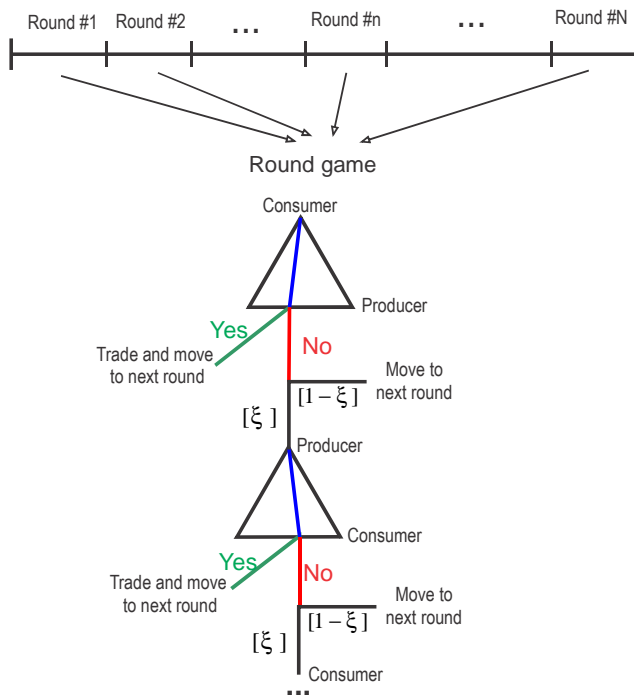


Figure 4: Game tree with alternating offers in each round

Proposition 3 (Repeated Rubinsten game.) *There exists a SPE of the repeated Rubinstein game when taking limits according to the order $\xi_N \rightarrow 1, \xi_{N-1} \rightarrow 1, \dots, \xi_1 \rightarrow 1$, characterized by a sequence of interme-*

diate allocations, $\{(y_n, p_n)\}_{n=0}^N$, solution to:

$$(y_n, p_n) \in \arg \max_{y, p} [u(y) - p - u(y_{n-1}) + p_{n-1}] [-v(y) + p + v(y_{n-1}) - p_{n-1}] \quad \text{s.t.} \quad p \leq \frac{nz}{N}, \quad (17)$$

for all $n \in \{1, \dots, N\}$ with $(y_0, p_0) = (0, 0)$. As $N \rightarrow +\infty$ the solution converges to the solution of the alternating-ultimatum-offer game characterized in Proposition 2.

The intermediate allocations in each round, given by (17), maximize the Nash product of agents' surpluses where the endogenous disagreement points are the intermediate payoffs of the previous round. The proof (in Appendix C) is based on backward induction. Consider the last round with some intermediate agreement (u_{N-1}^b, u_{N-1}^s) . The outcome of the Rubinstein game as the risk of breakdown goes to zero corresponds to the Nash solution with disagreement point (u_{N-1}^b, u_{N-1}^s) . Next, consider round $N-2$ with intermediate payoffs (u_{N-2}^b, u_{N-2}^s) . The relevant disagreement points, $(\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s)$, are given by the outcome of the negotiation in round $N-1$ if there is no agreement in round $N-2$, i.e., $(\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s)$ maximizes the Nash product $(\tilde{u}_{N-1}^b - u_{N-2}^b)(\tilde{u}_{N-1}^s - u_{N-2}^s)$. Given $(\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s)$, the negotiation in round $N-2$, which is forward looking, determines the final payoffs. As the risk of breakdown vanishes, these payoffs, (u_N^b, u_N^s) , coincide with the Nash solution, i.e., they maximize $(u_N^b - \tilde{u}_{N-1}^b)(u_N^s - \tilde{u}_{N-1}^s)$. For any given initial condition (u_0^b, u_0^s) , this iterative procedure pins down the terminal payoffs. Once terminal payoffs are determined, we use a second backward induction to find the sequence of intermediate payoffs. Intermediate payoffs in round $N-1$, (u_{N-1}^b, u_{N-1}^s) , correspond to the disagreement points of the Nash solution that generates the terminal payoffs, i.e., $(u_{N-1}^b, u_{N-1}^s) = (\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s)$. And so on. The determination of payoffs is illustrated in Figure 5.

From (17) the intermediate allocations, $\{(y_n, p_n)\}_{n=0}^N$, solve:

$$\int_{y_{n-1}}^{y_n} \frac{v'(y_n)u'(x) + u'(y_n)v'(x)}{u'(y_n) + v'(y_n)} dx \leq \frac{z}{N} \quad " = " \text{ if } y_n < y^*, \quad (18)$$

$$p_n - p_{n-1} = \min \left\{ \frac{[u(y^*) - u(y_{n-1})] + [v(y^*) - v(y_{n-1})]}{2}, \frac{z}{N} \right\},$$

with $y_0 = 0$. From (18), when the liquidity constraint, $p_n \leq nz/N$, binds, then the payment for $y_n - y_{n-1}$ units of DM goods is equal to a weighted sum of the marginal utilities of consumption and the marginal disutilities of production. If $N = 1$ then (18) corresponds to symmetric Nash. Summing (18) across n and taking the limit as N goes to $+\infty$ gives the gradual solution.

In the following proposition we let consumers (asset owners) choose the number of rounds of the negotiation, N . The key observation from (18) is that the consumer's share in the surplus of the n^{th} round, $u'(y_n)/[u'(y_n) + v'(y_n)]$, decreases with y_n .

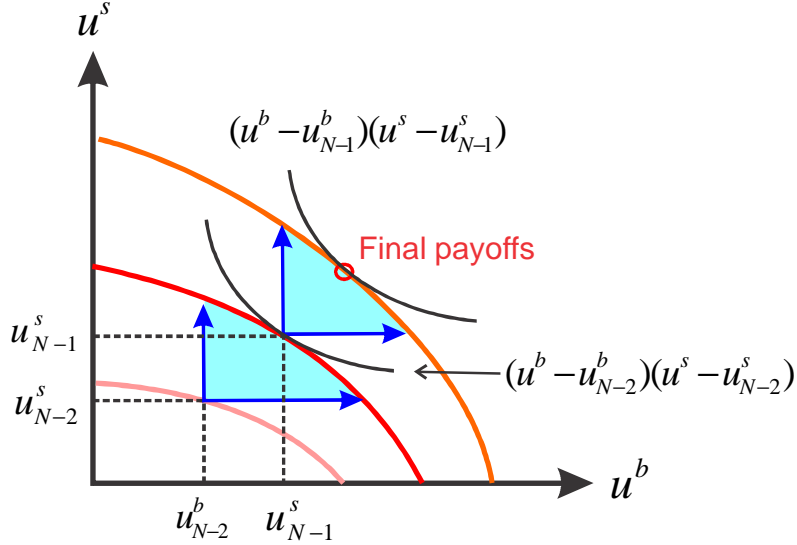


Figure 5: Computing terminal payoffs from round $N - 2$

Proposition 4 (Optimal gradualism) *Consumers obtain their highest surplus by negotiating the sale of their assets one infinitesimal unit at a time, $N = +\infty$.*

The agenda underlying the Nash solution ($N = 1$) is suboptimal from the standpoint of asset owners. They strictly prefer to sell their assets gradually over time. The consumer gain from bargaining gradually is

$$p_1(y) - p_\infty(y) = \int_0^y \left[\frac{v'(y)}{u'(y) + v'(y)} - \frac{v'(x)}{u'(x) + v'(x)} \right] [u'(x) - v'(x)] dx,$$

where $p_N(y)$ is the amount of assets in exchange for y units of DM goods if the negotiation takes place in N rounds. Under Nash bargaining the producer's share in each increment of the match surplus is constant and equal to $v'(y)/[u'(y) + v'(y)]$, which is larger than the variable share, $v'(x)/[u'(x) + v'(x)]$ for all $x < y$, under gradual bargaining. Intuitively, selling all the assets at once has a negative impact on the consumer's surplus share that can be reduced by selling them through small quantities — a form of dynamic price discrimination.

4.2 Gradual bargaining in OTC markets

In Example 1 of Section 2, we interpret y as an illiquid asset traded over the counter, as in Duffie et al. (2005) and Lagos and Zhang (2018). Agents trade a liquid asset that has a common value for an illiquid asset that is valued differently by buyers and sellers. Even if we focus on gradual solutions, the agenda of the negotiation can take two forms: agents can bargain gradually over the liquid asset, as described above, or they can bargain gradually over the illiquid asset. We now describe the outcome of the latter agenda,

i.e., agents add the illiquid asset, y , on the negotiation table gradually over time and bargain over the price of each unit in terms of the liquid asset. In that case each Pareto frontier in the definition of the gradual bargaining problem is indexed by the amount of illiquid asset, \bar{y} , that has been up for negotiation at a given point of time. With no loss in generality we normalize $u_0^b = u_0^s = 0$.

Lemma 2 (Pareto frontiers when bargaining over DM goods.) For a given asset holding z , the bargaining problem is a collection of Pareto frontiers, $\langle H(u^b, u^s, \bar{y}) = 0, \bar{y} \in [0, y^*] \rangle$ where:

$$H(u^b, u^s, \bar{y}) = \begin{cases} u(\bar{y}) - v(\bar{y}) - u^b - u^s & \text{if } u^s \leq z - v(\bar{y}) \\ z - v \circ u^{-1}(u^b + z) - u^s & \text{otherwise} \end{cases}, \quad (19)$$

for all $u^s \leq \min \{u(\bar{y}) - v(\bar{y}), z - v \circ u^{-1}(z)\}$.

As long as the DM output to be negotiated is sufficiently small relative to the consumer's real balances, $z \geq u(\bar{y})$, then the Pareto frontier is entirely linear (see Figure 6). In contrast, if $z < u(\bar{y})$ then the payment constraint binds if the producer receives a sufficiently large surplus, in which case the Pareto frontier is strictly concave over some range.

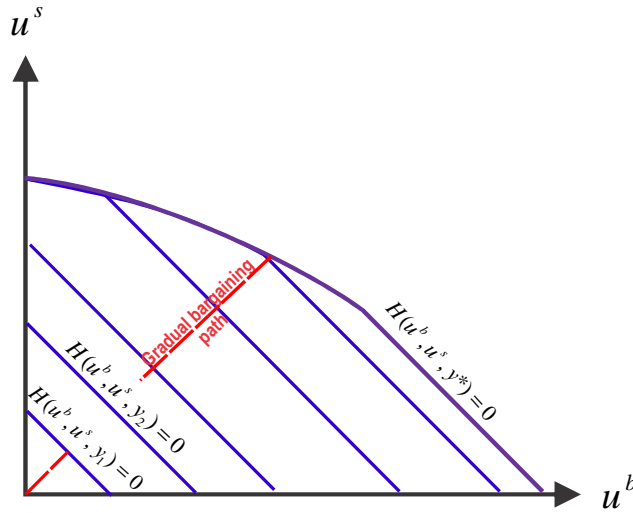


Figure 6: Bargaining gradually over DM goods

The alternating-ultimatum-offer game associated with this agenda is analogous to the one described in Section 3.1. The producer can now transfer at most y^*/N units of DM goods for some liquid assets in each round. The transfer of liquid asset is also subject to a feasibility constraint according to which the consumer cannot transfer more liquid asset than what he holds in a given round (taking into account the assets spent in earlier rounds). So the game ends when either the N^{th} round has been reached or the liquid assets of the

consumer have been depleted. The identity of the proposer (the consumer or the producer) alternates across rounds.

Proposition 5 (*Gradual bargaining over DM output*) *The gradual limit (as N tends to $+\infty$) of the SPE of the alternating-ultimatum-offer game where agents bargain gradually over the DM output is such that the total payment function is*

$$p^{DM}(y) = \frac{1}{2} [u(y) + v(y)],$$

and DM output solves $p^{DM}(y) = \min \{z, p^{DM}(y^)\}$. It also corresponds to the ordinal solution of O’Neill et al. (2004) when the agenda is given by (19).*

Proposition 5 (whose proof can be found in Appendix B) shows that the payment made by the consumer is the arithmetic mean of the utility of the consumer and the cost of the producer. As a result, the surplus is shared equally between the consumer and the producer and the gradual bargaining path is linear, in accordance with the proportional solution of Kalai (1977).

The proportional solution has been used extensively in the monetary literature since Aruoba et al. (2007) because of its tractability and strong monotonicity property. However, two types of criticisms have been formulated against it. First, it is not scale invariant. Second, it does not have strategic foundations in terms of an extensive form game. While these two criticisms are legitimate in general, Proposition 5 shows that they are unwarranted in the context of decentralized asset markets under quasi-linear preferences since our solution is ordinal and has strategic foundations in terms of an alternating-offer game.¹²

We now endogenize the agenda by adding a stage prior to the negotiation where one of the players is picked at random to choose whether to bargain gradually over the DM good or the asset. We maintain for now the assumption that there is no constraint on the horizon of the negotiation.

Proposition 6 (*Endogenous agenda*). *Suppose that either the consumer or the producer of the DM good has to choose the agenda of the negotiation. The consumer chooses to bargain gradually over his holdings of Lucas trees while the producer chooses to bargain gradually over the DM good.*

If the asset owner (the consumer) chooses the agenda of the negotiation, then he bargains gradually over his asset holdings. In contrast, the producer chooses to bargain gradually over the DM good. In both cases, each agent wants to sell gradually the commodity or asset he is offering in the negotiation.¹³

¹²Dutta (2012) also proposes non-cooperative foundations for the Kalai solution, however not in the spirit of Rubinstein’s alternating-offers game since players must simultaneously coordinate on an allocation.

¹³In Appendix D we use this result to describe a two-sided OTC market where asset trades are intermediated by dealers. We let investors choose the agenda of the negotiation so that buyers bargain gradually over the liquid asset while sellers bargain gradually over the illiquid asset.

5 Asset prices and negotiability

The second part of the paper studies the general equilibrium implications of the gradual bargaining solutions for asset prices, allocations, and welfare. We first study the pricing of Lucas trees in a New-Monetarist model with idiosyncratic spending opportunities (e.g., Geromichalos et al., 2007; Lagos, 2010) taking the negotiability of assets, δ , as exogenous. In order to contrast the findings under Nash and gradual bargaining, we also investigate an OTC version of the model with linear payoffs where agents trade an illiquid short-term asset with money (as, e.g., Lagos and Zhang, 2018). In the second part we endogenize negotiability as a costly investment decision.

Agendas have a natural time dimension. In extensive-form games, the different items of the agenda are negotiated sequentially at different points in time. In the axiomatic definition of O’Neill et al. (2004), the agenda is a collection of Pareto frontiers indexed by time. In order for time to matter, we now assume that the amount of time allocated to the negotiation, $\bar{\tau}$, is a random variable exponentially distributed with mean $1/\lambda$ and realized at the beginning of a match. This assumption captures the idea that agents might have more or less time to negotiate the sale of their assets. Throughout the section we assume that the consumer’s bargaining power is $\theta \in [0, 1]$.

5.1 Negotiability, asset prices, and welfare

The lifetime expected utility of a consumer bringing z assets to the DM solves

$$V^b(z) = \alpha \int_0^{+\infty} \tau e^{-\lambda\tau} \{u[y(z; \tau)] + W^b[z - p[y(z; \tau)]]\} d\tau + (1 - \alpha) W^b(z), \quad (20)$$

where $y(z; \tau)$ is the consumer’s consumption and $p[y(z; \tau)]$ is his sale of Lucas trees in the DM in terms of numeraire if the time to negotiate is τ . According to (20) a consumer meets a producer with probability α , in which case τ is drawn from an exponential distribution. The consumer enjoys y units of DM consumption in exchange for p units of assets. With probability $1 - \alpha$ the consumer is unmatched and enters the CM with z units of asset. Substituting $V^b(z)$ with its expression given by (20), the consumer’s choice of asset holdings solves

$$\max_{z \geq 0} \left\{ -sz + \alpha \int_0^{+\infty} \tau e^{-\lambda\tau} \{u[y(z; \tau)] - p[y(z; \tau)]\} d\tau \right\}, \quad (21)$$

where s is the spread between the rate of time preference and the real rate on liquid Lucas trees,

$$s = \frac{\rho - r}{R} \geq 0. \quad (22)$$

We rewrite the portfolio problem, (21), as a choice of DM consumption, taking into account that the payment function, $p(y)$, is given by (16). It becomes:

$$\max_{y \in [0, y^*]} \left\{ -sp(y) + \alpha \int_0^y e^{-\frac{\lambda}{\delta} p(x)} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta)v'(x)} dx \right\}. \quad (23)$$

The second term in the objective function is the consumer's expected surplus from a DM trade (see the appendix for the full derivation); note that we can restrict the choice of y to $[0, y^*]$ since that term becomes constant for $y > y^*$. The objective function is continuous and strictly concave for all $y \in (0, y^*]$.

By market clearing,

$$p(y) \leq \left(\frac{1 + \rho}{\rho - s} \right) Ad, \quad " = " \quad \text{if } s > 0, \quad (24)$$

where we have used that the cum-dividend price of the asset is $\phi + d = (1 + \rho)d/(\rho - s)$. When $s > 0$, consumers hold exactly $p(y) = (\phi + d)A$. If $s = 0$, then from (26) $y = y^*$. The total supply of the asset, $(\phi + d)A$, is larger or equal than $p(y^*)$ since assets can also be held as a pure store of value. An equilibrium can be reduced to a pair (s, y) that solves (23) and (24). We measure social welfare as the sum of surpluses in pairwise meetings but we do not include the output from Lucas trees, Ad :

$$\mathcal{W} = \alpha \int_0^y e^{-\frac{\lambda}{\delta} p(x)} [u'(x) - v'(x)] dx. \quad (25)$$

Proposition 7 (*Asset prices and welfare.*) *An equilibrium exists and is unique.*

1. *If $Ad \geq \rho p(y^*)/(1 + \rho)$ then $s = 0$ and y^* is implemented in a fraction $e^{-\frac{\lambda}{\delta} p(y^*)}$ of all matches. Social welfare is independent of Ad but it increases with δ and decreases with λ .*
2. *If $Ad < \rho p(y^*)/(1 + \rho)$ then*

$$s = \alpha \theta e^{-\frac{\lambda}{\delta} p(y)} \ell(y) > 0, \quad (26)$$

where $\ell(y) = u'(y)/v'(y) - 1$, and y^ is never implemented. The asset spread, s , decreases with Ad and λ but increases with δ . Social welfare increases with Ad and δ but decreases with λ .*

3. *Suppose $\lambda = 0$ and $\theta = 1/2$. If $Ad \geq \rho p(y^*)/(1 + \rho)$, then the equilibrium under gradual bargaining implements the first best. In contrast, the equilibrium under Nash bargaining never implements the first best, i.e., $y < y^*$ for all $A > 0$.*

Proposition 7 identifies two regimes. In the first regime consumers hold enough wealth to buy y^* provided that the negotiation lasts long enough, which happens with probability $e^{-\frac{\lambda}{\delta} p(y^*)}$. As λ decreases or δ increases, the fraction of matches where y^* is implemented increases and welfare increases. The asset price,

however, is not affected by λ or δ . In the second regime consumers hold less than $p(y^*)$ and hence trades are inefficient in all matches.

From (26) the interest rate spread is the product of four components: the search friction, α , the bargaining power, θ , the negotiability friction, $e^{-\frac{\lambda}{\delta}p(y)}$, and the marginal value of wealth in the DM, $\ell(y)$. The negotiability term is akin to a stochastic pledgeability coefficient, but there are noteworthy differences. First, this negotiability term is endogenous and depends on the time it takes to negotiate assets, the stochastic time horizon of the negotiation, and the bargaining protocol as represented by $p(y)$. Second, the comparative statics for asset prices differ from the ones with a fixed pledgeability coefficient. Indeed, the asset spread decreases with A , and it increases with both α and δ . In contrast, asset prices vary in a non-monotone fashion with constant pledgeability coefficients. Overall, the expression for the interest rate spread captures the fact that bargaining frictions affect asset prices through two channels: traders' bargaining powers and the time to negotiate asset sales.

The last part of Proposition 7 compares equilibria under symmetric Nash bargaining and equilibria under symmetric gradual bargaining when $\lambda = 0$ so that the negotiability constraint does not bind. Under gradual bargaining, if A is sufficiently large, then $y = y^*$. In contrast, under Nash bargaining, the equilibrium never achieves first best. The non-monotonicity of the Nash solution generates asset misallocation by preventing the market from clearing if all the asset supply is held by consumers. As a result, a fraction of A is held by producers even though they have no liquidity needs while consumers are liquidity-constrained. This result shows that gradual bargaining is not only desirable for asset owners to increase their surplus (Proposition 4), it is also socially desirable to avoid the misallocation of assets.

5.2 An OTC market with linear payoffs

In order to illustrate the last part of Proposition 7, we provide a stark example of an OTC market where Nash bargaining delivers the worst possible allocation while gradual bargaining delivers the first best. We adopt a specification with linear payoffs, similar to Lagos and Zhang (2018). As in Example 1, at the beginning of each period producers (sellers) are endowed with Ω units of DM goods (interpreted as short-lived assets) and have a linear technology to transform each unit of the DM good into $\varepsilon_\ell > 0$ units of numeraire. Consumers (buyers) receive no endowment but can transform the DM good into $\varepsilon_h > \varepsilon_\ell$ units of numeraire. Hence, $u(y) = \varepsilon_h y$ and $v(y) = \varepsilon_\ell y$. Producers choose the quantity of DM goods, $\omega \leq \Omega$, to bring into a bilateral match.¹⁴ Lucas trees are now interpreted as fiat money by setting $d = 0$. The spread between the rate

¹⁴The assumption according to which agents can choose to bring only a fraction of their asset holdings in a match was introduced by Berentsen and Rocheteau (2003), Lagos and Rocheteau (2008), and Lagos (2010). This assumption addresses the fact that under Nash bargaining agents might have incentives to hide some of their assets.

of return of money ($r = -\pi/(1 + \pi)$ where π is the money growth rate implemented through lump-sum transfers) and the rate of time preference, given by (22), is denoted i . We set $\lambda = 0$ so that the negotiability constraint is never binding.

Suppose first that agents negotiate according to Nash. The outcome in a match where the buyer holds z and the seller holds ω is given by:

$$\max_{y,p} (\varepsilon_h y - p)(p - \varepsilon_\ell y) \quad \text{s.t.} \quad p \leq z \quad \text{and} \quad y \leq \omega.$$

If $p \leq z$ does not bind, then the solution is $y = \omega$ and $p = (\varepsilon_h + \varepsilon_\ell)\omega/2$. Buyers purchase all the DM goods, which is socially efficient, and a payment is made to divide the match surplus evenly. It requires $(\varepsilon_h + \varepsilon_\ell)\omega/2 \leq z$. If $p \leq z$ binds then $y = \omega$ and $p = z$ if $(\varepsilon_h + \varepsilon_\ell)z \geq 2\varepsilon_h\varepsilon_\ell\omega$. Otherwise, if $(\varepsilon_h + \varepsilon_\ell)z < 2\varepsilon_h\varepsilon_\ell\omega$, then $p = z$ and $y = (\varepsilon_h + \varepsilon_\ell)z/(2\varepsilon_\ell\varepsilon_h)$. The seller's surplus, $u^s(\omega, z) \equiv p(\omega, z) - \varepsilon_\ell y(\omega, z)$, is piecewise linear and non-monotone in ω . It reaches a maximum for $\omega = 2z/(\varepsilon_h + \varepsilon_\ell)$. Similarly, the buyer's surplus, $u^b(\omega, z) \equiv \varepsilon_h y(\omega, z) - p(\omega, z)$, is piecewise linear, non-monotone in z , and reaches a maximum when $z = 2\varepsilon_h\varepsilon_\ell\omega/(\varepsilon_h + \varepsilon_\ell)$.

Alternatively, if agents bargain gradually over real balances, then, from (9), $u^{b'}(z) = (\varepsilon_h - \varepsilon_\ell)/(2\varepsilon_\ell)$ if $y \leq \omega$ does not bind. The buyer's surplus is monotone increasing in his real balances. Similarly, the seller's surplus is monotone in ω and the payment for y units of DM goods is $p(y) = 2\varepsilon_h\varepsilon_\ell y/(\varepsilon_h + \varepsilon_\ell)$.

Irrespective of the bargaining solution, the seller chooses $\omega \leq \Omega$ to maximize $\int u^s(\omega, z)dF^b(z)$, where $F^b(z)$ is the distribution of real balances across buyers. The buyer's problem consists in choosing z in order to maximize $-iz + \alpha \int u^b(z, \omega)dF^s(\omega)$ where $F^s(\omega)$ is the distribution of inventories held by sellers in DM matches. We characterize equilibrium allocations in the following proposition.

Proposition 8 (Allocations in OTC markets.) *Suppose sellers are endowed with Ω units of DM goods and preferences are given by $u(y) = \varepsilon_h y$ and $v(y) = \varepsilon_\ell y$. The liquid asset takes the form of fiat money, $d = 0$, with spread i . Under Nash bargaining, for all $i > 0$, there is no monetary equilibrium and the OTC market is inactive. Under gradual bargaining, if $i \leq \frac{\alpha(\varepsilon_h - \varepsilon_\ell)}{2\varepsilon_\ell}$ then there exists a monetary equilibrium implementing the first best.*

Proposition 8 provides a stark illustration of the importance of the agendas of the bilateral negotiations in OTC markets. If agents bargain according to Nash, then the OTC market is inactive and money is not valued for all $i \geq 0$, even at the Friedman rule. All assets are held by the least productive agents, which corresponds to the worst allocation.¹⁵ We represent the seller's and buyer's best-response functions, ω^{BR}

¹⁵This result does not rely on preferences being linear and is robust to various alternative assumptions. See Lebeau (2018) for details. In Appendix D we study a version of this model with strictly concave payoffs and dealers, as in Duffie et al. (2005).

and z^{BR} , for symmetric equilibria in the left panel of Figure 7. The only intersection is when $z = \omega = 0$. If sellers bring Ω in the match, then buyers bring at most $z = 2\varepsilon_h\varepsilon_\ell\Omega/(\varepsilon_h + \varepsilon_\ell)$ real balances in order to maximize their surplus. But if sellers anticipate this amount of real balances, they will bring at most $\omega = 4\varepsilon_h\varepsilon_\ell\Omega/(\varepsilon_h + \varepsilon_\ell)^2 < \Omega$. And so on. The process unravels until neither the buyer nor the seller brings anything to trade.

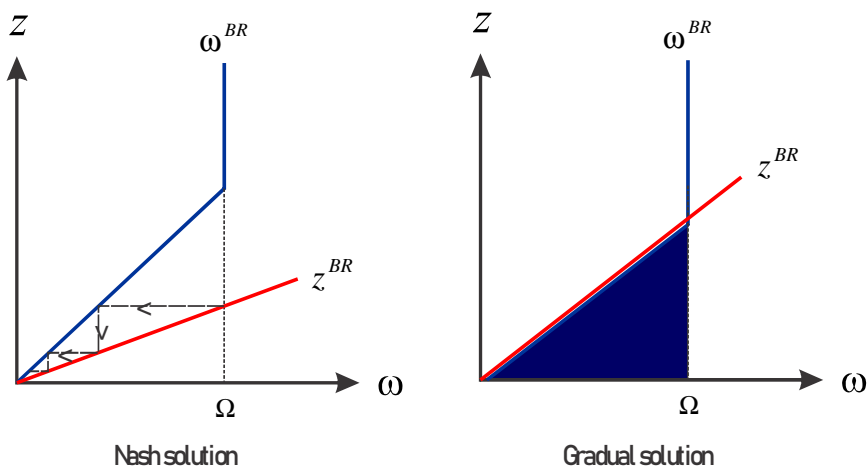


Figure 7: Symmetric best-response correspondences under Nash (left) and gradual (right) bargaining.

In contrast, if agents bargain gradually, then the first-best trades are implemented in all matches provided that i is not too high. (This result is also true if agents bargain gradually over the DM good.) We represent the best response correspondences under gradual bargaining and assuming symmetry across agents in the right panel of Figure 7. Note that sellers bring at the minimum the amount of DM goods corresponding to what buyers can pay for and they can bring up to their full endowment Ω (i.e., their best response is an interval). For low interest rates, there exists a Nash equilibrium where $\omega = \Omega$ and $z = 2\varepsilon_h\varepsilon_\ell\Omega/(\varepsilon_h + \varepsilon_\ell)$. The OTC market is active and it achieves the first best where in all matches sellers transfer all their endowments of DM goods to buyers. The unravelling that occurs under the Nash solution is avoided precisely because agents' surpluses are monotone increasing in the goods or assets that agents bring in a match.

5.3 Endogenous negotiability

We now endogenize the negotiability of assets, δ , by allowing consumers to choose the speed at which their assets are negotiated and transferred. For example, Bitcoin sellers can choose among a menu of fees to remunerate the third parties who will confirm their transactions. As evidenced in Figure 8, where we plot estimates of the transaction fees paid by sellers as a function of confirmation time for a median-sized Bitcoin transaction, the shorter the desired confirmation time, the higher the fee.

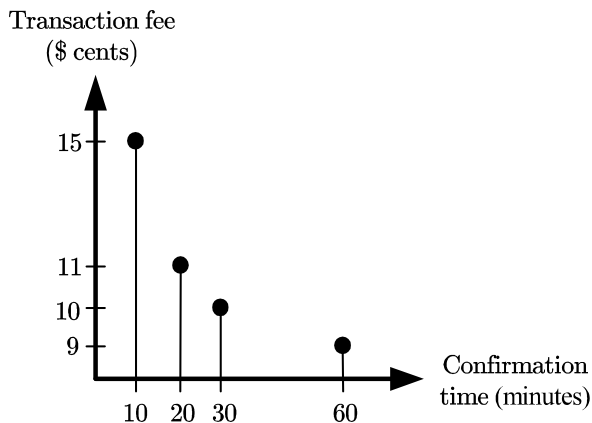


Figure 8: Estimated transaction fee vs confirmation time for median Bitcoin transaction (225 bytes). Source: <https://bitcoinfees.github.io/>, consulted August 1, 2018.

Consumers choose δ when a match is formed but before $\bar{\tau}$ is realized, where $\bar{\tau}$ is exponentially distributed with mean $1/\lambda$. There is a cost, $\psi(\delta)$, associated with the speed of the transaction, where $\psi(0) = \psi'(0) = 0$, $\psi'(\delta) > 0$ and $\psi''(\delta) > 0$. We can think of it as the cost of computer power to execute a trade and transfer assets safely.

The consumer's choice of asset holdings and speed of negotiation can be written compactly as:

$$\max_{z^b, \delta} \{-sz^b + \alpha [-\psi(\delta) + S^b(z^b, \delta)]\}, \quad \text{where } S^b(z^b, \delta) = \int_0^y e^{-\lambda \frac{p(x)}{\delta}} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1-\theta)v'(x)} dx, \quad (27)$$

i.e. $S^b(z^b, \delta)$ is the expected surplus of a consumer holding z^b assets when the speed of negotiation is δ and $y = p^{-1}(z^b)$. The novelty in (27) is the first term in squared brackets that represents the cost to invest in a technology to negotiate assets at speed δ . Despite the lack of concavity of the problem we can still fully characterize its solution and in the Appendix (see Lemma 5) we show that it is generically unique. Moreover, as the cost of holding assets, s , increases consumers reduce both their asset holdings and the speed of negotiation. A reduction in search frictions raises the demand for assets and the speed of negotiation.

The following proposition shows the existence of a unique general equilibrium with endogenous negotiability, and compares the equilibrium outcome to the constrained efficient δ . The speed of negotiation is constrained-efficient if it maximizes the social welfare subject to the same cost as private agents, $\psi(\delta)$, and subject to the same trading protocols in the DM and CM. It means that the pricing in the DM is given by $p(y)$ and the asset spread in the CM is a market clearing price.

Proposition 9 (*Equilibrium with endogenous negotiability.*)

1. *There exists a steady-state equilibrium and the equilibrium spread, s , is uniquely determined. If $Ad \geq \rho p(y^*)/(1 + \rho)$ then $s = 0$ and δ is maximum. If $Ad < \rho p(y^*)/(1 + \rho)$ then an increase in A reduces s , but raises δ .*
2. *Asset negotiability is constrained-efficient if and only if $Ad \geq \rho p(y^*)/(1 + \rho)$ and $\theta = 1$. If $Ad < \rho p(y^*)/(1 + \rho)$ then δ is inefficiently low for all θ .*

The first part of Proposition 9 shows that an increase in A reduces the spread s , which leads to a higher δ . Intuitively, if consumers have to sell more assets, they will find it worthwhile to increase the speed at which they can negotiate those assets. The second part shows that equilibrium negotiability is constrained-efficient if and only if A is abundant, so that $s = 0$, and consumers have all the bargaining power. This result is intuitive since the costly investment in asset negotiability creates a holdup problem that can only be solved by having the ones making the investment receive the whole match surplus. However, if A is low so that $s > 0$, then the investment in δ is inefficiently low even when $\theta = 1$. This inefficiency occurs because of a pecuniary externality according to which the demand for the asset, and hence its price, increases with δ . The planner understands this externality and hence chooses a δ larger than the one that consumers would choose even if they had all the bargaining power.

6 Bargaining with multiple assets and endogenous agenda

We extend our model to have multiple (liquid) assets in order to investigate cross-sectional differences in asset prices. We will show that our model generates a pecking-order theory of asset sales and rate-of-return differences across assets.

There are J types of one-period lived Lucas trees indexed by $j \in \{1, \dots, J\}$, where each Lucas tree born in $t - 1$ pays off one unit of numeraire in the CM of t . The supply of each Lucas tree is denoted A_j and the new Lucas trees are received by consumers in a lump-sum fashion at the beginning of each CM. We index fiat money by $j = 0$. We rank assets according to their negotiability, $\delta_0 \geq \delta_1 \geq \delta_2 \geq \dots \geq \delta_J$, that we interpret as a parameter of the technology to transfer asset ownership (e.g., physical transfer, a ledger, a blockchain technology).¹⁶ We assume that fiat money is the most negotiable asset because it is a tangible object whose ownership is asserted by simply carrying it and it can be authenticated with relatively small effort. It takes more time to transfer and verify the ownership of non-tangible assets (e.g., crypto-currencies), making them less negotiable. Complex financial securities take even more time to be authenticated and evaluated.

¹⁶In that regard, our theory complies with the Wallace (1998) dictum in that it specifies assets by how their physical properties determine the technology to transfer their ownership, which permits the assets' role in exchange to be endogenous.

In Figure 9 we provide some evidence based on Pagnotta and Philippon (2018) and O’Keeffe (2018) that transaction times vary for different classes of assets.¹⁷ In each pairwise meeting, the negotiation ends at time $\bar{\tau}$ where $\bar{\tau}$ is exponentially distributed with mean $1/\lambda$. The consumer’s bargaining power is θ .¹⁸

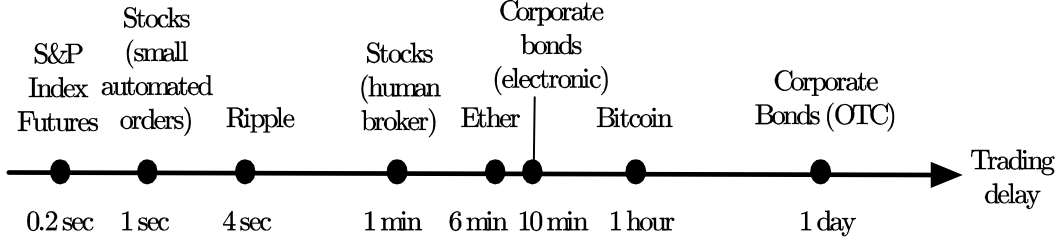


Figure 9: Trading delays by asset classes. Sources: Pagnotta and Philippon (2018), O’Keeffe (2018).

We let consumers choose the order according to which assets are sold (after $\bar{\tau}$ has been realized). The cumulative amount of asset of type j that has been up for negotiation at time τ is denoted $\omega_j(\tau)$ and $\omega(\tau) = \sum_{j=0}^J \omega_j(\tau)$. It obeys the following law of motion:

$$\omega'_j(\tau) = \delta_j \sigma_j(\tau) \quad \text{for all } j \in \{0, 1, \dots, J\}, \quad (28)$$

where $\sigma_j(\tau) \in [0, 1]$ is the fraction of time devoted to the sale of asset j at time τ and $\sum_{j=0}^J \sigma_j(\tau) = 1$. Moreover, feasibility implies $\sigma_j(\tau) \in [0, 1]$ if $\omega_j(\tau) < a_j$ and $\sigma_j(\tau) = 0$ otherwise. In words, an agent can add asset j on the negotiating table at time τ only if he has not sold all his holdings of asset j prior to τ . Replacing δ by ω'_j in (15), the change in the consumer’s consumption and the change in the overall payment over time are

$$y'(\tau) = \frac{\theta u'(y) + (1 - \theta) v'(y)}{u'(y)v'(y)} \omega'(\tau) \quad (29)$$

$$p'(\tau) = \omega'(\tau), \quad (30)$$

if $y(\tau) < y^*$ and $y'(\tau) = p'(\tau) = 0$ otherwise.

The surplus of a consumer in a DM match with portfolio $\mathbf{a} = [a_j]_{j=0}^J$, agenda $\boldsymbol{\sigma} = [\sigma_j]_{j=0}^J$, and time to negotiate $\bar{\tau}$ is:

$$S(\mathbf{a}, \boldsymbol{\sigma}, \bar{\tau}) = \theta \int_0^{\bar{\tau}} \ell[y(\tau)] \omega'(\tau) d\tau = \theta \int_0^{\omega(\bar{\tau})} \ell[y(\omega)] d\omega. \quad (31)$$

¹⁷As mentioned earlier, it is hard to disentangle the different sources of delays in asset transactions (see, e.g., Duffie, 2012) but there is strong evidence that those delays vary across assets. In our model, we keep search frictions the same across assets and attribute all the differences to the negotiation process and the time to transfer ownership.

¹⁸One could allow θ to be a function of τ , which would not affect our results qualitatively. One could also assume that θ varies with the type of asset that is currently under negotiation. Such extension would allow our theory to encompass the explanations for rate-of-return differences across assets by Zhu and Wallace (2007) and Rocheteau and Nosal (2017).

Over a small time interval of length $d\tau$ the consumer sells $\omega'(\tau)$ units of assets where each unit generates a marginal surplus equal to $\theta\ell(y)$. The right side of (31) is obtained by adopting the change of variable $\omega = \omega(\tau)$. It follows that the consumer surplus depends on the agenda $\boldsymbol{\sigma}$ only through the amount of assets that can be negotiated up to $\bar{\tau}$, $\omega(\bar{\tau})$. From (28) $\omega(\bar{\tau}) = \int_0^{\bar{\tau}} \sum_{j=0}^J \delta_j \sigma_j(\tau) d\tau$. In order to characterize the optimal strategy to maximize $\omega(\bar{\tau})$ we denote $T_0 = 0$ and

$$T_j(\mathbf{a}) = \sum_{k=0}^{j-1} \frac{a_k}{\delta_k} \text{ for all } j \in \{1, 2, \dots, J+1\}. \quad (32)$$

So T_j is the time that it takes to sell the first $j-1$ most negotiable assets.

Lemma 3 (Pecking order) *For any portfolio \mathbf{a} and any realization of $\bar{\tau}$, the optimal choice $\boldsymbol{\sigma}^* = [\sigma_j^*]$ is given by*

$$\sigma_j^*(\tau) = \begin{cases} 1 & \text{if } T_j < \tau \leq T_{j+1} \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 3 shows that it is optimal to adopt a pecking order to sell assets.¹⁹ Consumers start paying with money. When their money holdings are exhausted, they start selling asset 1, etc. Hence, in a fraction $1 - e^{-\lambda T_1}$ of matches only money is used to finance consumption, where T_1 is endogenous. In a fraction $e^{-\lambda T_1} - e^{-\lambda T_2}$ of matches both money and type-1 Lucas trees serve as means of payments. And so on. Given this pecking order, the expected maximized surplus of the consumer is:

$$S(\mathbf{a}) = \int_0^{+\infty} \lambda e^{-\lambda\tau} S(\mathbf{a}, \boldsymbol{\sigma}^*, \tau) d\tau = \theta \sum_{j=0}^J \delta_j \int_{T_j}^{T_{j+1}} e^{-\lambda\tau} \ell[y(\tau)] d\tau. \quad (33)$$

Over the time interval $[T_j, T_{j+1}]$ agents negotiate asset j where the speed of the negotiation is given by δ_j .

We now turn to the asset pricing implications of this pecking order. The portfolio problem in the CM is given by

$$\max_{\mathbf{a} \geq \mathbf{0}} \{-\mathbf{s}\mathbf{a} + \alpha S(\mathbf{a})\}, \quad (34)$$

where $\mathbf{s} = [s_j]$ is the vector of asset spreads, i.e., $s_j = (i - i_j) / (1 + i_j)$ where the nominal interest rate of asset j is i_j . For fiat money, $i_0 = 0$ and $s_0 = i$. According to (34) the consumer maximizes his expected DM surplus net of the costs of holding assets as measured by the spreads $[s_j]$. The FOCs of the maximization problem (34) are:

$$s_j = \alpha \frac{\partial S(\mathbf{a})}{\partial a_j}. \quad (35)$$

The left side of (35) is the opportunity cost of holding asset j . The right side is the probability α that the consumer receives an opportunity to spend, α , times the marginal liquidity value from holding asset j . The expression of this last term is given in the following lemma.

¹⁹For a pecking-order theory of payments based on informational asymmetries between consumers and producers, see Rocheteau (2011).

Lemma 4 *The marginal value of asset j to a consumer with portfolio \mathbf{a} is*

$$\frac{\partial S(\mathbf{a})}{\partial a_j} = \theta \lambda \overbrace{\sum_{k=j+1}^J \int_{T_k}^{T_{k+1}} \frac{(\delta_j - \delta_k)}{\delta_j} e^{-\lambda \tau} \ell[y(\tau)] d\tau}^{\text{negotiability value}} + \overbrace{\theta e^{-\lambda T_{J+1}} \ell[y(T_{J+1})]}^{\text{liquidity value}}. \quad (36)$$

From (36), holding an additional unit of a_j has two benefits to the consumer. First, there is a liquidity benefit according to which the consumer has more wealth, which relaxes his liquidity constraint and allows him to consume more if the negotiation is not terminated before the whole portfolio has been sold. This effect is captured by the last term on the right side and is analogous to (26). Second, there is a negotiability benefit according to which asset j speeds up the negotiation relative to less negotiable assets of types $j+k$. This first term on the right side of (36) is asset specific, as it depends on δ_j .

By market clearing $a_j = A_j$ for all $j \geq 1$. Hence, an equilibrium can be reduced to a list $\langle a_0, \{s_j\}_{j=1}^J \rangle$ that solves (35). In the following proposition we measure the liquidity of an asset by its velocity or turnover defined as

$$\mathcal{V}_j \equiv \frac{\alpha \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \omega'_j(\tau) \mathbf{1}_{\{\omega(\tau) < p(y^*)\}} d\tau dx}{A_j}. \quad (37)$$

The numerator corresponds to the aggregate quantity of asset j sold in pairwise meetings while the denominator is the supply of the asset.

Proposition 10 *(The negotiability structure of asset yields.) For all $\{A_j\}_{j=1}^J$ if $\delta_0 > \delta_1$ then there is a $\bar{i} > 0$ such that for all $i < \bar{i}$ there exists a unique steady-state monetary equilibrium with aggregate real balances $A_0(i) > 0$. Let $\Omega_1 = A_0(i)$ and for each $j = 2, \dots, J$, let $\Omega_j = A_0(i) + \sum_{k=1}^{j-1} A_k$.*

1. If $\Omega_{j+1} < p(y^*)$ and $\delta_j > \delta_{j+1}$, then $s_j > s_{j+1}$. If $\Omega_{j+1} \geq p(y^*)$, then $s_{j+k} = 0$ for all $k \geq 0$.
2. If $\delta_j > \delta_{j+1}$ and $p(y^*) > \Omega_j$, then $\mathcal{V}_j > \mathcal{V}_{j+1}$. If $p(y^*) \leq \Omega_j$ then $\mathcal{V}_j = 0$.
3. As λ approaches 0, $|s_j - s_{j'}|$ approaches 0 for all $j, j' \in \{0, \dots, J\}$. Asset velocity, \mathcal{V}_j , approaches α for all j such that $\Omega_j \leq p(y^*)$, 0 for all j such that $\Omega_j \geq p(y^*)$, and $\alpha [p(y^*) - \Omega_j] / A_j$ for j such that $p(y^*) \in (\Omega_j, \Omega_{j+1})$.

Proposition 10 has several implications. First, fiat money is valued for low i irrespective of the supply of Lucas trees. Even if the capitalization of all Lucas trees, $\sum_{k=1}^J A_k$, is larger than liquidity needs, $p(y^*)$, money is useful because it can be negotiated faster, thereby allowing agents to finance a larger consumption when $\bar{\tau}$ is low.

Second, even though all Lucas trees yield identical dividend streams, the equilibrium features rate-of-return differences across assets. Provided that asset supplies are not too large, assets with a high negotiability

command a lower interest rate than assets with a low negotiability, i.e., $i_j < i_{j+1}$ if $\delta_j > \delta_{j+1}$. The key components of our theory is that negotiation takes time as assets are sold gradually, and not all assets can be sold at equal speed due to technological differences to authenticate and transfer assets. Part 2 of Proposition 10 shows that assets that are more negotiable have a higher velocity, which is a consequence of the endogenous pecking order. As a result, there is a positive correlation between velocity and asset prices.

Finally, Part 3 of Proposition 10 considers the limit when the expected time horizon of the negotiation becomes arbitrarily large. If the risk that the negotiation ends before the portfolio of assets has been sold goes to zero, then the rates of return of all assets converge to the same value, i.e., there is rate of return equality. In that case the negotiability of assets, and the order according to which they are negotiated, does not affect their rates of return. The order at which assets are sold, however, matters for velocities. Indeed, only a fraction of assets are used for transactions and those assets have a maximum velocity equal to α .

7 Two applications

We propose two applications of the model in Section 6 to address puzzles in monetary theory and finance, e.g., the rate of return dominance and the indeterminacy of nominal exchange rates. In the first application, we study OMOs and anticipated inflation in economies where money and interest-bearing government bonds coexist. The second application describes an economy with two currencies that differ by their inflation rate and their negotiability, allowing us to break exchange rate indeterminacy.

7.1 Money and bonds

We illustrate some novel comparative statics of our model regarding the effects of OMOs on aggregate output. We consider the case where $J = 1$ with asset 1 being interpreted as short-term, real government bonds. We start with the case where $\bar{\tau}$ is deterministic and we will return to the case where $\bar{\tau}$ is stochastic later. The consumer's portfolio problem in the CM is given by

$$\max_{(a_0, a_1)} -ia_0 - s_1 a_1 + \alpha\{u[y(a_0, a_1)] - p[y(a_0, a_1)]\},$$

where DM output is

$$y(a_0, a_1) = \begin{cases} p^{-1}(\delta_0 \bar{\tau}) & \leq a_0/\delta_0 \\ p^{-1}[a_0(1 - \delta_1/\delta_0) + \delta_1 \bar{\tau}] & \text{if } \bar{\tau} \in (a_0/\delta_0, a_0/\delta_0 + a_1/\delta_1] \\ p^{-1}(a_0 + a_1) & \geq a_0/\delta_0 + a_1/\delta_1 \end{cases} \quad (38)$$

While $a_1 = A_1$ by market clearing, a_0 is endogenous and depends on policy through both i and A_1 . We interpret an open-market operation as a change in A_1 associated with a change of opposite sign of the money supply. Because money is neutral, only the change in A_1 matters (e.g., Rocheteau et al., 2018).

We distinguish four regimes, represented in the parameter space $(\bar{\tau}, A_1)$ in Figure 10, where y_1 satisfies $i = \alpha\theta[(\delta_0 - \delta_1)/\delta_0]\ell(y_1)$ and y_2 satisfies $i = \alpha\theta\ell(y_2)$.

In regime I, $\bar{\tau} = T_1$, the consumer holds just enough real balances to spend them all by the time the negotiation ends. In such an endogenous "cash-in-advance" regime, $y = p^{-1}(a_0)$, $i = \alpha\theta\ell(y)$, and $s_1 = 0$. In regime II, $\bar{\tau} \in (T_1, T_2)$, only a fraction of bonds can be sold before the negotiation ends. Hence, $s_1 = 0$. Output and real balances solve $y = p^{-1}[a_0(1 - \delta_1/\delta_0) + \delta_1\bar{\tau}]$ and $i = (\delta_0 - \delta_1)\alpha\theta\ell(y)/\delta_0$. In both regimes I and II a change in A_1 has no effect on interest rates and output. In regime IV, $\bar{\tau} > T_2$, the negotiability constraint does not bind. Hence, $y = \min\{p^{-1}(a_0 + a_1), y^*\}$, $s_1 = i$, and $i_1 = 0$. Changes in A_1 are ineffective because money and bonds are perfect substitutes.

We now focus on regime III, $T_2 = \bar{\tau}$, where the consumer's portfolio is sold in exactly $\bar{\tau}$ units of time. Such equilibria feature rate of return dominance, $s_1 \in (0, i)$, and OMOs are effective.

Proposition 11 (Coexistence of money and interest-bearing bonds and policy.) *A monetary equilibrium with $T_2 = \bar{\tau}$ exists if*

$$\frac{\delta_1 [\delta_0\bar{\tau} - p(y_2)]}{\delta_0 - \delta_1} < A_1 < \min\left\{\frac{\delta_1 [\delta_0\bar{\tau} - p(y_1)]}{\delta_0 - \delta_1}, \delta_1\bar{\tau}\right\} \quad \text{and} \quad \frac{p(y_1)}{\delta_0} < \bar{\tau} < \frac{p(y_2)}{\delta_1}.$$

Output and the interest rate spread are determined recursively according to:

$$y = p^{-1}\left[\delta_0\bar{\tau} - \left(\frac{\delta_0 - \delta_1}{\delta_1}\right)A_1\right] \quad (39)$$

$$s_1 = \frac{\delta_0}{\delta_1}i - \alpha\theta\left(\frac{\delta_0 - \delta_1}{\delta_1}\right)\ell(y). \quad (40)$$

An open-market sale of bonds raises i_1 and reduces y . An increase in the money growth rate has no effect on output. Assuming ρ and π are close to 0, money growth affects nominal interest rates according to

$$\frac{\partial i}{\partial \pi} \approx 1, \quad \frac{\partial i_1}{\partial \pi} \approx \frac{\delta_1 - \delta_0}{\delta_1} < 0.$$

As A_1 increases, consumers reduce a_0 in order to be able to sell their whole portfolio in $\bar{\tau}$ units of time. But bonds take more time than money to be sold, and hence consumption decreases. Formally, $a_0/\delta_0 + A_1/\delta_1 = \bar{\tau}$ and hence $\partial a_0/\partial A_1 = -\delta_0/\delta_1 < -1$. An open market sale of bonds decreases output by crowding out a highly negotiable asset, money, with a less negotiable asset, bonds.

An increase in the money growth rate does not affect real balances in equilibria where $T_2 = \bar{\tau}$ because agents would hold more real balances if they were not constrained by $\bar{\tau}$. The interest rate on illiquid bonds increases one-to-one with money growth by the Fisher effect. The interest rate on liquid government bonds, $i_1 \approx \left(\frac{\delta_0 - \delta_1}{\delta_1}\right)[\alpha\theta\ell(y) - i]$, decreases with inflation according to the Mundell-Tobin effect.

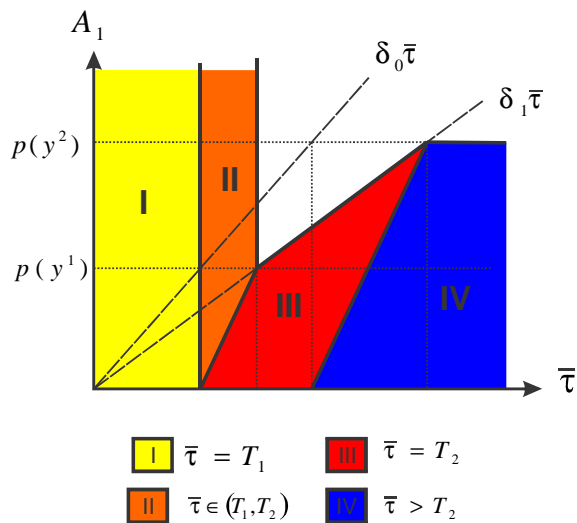


Figure 10: Typology of equilibria with money and bonds

Figure 11 presents a numerical example where $\bar{\tau}$ is exponentially distributed.²⁰ The black line on Figure 11 plots y as a function of $\bar{\tau}$ when $A_1 = 0.15$. Following an increase in A_1 (from 0.15 to 0.35) consumers

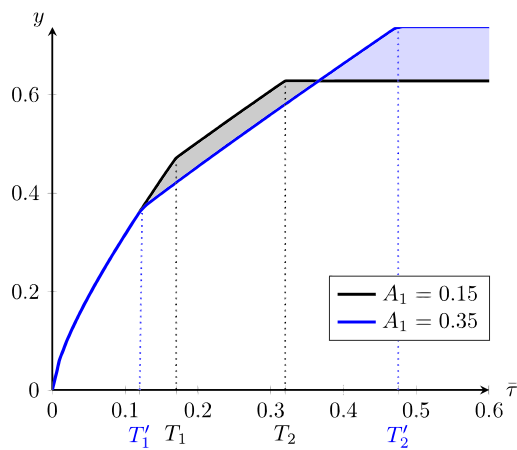


Figure 11: Stochastic case with money and bonds: Output distribution

reduce their real balances but T_2 increases. For low values of $\bar{\tau}$, y is not affected by the change in A_1 . If $\bar{\tau}$ falls into an intermediate range, then output is lower. This negotiability effect of OMOs corresponds to the gray region in Figure 11. If $\bar{\tau}$ is big enough, output is higher through a liquidity effect, as visible in the blue region. The impact on aggregate output (across all pairwise meetings) depends on the relative sizes of the negotiability and liquidity effects, which are ultimately determined by the distribution of $\bar{\tau}$. In

²⁰Preferences are defined by $u(y) = 2\sqrt{y}$ and $v(y) = (2/3)y^{3/2}$, thus $y^* = 1$. We pick $\alpha = 1$, $\theta = 0.5$, $\delta_0 = 2$, $\delta_1 = 1$, $i = 0.15$. $\lambda = 3.33$, so that the mean time horizon is 0.3.

our example, the weight on the blue region is high enough for the liquidity effect to dominate, causing the expected aggregate output to increase.

7.2 Multiple (crypto-)currencies

Our model can be applied to economies with multiple fiat monies in order to address the nominal exchange rate indeterminacy puzzle. This application is topical given the development of multiple crypto-currencies, such as Bitcoins, Litecoin, Ethereum, and others. A transaction with crypto-currencies requires confirmation that takes time, and confirmation times vary across currencies as illustrated in Figure 12. Our negotiability parameter, δ , is a proxy for the time it takes to transfer the ownership of virtual coins.

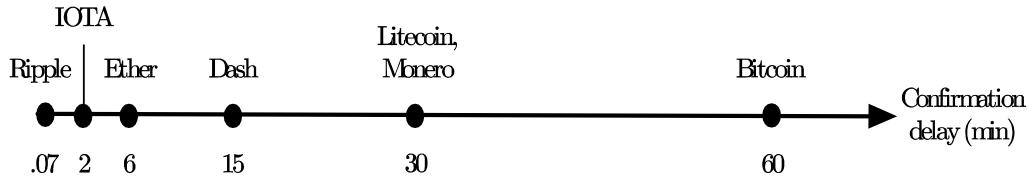


Figure 12: Confirmation times for different cryptocurrencies. Source: O’Keeffe (2018).

We now consider an economy with two currencies, currency 0, the supply of which grows at rate π_0 and currency 1, the supply of which grows at rate π_1 . Currency 0 has lower confirmation times and can be transferred faster than currency 1, i.e., $\delta_0 > \delta_1$. However, the supply of currency 0 grows faster than the supply of currency 1, $\pi_0 > \pi_1$. We focus on steady-state equilibria where the rate of return and the aggregate real supply of each currency are constant.

We start with the simple case where $\bar{\tau}$ is deterministic. For the two monies to coexist the equilibrium must feature $\bar{\tau} \geq T_2$. The FOCs are

$$-i_0 - \frac{\mu}{\delta_0} + \alpha\theta\ell(y) \leq 0, \text{ “=” if } a_0 > 0 \quad (41)$$

$$-i_1 - \frac{\mu}{\delta_1} + \alpha\theta\ell(y) \leq 0, \text{ “=” if } a_1 > 0. \quad (42)$$

where $\mu \geq 0$ is the Lagrange multiplier associated with the negotiability constraint. If the negotiability constraint does not bind, $\mu = 0$, then (41)-(42) imply $i_0 = i_1$. The two currencies must have the same rate of return, which requires $\pi_0 = \pi_1$ in a steady-state equilibrium. If the negotiability constraint binds, $\mu > 0$, then the two currencies will be held only if $i_0 > i_1$. Moreover, by market clearing, the values of the two currencies solve $\phi_{0,t}A_{0,t} + \phi_{1,t}A_{1,t} = p(y)$ and the nominal exchange rate is $e_t = \phi_{0,t}/\phi_{1,t}$.

Proposition 12 (*Dual currency economy.*)

1. Suppose $i_0 = i_1 = i$. If $\delta_0 \bar{\tau} > p(y)$, where $y = \ell^{-1}(i/\alpha\theta)$, then there exists a steady-state equilibrium where currencies 0 and 1 are valued. If $\delta_1 \bar{\tau} \geq p(y)$, then any $e \in (0, +\infty)$ is an equilibrium exchange rate. If $\delta_1 \bar{\tau} < p(y)$ then there is a positive lower bound for the exchange rate equal to

$$\underline{e} = \frac{A_{1,t} \delta_0 [p(y) - \delta_1 \bar{\tau}]}{A_{0,t} \delta_1 [\delta_0 \bar{\tau} - p(y)]}. \quad (43)$$

2. Suppose $i_0 > i_1$. There are two thresholds $0 < \bar{\tau}_0 < \bar{\tau}_1$ such that for all $\tau \in (\bar{\tau}_0, \bar{\tau}_1)$, there exists a unique steady-state equilibrium where both currencies 0 and 1 are valued and output solves

$$\frac{i_0 \delta_0 - i_1 \delta_1}{\delta_0 - \delta_1} = \alpha \theta \ell(y). \quad (44)$$

Inflation rates affect output according to $\partial y / \partial \pi_0 < 0$ and $\partial y / \partial \pi_1 > 0$. Moreover, currency 0 appreciates vis-a-vis currency 1 as α or θ increases or as $\bar{\tau}$ decreases.

The first part of Proposition 12 assumes equal money growth rates across currencies. If currency 0 is sufficiently negotiable, then there exists an equilibrium where both currencies are valued. Moreover, if currency 1 is also highly negotiable, then the nominal exchange rate between the two currencies, $e = \phi_0 / \phi_1$, can be anything, in accordance with the indeterminacy result of Kareken and Wallace (1981). However, if the negotiability of currency 1 is limited, then the range of equilibrium values for e is reduced, i.e., there is a lower bound for the exchange rate.

The second part of Proposition 12 assumes different money growth rates and focuses on equilibria where $\mu > 0$ and $T_2 = \bar{\tau}$, i.e., $a_0 / \delta_0 + a_1 / \delta_1 = \bar{\tau}$ and $p(y) = a_0 + a_1$. The determination of a dual currency equilibrium is illustrated in Figure 13. The condition $a_0 + a_1 = p(y)$ is represented by the red line while the negotiability constraint, $a_0 / \delta_0 + a_1 / \delta_1 = \bar{\tau}$, is represented by the blue line. If an intersection exists, then it is unique. There exists an equilibrium where the two currencies coexist provided that the time allocated to the negotiation, $\bar{\tau}$, is neither too small nor too large. If $\bar{\tau}$ is small, agents will choose to trade with the most negotiable currency only. If $\bar{\tau}$ is large, agents will choose to only hold the currency with the lowest inflation rate. For intermediate values for $\bar{\tau}$ agents choose a diversified portfolio of currencies. One can interpret such an equilibrium as one where different crypto-currencies with different technologies to record asset transfers coexist. One can also think of a dollarization equilibrium where the high-inflation domestic currency coexists with the low-inflation foreign currency.

If the inflation rate of the most negotiable currency increases, then output decreases, in accordance with textbook comparative statics. However, as π_1 increases, agents find it optimal to reduce their holdings of currency 1 and raise their holdings of currency 0. As a result, they can buy more output over the time

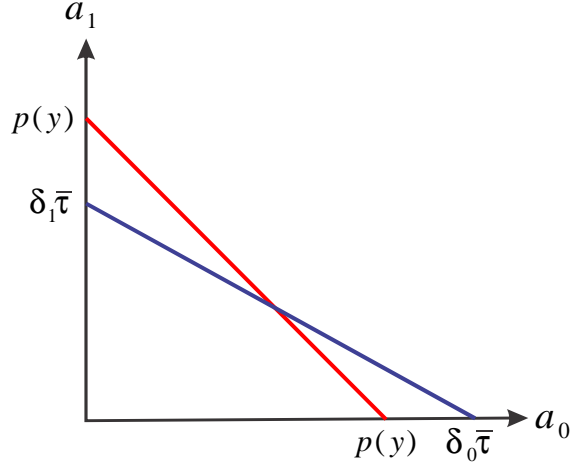


Figure 13: Equilibrium with 2 currencies

horizon $\bar{\tau}$. In the context of a dollarization equilibrium this would mean that an increase of the inflation rate of the foreign currency raises output by reverting the dollarization process.

Our model provides a resolution to the Kareken-Wallace indeterminacy result.²¹ In a two-currency equilibrium the nominal exchange rate is uniquely determined and given by

$$e_t = \frac{\delta_0 A_{1,t} p(y) - \delta_1 \bar{\tau}}{\delta_1 A_{0,t} \delta_0 \bar{\tau} - p(y)}. \quad (45)$$

It depends on the ratio of the negotiability parameters and the ratio of the money supplies. As the frequency of trading opportunities or the consumer's bargaining power increase, consumers shift their portfolios toward the most negotiable currency, which leads to an appreciation of the exchange rate. Conversely, as the time to negotiate increases, agents reallocate their portfolios toward the currency with the highest rate of return, and hence the exchange rate depreciates.

8 Conclusion

The objective of this paper was to introduce a new approach to bargaining into models of decentralized asset market. More than a new solution, we advocate for a new definition to the bargaining problem for negotiations over unrestricted asset portfolios. This new definition is a natural extension of existing bargaining theories (e.g., Osborne and Rubinstein, 1990) for a new class of models of decentralized markets with richer asset holdings. It includes as a primitive the agenda of the negotiation, i.e., a partition of the portfolio into asset bundles to be sold sequentially. Each portfolio negotiation has an agenda: we require

²¹For information-based theories of the determinacy of the nominal exchange rate, see Zhang (2014) based on Lester et al. (2012) and Gomis-Porqueras et al. (2017) based on Li et al. (2012). Garratt and Wallace (2018) study an overlapping generation model with fiat money and crypto-currencies and show coexistence when fiat money is subject to a storage cost.

making it explicit. The notion of agenda acknowledges the size and multi-dimensionality of asset portfolios and its importance for trading outcomes.

Our approach complies with the Nash program: it has (multiple) strategic foundations, in the form of alternating-offer games, and axiomatic foundations. It encompasses existing bargaining solutions; we provided specific agendas for which our model generates the Nash or Kalai solutions. We showed through several examples that the choice of the agenda is crucial for allocations and welfare. We offered insights on how to pick, or endogenize agendas.

The second part of the paper consisted in introducing bargaining solutions with an agenda into off-the-shelves models of decentralized or OTC markets. Our results demonstrate that these bargaining solutions are very tractable and generate novel normative and positive implications for asset markets. For instance, the choice of the agenda can have dramatic implications for the functioning of OTC markets with outcomes varying from a complete break-down to the implementation of first-best trades. We provided a theory to endogenize agendas in the presence of multiple assets, which gave rise to a pecking order for asset sales and a non-degenerate distribution of asset returns. We used our model to account for the rate-of-return dominance puzzle in monetary theory and for the determinacy of the nominal exchange rate between two fiat currencies.

In terms of future research, we want to explore a continuous-time version of the alternating ultimatum offer bargaining games with endogenous risk of breakdown determined by the random arrivals of potential trading partners. More can be done to endogenize agendas in the presence of multiple assets, e.g., by introducing informational frictions. It would also be realistic to allow for multilateral matching and sequential negotiations with multiple trading partners.

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Appendix A: Proofs of Lemmas and Propositions

Proof of Lemmas 1 and 2. Both lemmas are concerned with the derivation of Pareto frontiers. We use u^b to denote the consumer's payoff and u^s the producer's payoff. First we begin with bargaining over assets, and the Pareto frontier is derived from the program

$$u^b = \max_{y \geq 0} \{u(y) - p + u_0^b\} \quad \text{s.t.} \quad p - v(y) + u_0^s \geq u^s, \quad p \leq \delta\tau.$$

The consumer chooses the terms of trade, (y, p) , to maximize his utility subject the constraint that he must guarantee some utility level u^s to the producer. If $\delta\tau \geq u^s - u_0^s + v(y^*)$, then $y = y^*$ and $p = u^s - u_0^s + v(y^*)$. Moreover, $u^b + u^s = u(y^*) - v(y^*) + u_0^b + u_0^s$. If $\delta\tau < u^s - u_0^s + v(y^*)$, then $p = \delta\tau = u^s - u_0^s + v(y)$, i.e., $y = v^{-1}(\delta\tau - u^s + u_0^s)$.

Now we turn to bargaining over DM goods. The Pareto frontier solves

$$u^b = \max_{y, p} \{u(y) - p\} \quad \text{s.t.} \quad -v(y) + p = u^s, \quad p \leq z, \quad y \leq \bar{y}.$$

The payment cannot be greater than the consumer's asset holdings and the output is not greater than the upper bound \bar{y} . Substitute $p = u^s + v(y)$ into the constraints and rewrite the problem as:

$$u^b = \max_{y, p} \{u(y) - v(y) - u^s\} \quad \text{s.t.} \quad y \leq \min \{v^{-1}(p - u^s), \bar{y}\}.$$

If $u^s \leq z - v(\bar{y})$ then $y = \bar{y}$ (note that we assume $\bar{y} \leq y^*$) and the equation of the Pareto frontier is simply

$$u^b + u^s = u(\bar{y}) - v(\bar{y}).$$

If the payment constraint binds then $y = v^{-1}(z - u^s)$ and

$$u^b = u \circ v^{-1}(z - u^s) - z.$$

This gives a negative relationship between u^b and u^s since $\partial u^b / \partial u^s = -u'(y)/v'(y)$. Moreover, $\partial^2 u^b / (\partial u^s)^2 < 0$, i.e., the Pareto frontier is strictly concave. ■

Proof of Proposition 2. By the definition of the consumer's utility, $u^b(\tau) = u_0^b + u[y(\tau)] - \delta\tau$, it follows that

$$u^b(\tau) = u'(y) y'(\tau) - \delta. \tag{46}$$

The change in the consumer's utility along the gradual bargaining path the change in DM consumption as the consumer adds assets to the negotiating table, net of the asset transfer (the second term on the right side). From (9) and (46), we obtain (11). The total transfer of assets is $p(y) = \int_0^y \delta \frac{\partial \tau}{\partial x} dx$ where from (11) $\partial \tau / \partial x$ coincides with $1/y'(\tau)$ evaluated at x . ■

Proof of Proposition 4. We assume that, with no loss of generality, $z \leq p_\infty(y^*)$. This also allows us to assume that (18) has interior solutions, and, summing (18) from $n = 1$ to N :

$$\sum_{n=1}^N \left[\int_{y_{n-1}}^{y_n} \frac{v'(y_n)}{u'(y_n) + v'(y_n)} u'(x) dx + \int_{y_{n-1}}^{y_n} \frac{u'(y_n)}{u'(y_n) + v'(y_n)} v'(x) dx \right] = z.$$

It can be expressed more compactly as

$$\int_0^{y_N} \left[1 - \Theta \left(x; \frac{z}{N} \right) \right] u'(x) + \Theta \left(x; \frac{z}{N} \right) v'(x) dx = z,$$

where

$$\Theta \left(x; \frac{z}{N} \right) = \sum_{n=1}^N \frac{u'(y_n)}{u'(y_n) + v'(y_n)} 1_{(y_{n-1}, y_n]}(x)$$

and $1_{(y_{n-1}, y_n]}(x)$ is the indicator function for the interval $(y_{n-1}, y_n]$. Note that for all $N < +\infty$ and for all $x \notin \{y_n\}$,

$$\Theta \left(x; \frac{z}{N} \right) < \frac{u'(x)}{u'(x) + v'(x)}.$$

Hence,

$$\int_0^{y_N} \left[1 - \Theta \left(x; \frac{z}{N} \right) \right] u'(x) + \Theta \left(x; \frac{z}{N} \right) v'(x) dx > \int_0^{y_N} \frac{2v'(x)u'(x)}{u'(x) + v'(x)} dx.$$

So for all $N < +\infty$, the payment to finance y_N units of consumption, the left side of the inequality, is larger than the one when $N = +\infty$, the right side of the inequality. Hence, the consumer extracts the largest surplus when $N = +\infty$. ■

Derivation of consumer surpluses (23) and (27). First note that $y(z; \tau) = p^{-1}(\min\{\delta\tau, z\})$ and $p(z; \tau) = \min\{\delta\tau, z\}$ if $\min\{\delta\tau, z\} \leq p(y^*)$, and $y(z; \tau) = y^*$ and $p(z; \tau) = p(y^*)$ otherwise, where p is given by (16). Now, assuming that $z \leq p(y^*)$, the expected surplus is given by

$$\begin{aligned} S^b(z, \delta) &= \int_0^\infty \lambda e^{-\lambda\tau} \{u[y(z; \tau)] - p(z; \tau)\} d\tau \\ &= \int_0^\infty \lambda e^{-\lambda\tau} \int_0^{p^{-1}(\min\{\delta\tau, z\})} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1-\theta)v'(x)} dx d\tau \\ &= \int_0^\infty \int_0^{p^{-1}(\min\{\delta\tau, z\})} \lambda e^{-\lambda\tau} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1-\theta)v'(x)} dx d\tau \\ &= \int_0^{p^{-1}(z)} \int_{p(x)/\delta}^\infty \lambda e^{-\lambda\tau} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1-\theta)v'(x)} d\tau dx \\ &= \int_0^y e^{-\frac{\lambda}{\delta} p(x)} \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1-\theta)v'(x)} dx, \end{aligned}$$

where $y = p^{-1}(z)$, the second equality follows from (16), the fourth uses iterated integral, and in the fifth we integrate out τ . Moreover, $S^b(z, \delta) = S^b(p(y^*), \delta)$ for all $z > p(y^*)$. The derivation for social welfare follows exactly the same steps except for replacing $\frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1-\theta)v'(x)}$ by $[u'(x) - v'(x)]$. ■

Proof of Proposition 6. If agents bargain gradually over the asset then the payment function is:

$$p(y) = \int_0^y \frac{2u'(x)v'(x)}{u'(x)+v'(x)} dx.$$

Using that

$$\frac{2u'(x)v'(x)}{u'(x)+v'(x)} = \left[\frac{v'(x)}{u'(x)+v'(x)}u'(x) + \frac{u'(x)}{u'(x)+v'(x)}v'(x) \right] < \frac{u'(x)+v'(x)}{2},$$

for all $x < y^*$ since $u'(x) > v'(x)$, we obtain the following inequality:

$$p(y) < \int_0^y \frac{1}{2} [u'(x) + v'(x)] dx = \frac{u(y) + v(y)}{2} = p^{DM}(y),$$

where $p^{DM}(y)$ is the payment function if agents bargain gradually over the DM good. We denote $y^1(z)$ as the solution to $p(y^1) = \min\{z, p(y^*)\}$ and $y^{DM}(z)$ as the solution to $p^{DM}(y^{DM}) = \min\{z, p^{DM}(y^*)\}$. Using the inequality above it follows that $y^1(z) > y^{DM}(z)$ for all z such that $y^{DM}(z) < y^*$. We can now compare the consumer's surpluses under the two agendas: $u(y) - p(y) > u(y) - p^{DM}(y)$ and $y^1(z) \geq y^{DM}(z)$ for all z . Using that surpluses are monotone increasing in y it follows that consumers are better off with the first agenda than the second. We now compare the producer's surpluses under the two agendas. Let $p^* = p(y^*)$ and $p^{DM*} = p^{DM}(y^*)$. For all $z \leq p^*$, $z - v[y^{DM}(z)] > z - v[y^1(z)]$ since $y^{DM}(z) < y^1(z)$. For all $z \in (p^*, p^{DM*})$, $z - v[y^{DM}(z)] > p^* - v(y^*)$ since the bargaining solution is monotone. It follows that producers always prefer to bargain gradually over the DM good. ■

Proof of Proposition 7. For each $y \in (0, y^*]$, equation (26) gives a negative relationship between s and y , denoted by $s = s(y)$, with $\lim_{y \rightarrow 0} s(y) = +\infty$, and $s(y)$ is strictly decreasing. Given this function, equilibrium is given by y that satisfies (24). Since the left side of (24) is strictly increasing in y and the right side is strictly increasing in s and hence strictly decreasing in y , with $s = s(y)$, and since the right side of (24) is positive at $y = 0$, there is unique y that satisfies (24).

(1) Since $p(y^*) \leq (1 + \rho)Ad/\rho$ and $s^r(y^*) = 0$, $y = y^*$ is the unique equilibrium. In this equilibrium, the time it takes to sell $p(y^*)$ units of wealth is $\tau^* = p(y^*)/\delta$ and the probability that $\bar{\tau} \geq \tau^*$ is $e^{-\frac{\lambda}{\delta}p(y^*)}$. From (25), social welfare is

$$\mathcal{W} = \alpha \int_0^{y^*} e^{-\frac{\lambda}{\delta}p(x)} [u'(x) - v'(x)] dx,$$

which is independent of Ad but decreasing with λ/δ .

(2) Since $p(y^*) > (1 + \rho)Ad/\rho$, the unique equilibrium features $y < y^*$ and $s > 0$. From (26) and (24) the spread is the unique $s \in (0, \rho)$ solution to

$$s = \alpha \theta e^{-\frac{\lambda}{\delta}(\frac{1+\rho}{\rho-s})Ad} \left\{ \ell \circ p^{-1} \left[\left(\frac{1+\rho}{\rho-s} \right) Ad \right] \right\}.$$

The right side is decreasing in Ad and λ/δ . Hence, s decreases with Ad and λ but increases with δ . From (26) y is a decreasing function of s , hence y increases with Ad and, from (25), social welfare increases with Ad . Similarly, y decreases with λ/δ and hence \mathcal{W} decreases with λ/δ .

(3) From (1), when $p(y^*) \leq (1 + \rho)Ad/\rho$, $s^r(y^*) = 0$, $y = y^*$. Thus, as λ approaches zero, the probability that $\bar{\tau} \geq \tau^*$ approaches 1, and hence the social welfare approaches the first best. ■

Proof of Proposition 8. The seller's surplus from a trade is:

$$u^s(\omega, z) = \begin{cases} \frac{\varepsilon_h - \varepsilon_\ell}{2} \omega & \text{if } \frac{z}{\omega} \geq \frac{\varepsilon_h + \varepsilon_\ell}{2} \\ z - \varepsilon_\ell \omega & \text{if } \frac{z}{\omega} \in \left[\frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}, \frac{\varepsilon_h + \varepsilon_\ell}{2} \right) \\ \frac{(\varepsilon_h - \varepsilon_\ell)z}{2\varepsilon_h} & \text{if } \frac{z}{\omega} < \frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}. \end{cases} \quad (47)$$

If z/ω is sufficiently high, then all DM goods are purchased by the buyer who only spends a fraction of his real balances. In that case, the seller's surplus increases with ω . If z/ω is in some intermediate range, then the buyer can still purchase all the DM goods of the seller but he has to spend all his real balances. In this case, the seller's surplus decreases with ω . Finally, if z/ω is low, then the buyer can only purchase a fraction of the seller's DM goods, and the seller's surplus is constant. As a result, the seller's surplus reaches a maximum when $p \leq z$ starts to bind, i.e., $\omega = 2z / (\varepsilon_h + \varepsilon_\ell)$. The surplus of a buyer in a bilateral match is

$$u^b(z, \omega) = \begin{cases} \frac{\varepsilon_h - \varepsilon_\ell}{2} \omega & \text{if } \frac{z}{\omega} \geq \frac{\varepsilon_h + \varepsilon_\ell}{2} \\ \varepsilon_h \omega - z & \text{if } \frac{z}{\omega} \in \left[\frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}, \frac{\varepsilon_h + \varepsilon_\ell}{2} \right) \\ \frac{(\varepsilon_h - \varepsilon_\ell)z}{2\varepsilon_\ell} & \text{if } \frac{z}{\omega} < \frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}. \end{cases} \quad (48)$$

Let \bar{z} denote the highest value on the support of $F^b(z)$. Then,

$$\omega \leq \min \left\{ \frac{2\bar{z}}{\varepsilon_h + \varepsilon_\ell}, \Omega \right\}. \quad (49)$$

Let $\bar{\omega}$ denote the highest value in the support of $F^s(\omega)$. The solution is such that

$$z \leq \frac{2\varepsilon_h \varepsilon_\ell \bar{\omega}}{\varepsilon_h + \varepsilon_\ell}. \quad (50)$$

It can be checked that $(\varepsilon_h + \varepsilon_\ell) / 2 > 2\varepsilon_h \varepsilon_\ell / (\varepsilon_h + \varepsilon_\ell)$, i.e., the intersection of the two best-response functions, (49) and (50), is such that the only Nash equilibrium is $\bar{z} = \bar{\omega} = 0$.

Under gradual bargaining, the Pareto frontier of the bargaining set, $u^b = \max(\varepsilon_h y - p)$ s.t. $p - \varepsilon_\ell y \geq u^s$, $p \leq z$, and $y \leq \omega$, is given by:

$$\begin{aligned} H(u^b, u^s, z, \omega) &= (\varepsilon_h - \varepsilon_\ell)\omega - u^b - u^s \text{ if } u^s \leq z - \varepsilon_\ell \omega \\ &= \frac{(\varepsilon_h - \varepsilon_\ell)z}{\varepsilon_\ell} - \frac{\varepsilon_h}{\varepsilon_\ell} u^s - u^b \text{ otherwise.} \end{aligned}$$

Hence, the gradual bargaining solution requires

$$u^{b'}(z) = -\frac{1}{2} \frac{\partial H / \partial z}{\partial H / \partial u^b} = \frac{1}{2} \frac{(\varepsilon_h - \varepsilon_\ell)}{\varepsilon_\ell},$$

and, by definition $u^{b'}(z) = \varepsilon_h \partial y / \partial z - 1$. It follows that gradual solution gives $\partial z / \partial y = 2\varepsilon_h \varepsilon_\ell / (\varepsilon_h + \varepsilon_\ell)$.

Integrating this expression, the payment function is $p(y) = \frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell} y$. The buyer's choice of y is given by:

$$\max_{y \in [0, \omega]} \left\{ -i \frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell} y + \alpha \left[\varepsilon_h y - \frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell} y \right] \right\}.$$

It can be re-expressed as:

$$\max_{y \in [0, \omega]} [-i2\varepsilon_\ell + \alpha(\varepsilon_h - \varepsilon_\ell)] y.$$

Provided that $i \leq \alpha(\varepsilon_h - \varepsilon_\ell) / (2\varepsilon_\ell)$, it is optimal to choose $y = \omega$ and to hold $z = p(\omega)$. The surplus of the seller is:

$$\begin{aligned} u^s(\omega, z) &= \min \left\{ p(\omega) - \varepsilon_\ell \omega, z - \frac{(\varepsilon_h + \varepsilon_\ell)}{2\varepsilon_h} z \right\} \\ &= \min \left\{ \varepsilon_\ell \left(\frac{\varepsilon_h - \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell} \right) \omega, \frac{\varepsilon_h - \varepsilon_\ell}{2\varepsilon_h} z \right\}. \end{aligned}$$

The seller's surplus is monotone (weakly) increasing in ω . Hence, $\omega = \Omega$ is a weakly dominant strategy. ■

Before the proof of Proposition 9, we need the following lemma, which characterizes the optimal speed of trade and asset holdings.

Lemma 5 *For each $s \geq 0$, there exists a solution, $[z^e(s), \delta^e(s)]$, to (27). The solution is generally not unique and hence $[z^e(s), \delta^e(s)]$ is a correspondence, which satisfies the following properties:*

- (a) *it is upper-hemi continuous and non-increasing in s , and non-decreasing in α ;*
- (b) *for any two spreads, $0 \leq s^1 < s^2$, and any $z^1 \in z^e(s^1)$ and $z^2 \in z^e(s^2)$, $z^1 \geq z^2$;*
- (c) *if $z_1, z_2 \in z^e(s)$ and $z_1 < z_2$, then for any $s' > s$, $z_2 \notin z^e(s')$, and for any $s'' < s$, $z_1 \notin z^e(s'')$;*
- (d) *for all but countably many s , $z^e(s)$ is a singleton set;*
- (e) *$z^e(0) = \{p(y^*)\}$; as s tends to infinity, the correspondence $[z^e(s), \delta^e(s)]$ converges to a singleton $\{(0, 0)\}$.*

Proof. First, we compute the partial derivatives of the consumer's surplus function:

$$\begin{aligned} \frac{\partial S^b(z^b, \delta)}{\partial z^b} &= e^{-\lambda \frac{z^b}{\delta}} \theta \ell(y) \geq 0, \\ \frac{\partial S^b(z^b, \delta)}{\partial \delta} &= \int_0^y \frac{\lambda}{\delta^2} e^{-\lambda \frac{p(x)}{\delta}} p(x) \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta)v'(x)} dx > 0, \text{ with } y = p^{-1}(z^b). \end{aligned}$$

Moreover, $\partial^2 S^b(z^b, \delta) / \partial z^b \partial \delta > 0$ if $y < y^*$, a fact that we will use later. So there are complementarities between the choice of asset holdings and the speed of negotiation. The first-order condition with respect to δ is then

$$\psi'(\delta) = \int_0^y \frac{\lambda}{\delta^2} e^{-\lambda \frac{p(x)}{\delta}} p(x) \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta)v'(x)} dx. \quad (51)$$

The consumer's surplus is bounded above by $u(y^*) - v(y^*)$. Hence, it is never optimal to choose a δ larger than $\bar{\delta} = \psi^{-1} [u(y^*) - v(y^*)]$. Similarly, for all $s > 0$ it is not optimal to accumulate more than $p(y^*)$ units of assets. Hence, with no loss in generality, we restrict the maximization problem to the compact set, $[0, \bar{\delta}] \times [0, p(y^*)]$. The objective in (27) is continuous. By the Theorem of the Maximum, a solution exists and it is upper hemi-continuous in s . We use $[z^e(s), \delta^e(s)]$ to denote the correspondence for the maximizers. Moreover, when $s = 0$, the solution is unique with $z^e(s) = \{p(y^*)\}$. This proves (a) and first part of (e).

To show generic uniqueness and monotonic comparative statics, consider the consumer problem in two steps. First, for any given $z^b \in [0, p(y^*)]$, consider

$$\bar{S}(z^b) = \max_{\delta \in [0, \bar{\delta}]} \{-\psi(\delta) + S^b(z^b, \delta)\} \quad (52)$$

The objective function has strictly increasing differences in (z^b, δ) since $\partial S^b(z^b, \delta) / \partial z^b$ is strictly increasing in δ . By Theorem 2.8.2 and 2.8.4 in Topkis (1998) $\arg \max_{\delta \in [0, \bar{\delta}]} \{-\psi(\delta) + S^b(z^b, \delta)\}$ is increasing in $z^b < p(y^*)$ and the set of maximizers is increasing in $z^b < p(y^*)$ as well. Now, for any z^b , the corresponding optimal δ solves (51) with $y = p^{-1}(z^b)$. Since the right side of (51) is strictly increasing in $z^b < p(y^*)$, the set of maximizers has to be strictly increasing as well; indeed, if $z_1^b < z_2^b$, and δ_1 and δ_2 are the corresponding maximizers, it must be the case that $\delta_1 \neq \delta_2$ as the same δ cannot satisfy the two FOC's at the same time. Now, if $\bar{S}(z_1^b) = \bar{S}(z_2^b)$, then

$$S^b(z_1^b, \delta_2) - S^b(z_1^b, \delta_1) < \psi(\delta_2) - \psi(\delta_1) = S^b(z_2^b, \delta_2) - S^b(z_1^b, \delta_1) < S^b(z_2^b, \delta_2) - S^b(z_2^b, \delta_1),$$

but the second inequality implies that $S^b(z_1^b, \delta_1) > S^b(z_2^b, \delta_1)$, a contradiction to the fact that $S^b(z^b, \delta_1)$ is increasing in z_b . Thus, $\bar{S}(z_1^b) < \bar{S}(z_2^b)$. Moreover, since $\bar{S}(z^b)$ is strictly increasing, it is also differentiable for all but at most a countably many points.

Thus, $z^e(s)$ consists of solutions to

$$\max_{z \geq 0} -sz + \alpha \bar{S}(z).$$

Now we prove (b) and (c). Property (b) shows that $z^e(s)$ is decreasing; (c) shows that overlapping can happen only at end points. For (b), it follows from $z^1 \in z^e(s^1)$ and $z^2 \in z^e(s^2)$ that:

$$\begin{aligned} -s^1 z^1 + \alpha \bar{S}(z^1) &\geq -s^1 z^2 + \alpha \bar{S}(z^2), \\ -s^2 z^2 + \alpha \bar{S}(z^2) &\geq -s^2 z^1 + \alpha \bar{S}(z^1). \end{aligned}$$

Rearrange these inequalities to obtain:

$$s^1 (z^1 - z^2) \leq \alpha [\bar{S}(z^1) - \bar{S}(z^2)] \leq s^2 (z^1 - z^2).$$

Using $s^2 > s^1$, it follows that $z^1 \geq z^2$. For (c), let s be given, and let $z_1, z_2 \in z^e(s)$ be such that $z_1 < z_2$. Suppose, by contradiction, that $z_2 \in z^e(s')$. Then,

$$\begin{aligned} -sz_1 + \alpha\bar{S}(z_1) &= -sz_2 + \alpha\bar{S}(z_2), \\ -s'z_2 + \alpha\bar{S}(z_2) &\geq -s'z_1 + \alpha\bar{S}(z_1). \end{aligned}$$

It then follows that

$$-s'z_2 + \alpha\bar{S}(z_2) \geq -(s' - s)z_1 - sz_2 + \alpha\bar{S}(z_2),$$

that is,

$$(s' - s)z_1 \geq (s' - s)z_2,$$

a contradiction to $z_1 < z_2$ and $s' > s$. The other case is similar.

To prove (d), if we let $\bar{z}^e(s) = \max z^e(s)$, then $\bar{z}^e(s)$ is a decreasing function, and hence has at most countably many gaps. Property (c) implies that only gaps in $\bar{z}^e(s)$ corresponds to nondegenerate values of $z^e(s)$. This proves (d). Now we prove the second part of (e). This directly follows from envelope theorem which implies that

$$\bar{S}'(z^b) = e^{-\lambda \frac{z^b}{s}} \theta \ell(y) \text{ with } y = p^{-1}(z^b) \text{ whenever it exists,}$$

$e^{-\lambda \frac{z^b}{s}}$ is bounded between $[0, 1]$, and the fact that $\ell'(y)$ is strictly decreasing in y and $\ell'(0) = \infty$. ■

Proof of Proposition 9. (1) By the Theorem of Maximum, $z^e(s)$ is upper hemi-continuous and compact-valued. To prove equilibrium existence, we need to convexify the correspondence $z^e(s)$ and thereby allow for asymmetric equilibria. To do so, we let $\underline{z}^e(s) = \min z^e(s)$ and let $\bar{z}^e(s) = \max z^e(s)$, and, for each s , consider the convex hull, $Z(s) = [\underline{z}^e(s), \bar{z}^e(s)]$, of $z^e(s)$. For any $z \in Z(s)$, we can interpret z as the aggregate optimal real balances from the consumers. The correspondence $Z(s)$ is then upper hemi-continuous, compact-valued, and convex-valued. Moreover, by Lemma 5 (c) we know that if $s > s'$, then $\underline{z}^e(s') \geq \bar{z}^e(s)$. Now, to define market-clearing in a unified way, we redefine $Z(0) = [p(y^*), \max\{\frac{1+\rho}{\rho} Ad, p(y^*)\}]$. We have an equilibrium iff we can find a spread s that satisfies the market clearing condition

$$\frac{1+\rho}{\rho-s} Ad \in Z(s) \tag{53}$$

for some $s \geq 0$. The properties on $Z(s)$ and the fact that $\bar{z}^e(0) \geq \frac{1+\rho}{\rho} Ad$ and $\underline{z}^e(\rho) < \infty = \lim_{s \rightarrow \rho} \frac{1+\rho}{\rho-s} Ad$ ensures that (53) is satisfied by some $s^* \in [0, \rho)$. Such s^* is unique since any selection of $Z(s)$ is decreasing and $\frac{1+\rho}{\rho-s} Ad$ is strictly increasing in s .

(2) The planner's problem solves:

$$\max_{z, \delta, s} \left\{ -\psi(\delta) + \int_0^y e^{-\lambda \frac{p(x)}{\delta}} [u'(x) - v'(x)] dx \right\} \quad (54)$$

$$\text{s.t. } z \in \arg \max_z \{ -sz + \alpha S^b(z, \delta) \} \quad (55)$$

$$p(y) \leq \left(\frac{1+\rho}{\rho-s} \right) Ad, \quad \text{"=" if } s > 0 \quad (56)$$

According to (54) the planner maximizes the expected surplus of each match net of the negotiability cost. It is subject to (55) according to which consumers choose their asset holdings optimally taking as given the negotiability of the asset and its cost (which is omitted from the consumer's objective). From (56) the spread, s , is consistent with market clearing.

From (1), if $Ad \geq \rho p(y^*)/(1+\rho)$ then equilibrium is such that $s = 0$ and $y = y^*$ irrespective of δ . Hence, the solution to (54) is

$$\psi'(\delta) = \int_0^{y^*} \frac{\lambda}{\delta^2} p(x) e^{-\lambda \frac{p(x)}{\delta}} [u'(x) - v'(x)] dx.$$

It coincides with (51) if and only if $\theta = 1$. If $\theta < 1$ then the decentralized choice of δ is smaller than the planner's choice. For the case $Ad < \rho p(y^*)/(1+\rho)$, we proved in Proposition 7 that s increases with δ . From market clearing $p(y) = \left(\frac{1+\rho}{\rho-s} \right) Ad$, and hence y is an increasing function of s . Hence, the solution to the planner's problem is:

$$\psi'(\delta) = \int_0^y \frac{\lambda}{\delta^2} p(x) e^{-\lambda \frac{p(x)}{\delta}} [u'(x) - v'(x)] dx + e^{-\lambda \frac{p(y)}{\delta}} [u'(y) - v'(y)] \frac{\partial y}{\partial \delta}.$$

The second term on the right side captures the effect of an increase of negotiability on the spread and hence y . Even if $\theta = 1$ this condition does not coincide with (51). ■

Derivation of (31). Note that by (29) and (30),

$$\begin{aligned} u'[y(\tau)] y'(\tau) - p'(\tau) &= u'[y(\tau)] \frac{\theta u'[y(\tau)] + (1-\theta) v'[y(\tau)]}{u'[y(\tau)] v'[y(\tau)]} \omega'(\tau) - \omega'(\tau) \\ &= \ell[y(\tau)] \omega'(\tau), \end{aligned}$$

and hence

$$\begin{aligned} S(\mathbf{a}, \boldsymbol{\sigma}, \bar{\tau}) &= \theta \int_0^{\bar{\tau}} \ell[y(\tau)] \omega'(\tau) d\tau = \theta \int_0^{\bar{\tau}} \ell[y(\tau)] \omega'(\tau) d\tau \\ &= \theta \int_0^{\omega(\bar{\tau})} \ell[y(\omega)] d\omega, \end{aligned}$$

where

$$\begin{aligned} y(\omega) &= p^{-1}(\omega) \text{ if } \omega \leq p(y^*) \\ y(\omega) &= y^* \text{ otherwise,} \end{aligned} \quad (57)$$

with p given by (16). ■

Proof of Lemma 3. By (31), an optimal $[\sigma_j]_{j=0}^J$ maximizes

$$\omega(\bar{\tau}) = \sum_{j=0}^J \int_0^{\bar{\tau}} \delta_j \sigma_j(x) dx = \sum_{j=0}^J \delta_j \Delta_j,$$

where $\Delta_j \equiv \int_0^{\bar{\tau}} \sigma_j(x) dx$, subject to feasibility. We can then rewrite this problem as

$$\max_{\Delta_j, j=0, \dots, J} \sum_{j=0}^J \delta_j \Delta_j, \text{ subject to } \sum_{j=0}^J \Delta_j = \bar{\tau} \text{ and } 0 \leq \Delta_j \leq a_j / \delta_j \text{ for all } j = 0, \dots, J,$$

where the constraints follow from feasibility requirement on $[\sigma_j]_{j=0}^J$. Now, let $\tilde{j} \geq 0$ satisfy

$$\sum_{j=0}^{\tilde{j}-1} a_j / \delta_j < \bar{\tau} \leq \sum_{j=0}^{\tilde{j}} a_j / \delta_j.$$

Since $\delta_0 \geq \delta_1 \geq \dots \geq \delta_J$, it is optimal to choose $\Delta_j = a_j / \delta_j$ for all $j = 0, \dots, \tilde{j} - 1$, $\Delta_{\tilde{j}} = \bar{\tau} - \sum_{j=0}^{\tilde{j}-1} a_j / \delta_j$, and $\Delta_j = 0$ for all $j > \tilde{j}$. Hence, $[\sigma_j^*]_{j=0}^J$ restricted to $[0, \bar{\tau}]$ is optimal. It is also uniquely optimal if $\delta_0 > \delta_1 > \dots > \delta_J$. ■

Proof of Lemma 4. Define $\Omega_j(\mathbf{a}) = \sum_{k=0}^{j-1} a_k$ for all $j = 1, \dots, J+1$ with $\Omega_0(\mathbf{a}) = 0$. We can then rewrite (33) as

$$S(\mathbf{a}) = \theta \sum_{j=0}^J \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[x(\omega)] d\omega,$$

where $x(\omega)$ is defined in (57) and we have changed the variable from τ to $\omega = \omega^*(\tau)$; note that for all $\omega \in (\Omega_j, \Omega_{j+1})$,

$$\begin{aligned} (\omega^*)^{-1}(\omega) &= \frac{(\omega - \Omega_j)}{\delta_j} + T_j, \\ \frac{d}{d\omega} (\omega^*)^{-1}(\omega) &= \frac{1}{\delta_j}. \end{aligned}$$

Now, let $k \geq 0$ be given. We shall compute the derivative of $S(\mathbf{a})$ w.r.t. a_k . We will compute it by grouping the terms inside the summation into three groups: terms with $j < k$, the term with $j = k$, and terms with $j > k$. Note that $S(\mathbf{a})$ depends on a_k through terms Ω_j and T_j with $j > k$ and hence, for $j < k$,

$$\frac{\partial}{\partial a_k} \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[y(\omega)] d\omega = 0,$$

for $j = k$,

$$\frac{\partial}{\partial a_k} \int_{\Omega_k}^{\Omega_{k+1}} e^{-\lambda \left[\frac{(\omega - \Omega_k)}{\delta_k} + T_k \right]} \ell[y(\omega)] d\omega = -e^{-\lambda T_k} \ell[y(\Omega_k)],$$

and for $j > k$,

$$\begin{aligned}
& \frac{\partial}{\partial a_k} \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[y(\omega)] d\omega \\
&= -e^{-\lambda T_j} \ell[y(\Omega_j)] + e^{-\lambda \left[\frac{(\Omega_{j+1} - \Omega_j)}{\delta_j} + T_j \right]} \ell[y(\Omega_{j+1})] + \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k} \right] e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[y(\omega)] d\omega \\
&= -e^{-\lambda T_j} \ell[y(\Omega_j)] + e^{-\lambda T_{j+1}} \ell[y(\Omega_{j+1})] + \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k} \right] e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[y(\omega)] d\omega.
\end{aligned}$$

Thus, adding the terms up across j , we obtain

$$\frac{\partial}{\partial a_k} S(\mathbf{a}) = \theta \sum_{j=k+1}^J \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k} \right] e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[y(\omega)] d\omega + e^{-\lambda T_{J+1}} \ell[y(\Omega_{J+1})],$$

where the terms $e^{-\lambda T_j} \ell[y(\Omega_j)]$ cancel one another except for the very last one. Equation (36) is obtained by another change of variable back to τ . ■

Proof of Proposition 10. (1) The equilibrium is solved recursively. The FOC (35) when $j = 0$ determines a_0 , which is equivalent to the following (again, by a change of variable):

$$i = \theta \sum_{j=1}^J \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_0} \right] e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[y(\omega)] d\omega + e^{-\lambda T_{J+1}} \ell[y(\Omega_{J+1})], \quad (58)$$

where $\Omega_j = \sum_{k=0}^{j-1} A_k$ by market clearing condition with $A_0 = a_0$. First we show that the right side of (58) is strictly decreasing in a_0 . Note that Ω_j is strictly increasing in a_0 , the range for $\frac{(\omega - \Omega_j)}{\delta_j} + T_j$ as a function of ω from Ω_j to Ω_{j+1} does not change when one changes a_0 , but $\ell[y(\omega)]$ is strictly decreasing in ω until it hits zero and stays there. Thus, the first term is strictly decreasing in a_0 . For the second term, note that both T_{J+1} and Ω_{J+1} are strictly increasing in a_0 but the term is strictly decreasing in both T_{J+1} and Ω_{J+1} . Now, the right side of (58) is also strictly positive at $a_0 = 0$ provided that $\delta_0 > \delta_1$ and equal to 0 as a_0 goes to ∞ . The threshold for the nominal interest rate below which a monetary equilibrium exists is

$$\bar{i} = \alpha \theta \lambda \sum_{k=1}^J \frac{(\delta_0 - \delta_k)}{\delta_0} \int_{T_k}^{T_{k+1}} e^{-\lambda \tau} \ell[y(\tau)] d\tau + \alpha \theta e^{-\lambda T_{J+1}} \ell[y(T_{J+1})],$$

where $T_1 = 0$, and $T_j = \sum_{k=1}^{j-1} A_k / \delta_k$ for all $j \in \{2, \dots, J+1\}$. Given a_0 , the spreads $\{s_j\}_{j=1}^J$ are determined by (35), with $A_0 = a_0$ and $T_j = \sum_{k=0}^{j-1} A_k / \delta_k$ for all $j \in \{1, \dots, J+1\}$. From (35) we can compute the difference between two consecutive spreads:

$$s_j - s_{j+1} = \alpha \theta \lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} \int_{T_{j+1}}^{T_{j+2}} e^{-\lambda \tau} \ell[y(\tau)] d\tau.$$

Hence, $s_j - s_{j+1} > 0$ requires $\delta_j - \delta_{j+1} > 0$ and $y(T_{j+1}) < y^*$, i.e., $\Omega_{j+1} = \sum_{k=0}^j A_k < p(y^*)$.

(2) We can simplify the expression of the velocity of asset j given by (37) as

$$\begin{aligned}
\mathcal{V}_j &\equiv \frac{\alpha \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \omega_j^{*'}(\tau) 1_{\{\omega^*(\tau) < p(y^*)\}} d\tau dx}{A_j} = \frac{\alpha \int_0^{+\infty} e^{-\lambda \tau} \omega_j^{*'}(\tau) 1_{\{\omega^*(\tau) < p(y^*)\}} d\tau}{A_j} \\
&= \frac{\alpha \int_{T_j}^{T_{j+1}} e^{-\lambda \tau} \delta_j 1_{\{\omega^*(\tau) < p(y^*)\}} d\tau}{A_j},
\end{aligned}$$

where the first equality changes the order of integration and the second uses the fact that $\omega_j^{*'}(\tau) = \delta_j \mathbf{1}_{\{T_j \leq \tau < T_{j+1}\}}$. Using the expressions for T_j and T_{j+1} we distinguish three cases:

$$\mathcal{V}_j = \begin{cases} A_j^{-1} \lambda^{-1} \alpha \delta_j e^{-\lambda T_j} \left(1 - e^{-\frac{\lambda}{\delta_j} A_j}\right) \\ A_j^{-1} \lambda^{-1} \alpha \delta_j e^{-\lambda T_j} \left[1 - e^{-\frac{\lambda}{\delta_j} [p(y^*) - \Omega_j]}\right] \\ 0 \end{cases} \quad \text{if } p(y^*) \begin{cases} \geq \Omega_{j+1} \\ \in (\Omega_j, \Omega_{j+1}) \\ \leq \Omega_j \end{cases} .$$

Thus, $\mathcal{V}_j > 0$ if and only if $p(y^*) > \Omega_j$. Moreover, for any j with $p(y^*) > \Omega_j$,

$$\begin{aligned} \mathcal{V}_j - \mathcal{V}_{j+1} &\geq A_j^{-1} \lambda^{-1} \alpha \delta_j e^{-\lambda T_j} \left(1 - e^{-\frac{\lambda}{\delta_j} A_j}\right) - A_{j+1}^{-1} \lambda^{-1} \alpha \delta_{j+1} e^{-\lambda T_{j+1}} \left(1 - e^{-\frac{\lambda}{\delta_{j+1}} A_{j+1}}\right) \\ &= \alpha e^{-\lambda T_{j+1}} \left[\delta_j A_j^{-1} \lambda^{-1} \left(e^{\frac{\lambda}{\delta_j} A_j} - 1\right) - \delta_{j+1} A_{j+1}^{-1} \lambda^{-1} \left(1 - e^{-\frac{\lambda}{\delta_{j+1}} A_{j+1}}\right) \right] > 0, \end{aligned}$$

where the inequality follows from the fact that

$$\frac{e^{\frac{\lambda}{\delta_j} A_j} - 1}{\frac{\lambda}{\delta_j} A_j} > 1 > \frac{1 - e^{-\frac{\lambda}{\delta_{j+1}} A_{j+1}}}{\frac{\lambda}{\delta_{j+1}} A_{j+1}}.$$

(3) It follows directly from (35) and the fact that:

$$\begin{aligned} |s_j - s_{j+1}| &= \alpha \theta \lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} \int_{T_{j+1}}^{T_{j+2}} e^{-\lambda \tau} \left[\frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right] d\tau \\ &\leq \alpha \theta \lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} e^{-\lambda T_{j+1}} \left[\frac{u'[y(T_{j+1})] - v'[y(T_{j+1})]}{v'[y(T_{j+1})]} \right] d\tau, \end{aligned} \quad (59)$$

which converges to zero as $\lambda \rightarrow \infty$. ■

Proof of Proposition 11. Since we focus on equilibria with $\bar{\tau} = T_2$, the consumer's portfolio problem can be written as

$$\max_{(a_0, a_1)} -i a_0 - s_1 a_1 + \alpha \{u[y(a_0, a_1)] - p[y(a_0, a_1)]\}, \quad (60)$$

where $y(a_0, a_1)$ is given by (38), and, by $\bar{\tau} = T_2$,

$$y(a_0, a_1) = p^{-1} [a_0 (1 - \delta_1/\delta_0) + \delta_1 \bar{\tau}], \quad (61)$$

$$a_1 = \delta_1 \bar{\tau} - \delta_1 a_0 / \delta_0. \quad (62)$$

Thus, the problem in (60) is reduced to choice of a_0 with the following FOC:

$$-i + s_1 \frac{\delta_1}{\delta_0} + \alpha \theta \ell(y) \left(1 - \frac{\delta_1}{\delta_0}\right) \leq 0 \quad (\text{with equality if } a_0 > 0). \quad (63)$$

Now, market clearing requires $a_1 = A_1$, and we look for an equilibrium with $a_0 > 0$ and $s_1 \in (0, i)$. Under these conditions, it follows from (62) and then (61) that

$$a_0 = \left(\bar{\tau} - \frac{A_1}{\delta_1}\right) \delta_0, \quad (64)$$

$$y = p^{-1} [a_0 (1 - \delta_1/\delta_0) + \delta_1 \bar{\tau}] = p^{-1} \left(\delta_0 \bar{\tau} - \frac{\delta_0 - \delta_1}{\delta_1} A_1\right). \quad (65)$$

Moreover, the FOC (63) with equality implies that

$$s_1 = \frac{\delta_0}{\delta_1} i - \alpha \theta \left(\frac{\delta_0 - \delta_1}{\delta_1} \right) \ell(y). \quad (66)$$

Given the assumption

$$\frac{\delta_1}{\delta_0 - \delta_1} [\delta_0 \bar{\tau} - p(y_2)] < A_1 < \min \left\{ \frac{\delta_1}{\delta_0 - \delta_1} [\delta_0 \bar{\tau} - p(y_1)], \delta_1 \bar{\tau} \right\}, \quad (67)$$

we show that $a_0 > 0$ and $s_1 \in (0, i)$. First, since by (67), $A_1 < \delta_1 \bar{\tau}$, (64) implies that $a_0 > 0$. Given (66), $s_1 \in (0, i)$ if and only if

$$0 < \frac{\delta_0}{\delta_1} i - \alpha \theta \left(\frac{\delta_0 - \delta_1}{\delta_1} \right) \ell(y) < i,$$

that is

$$\begin{aligned} i - \alpha \theta \left(\frac{\delta_0 - \delta_1}{\delta_0} \right) \ell(y) &> 0, \\ i - \alpha \theta \ell(y) &< 0. \end{aligned}$$

Recall that y_1 satisfies $i = \alpha \theta [(\delta_0 - \delta_1)/\delta_0] \ell(y_1)$ and y_2 satisfies $i = \alpha \theta \ell(y_2)$, and hence these inequalities are equivalent to $y_1 < y < y_2$. Now, by (65), $y > y_1$ if and only if

$$\bar{\tau} > \frac{p(y_1) - A_1}{\delta_0} + \frac{A_1}{\delta_1},$$

which is guaranteed by the second inequality in (67). Similarly, by (65), $y < y_2$ if and only if

$$\bar{\tau} < \frac{p(y_2) - A_1}{\delta_0} + \frac{A_1}{\delta_1},$$

which is guaranteed by the first inequality of (67).

Finally, since $i = (1 + \pi - \beta)/\beta$, $\frac{\partial}{\partial \pi} i = 1/\beta \approx 1$ when β is close to one (or, equivalently, when ρ is close to zero). Moreover, for s_1 given by (66),

$$\frac{\partial}{\partial \pi} s_1 = \frac{\partial}{\partial \pi} i \frac{\partial}{\partial i} s_1 = (1/\beta) \frac{\delta_0}{\delta_1}.$$

Now, since $i_1 = (i - s_1)/(1 + s_1)$ and since s_1 is close to zero as ρ is close to zero and π is close to zero,

$$\frac{\partial}{\partial \pi} i_1 = \frac{(1 + s_1) \frac{\partial}{\partial \pi} i - \frac{\partial}{\partial \pi} s_1 (1 + i)}{(1 + s_1)^2} \approx 1 - \frac{\delta_0}{\delta_1}.$$

■

Proof of Proposition 12. (1) Assume $i_0 = i_1 = i$. The two FOCs (41) and (42) hold at equality if and only if $\mu = 0$. Hence, y solves $u'(y)/v'(y) = 1 + i/\alpha \theta$. The negotiability constraint is slack if

$$\frac{e_t \phi_{1,t} A_{0,t}}{\delta_0} + \frac{\phi_{1,t} A_{1,t}}{\delta_1} \leq \bar{\tau},$$

where we used that $a_0 = \phi_0 A_0$ and $a_1 = \phi_1 A_1$ by market clearing. Moreover, the outcome of the negotiation is

$$e_t \phi_{1,t} A_{0,t} + \phi_{1,t} A_{1,t} = p(y).$$

Solving for $\phi_{1,t} = p(y)/(e_t A_{0,t} + A_{1,t})$ and substituting into the negotiability constraint we obtain:

$$\frac{p(y)}{e_t A_{0,t} + A_{1,t}} \left(\frac{e_t A_{0,t}}{\delta_0} + \frac{A_{1,t}}{\delta_1} \right) \leq \bar{\tau}.$$

We rearrange the inequality to obtain:

$$\frac{A_{1,t}}{e_t A_{0,t} + A_{1,t}} \leq \frac{\delta_1}{\delta_0 - \delta_1} \frac{\delta_0 \bar{\tau} - p(y)}{p(y)}.$$

There exists a $e_t > 0$ such that this inequality holds iff $\delta_0 \bar{\tau} - p(y) > 0$, and this is the necessary and sufficient condition for both currency to be valued in equilibrium. Moreover, given that the left side is decreasing in e_t , if the inequality holds for $e_t = 0$, then it holds for all $e_t > 0$. This is the case if $p(y) \leq \delta_1 \bar{\tau}$. If $p(y) > \delta_1 \bar{\tau}$ then there is a lowest value for e_t consistent with the inequality. This value \underline{e} is such that the inequality holds at equality.

(2) We have seen that for both currencies to be valued we need (44), which determines y . Equilibrium then requires $a_0/\delta_0 + a_1/\delta_1 = \bar{\tau}$ and $p(y) = a_0 + a_1$, which determine a_0 and a_1 :

$$\begin{aligned} a_1 &= \frac{\delta_1}{\delta_0 - \delta_1} [\delta_0 \bar{\tau} - p(y)] \\ a_0 &= \frac{\delta_0}{\delta_0 - \delta_1} [p(y) - \delta_1 \bar{\tau}]. \end{aligned}$$

Thus, to have both $a_0 > 0$ and $a_1 > 0$, it is necessary and sufficient that

$$\delta_1 \bar{\tau} < p(y) < \delta_0 \bar{\tau}.$$

This condition can be rewritten as $\bar{\tau} \in (\bar{\tau}_0, \bar{\tau}_1)$ where $\bar{\tau}_0 = p(y)/\delta_0$, $\bar{\tau}_1 = p(y)/\delta_1$. It is immediate from (44) that $\partial y/\partial i_0 < 0$ and $\partial y/\partial i_1 > 0$. Similarly, $\partial y/\partial \alpha > 0$ and $\partial y/\partial \theta > 0$ which from (45) gives $\partial e_t/\partial \alpha > 0$ and $\partial e_t/\partial \theta > 0$. ■

Appendix B: Proof of Propositions 1 and 5, and Extensions

As assumed in the main text, the number of bargaining rounds, N , is even, and the producer is the first to make an offer while the consumer is the last. We obtain essentially the same results for the other cases (either N is odd or the producer is making the last offer), as discussed in the proof. Here we also normalize $u_0^b = u_0^s = 0$. Also, with no loss of generality, we normalize δ to be one.

We define intermediate payoffs as the utilities that the players would enjoy based on the agreements reached up to some round $n \in \{1, \dots, N\}$. Let (y_n, p_n) denote the cumulative offers that are agreed upon up to round n . Feasibility requires $0 \leq p_n - p_{n-1} \leq z/N$ and $0 \leq y_n - y_{n-1}$ for all $n = 1, \dots, N$ and $p_0 = y_0 = 0$. From (3) and (4), we have $u_n^b = u(y_n) - p_n$ and $u_n^s = -v(y_n) + p_n$. The payoffs over terminal histories are simply u_N^b and u_N^s . If we restrict $y \in [0, y^*]$, then there is a one-to-one correspondence between the intermediate allocation (y, p) and the intermediate payoff (u^b, u^s) such that $H(u^b, u^s, p) = 0$.

The rest of the section consists in proving Proposition 1 followed by two extensions: one with explicit negotiation time limit and the other with asymmetric bargaining powers, and we give the proof of Proposition 5 in the end. The proof contains four parts: the first gives a full characterization of the equilibrium payoffs of any subgame; the second gives equilibrium intermediate payoffs; the third proves uniqueness; the fourth characterizes the solution as N goes to infinity.

Final equilibrium payoffs

To solve the game, we need to solve all possible subgames. A subgame is characterized by the intermediate payoffs, denoted by (u_0^b, u_0^s) with the corresponding allocation denoted by (y_0, p_0) , and the number of rounds remaining for bargaining, denoted by J . That is, the subgame begins at round $N - J + 1$, with the intermediate payoff (u_0^b, u_0^s) that results from the bargaining in the first $N - J$ rounds. (The entire game has $(u_0^b, u_0^s) = (0, 0)$ and $J = N$.) Feasibility requires $p_0 \leq (N - J)z/N$, and we only consider $y_0 < y^*$ so that there are still gains from trade to be exploited. Our first lemma describes the final payoffs of such a game. Let $S(y) = u(y) - v(y)$ and $S^* = S(y^*)$.

Lemma 6 *Consider a game $[(u_0^b, u_0^s), J]$ with $0 \leq u_0^b + u_0^s < S^*$, and $p_0 = u[S^{-1}(u_0^b + u_0^s)] - u_0^b = u_0^s + v[S^{-1}(u_0^b + u_0^s)]$. Equilibrium final payoffs, $(\tilde{u}_J^b, \tilde{u}_J^s)$, correspond to the last term of the sequence, $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$, defined as $(\tilde{u}_0^b, \tilde{u}_0^s) = (u_0^b, u_0^s)$,*

$$H(\tilde{u}_j^b, \tilde{u}_{j-1}^s, p_0 + jz/N) = 0 \text{ and } \tilde{u}_j^s = \tilde{u}_{j-1}^s, \text{ for } j \geq 1 \text{ odd}, \quad (68)$$

$$H(\tilde{u}_{j-1}^b, \tilde{u}_j^s, p_0 + jz/N) = 0 \text{ and } \tilde{u}_j^b = \tilde{u}_{j-1}^b, \text{ for } j \geq 2 \text{ even}. \quad (69)$$

The proof of Lemma 6 uses backward induction. When $J = 1$, the game $[(u_0^b, u_0^s), 1]$ is a standard take-it-or-leave-it offer game (with the consumer making the offer). In equilibrium, the consumer makes an offer that leaves the producer indifferent between rejecting or accepting, with the final payoff to the producer $\tilde{u}_1^s = u_0^s$. Taking this as given, the consumer spends up to z/N units of assets so that his final payoff \tilde{u}_1^b satisfies $H(\tilde{u}_1^b, u_0^s, p_0 + z/N) = 0$. (Note that the buyer will spend exactly z/N unless y^* is achieved with a slack liquidity constraint.) This proves (68) with $J = 1$.

Now consider $J = 2$, and the producer makes the first offer. If the consumer rejects the offer, the subgame becomes $[(u_0^b, u_0^s), 1]$, and the consumer can guarantee himself a final payoff of \tilde{u}_1^b , which we call the consumer's *reservation payoff*. Take this as given, the producer's offer is acceptable as long as the offer leads to a consumer final payoff no less than \tilde{u}_1^b . Thus, the producer's offer maximizes his final payoff, u_2^s , subject to $u_2^b \geq \tilde{u}_1^b$. Equivalently, the producer final payoff \tilde{u}_2^s solves $H(\tilde{u}_1^b, \tilde{u}_2^s, p_0 + 2z/N) = 0$. This proves (69) with $J = 2$. We illustrate this logic in Figure 14.

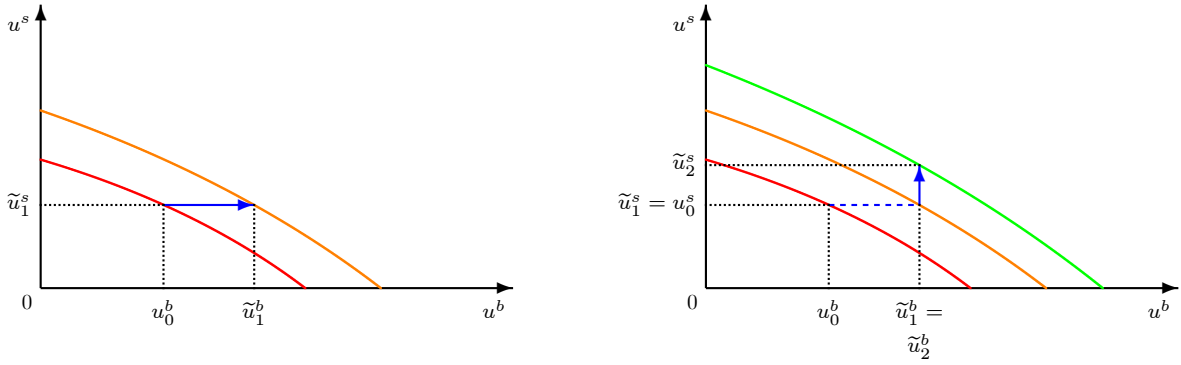


Figure 14: Construction of \tilde{u}_1^b and \tilde{u}_2^s

We continue this argument by induction. Suppose that the final payoffs are given by (68) and (69) for any game $[(u_0^b, u_0^s), J - 1]$ with $J \geq 3$ and consider a game $[(u_0^b, u_0^s), J]$ with J odd and the consumer is making the first offer. If the producer rejects the offer, his reservation payoff would be \tilde{u}_{J-1}^s . Following the same logic, the consumer's offer maximizes his final payoff u_b^J subject to the constraint that the producer's final payoff is no less than his reservation payoff, \tilde{u}_{J-1}^s . Thus, the final payoffs in the game $[(u_0^b, u_0^s), J]$, denoted by $(\tilde{u}_J^b, \tilde{u}_J^s)$, solve $H(\tilde{u}_J^b, \tilde{u}_{J-1}^s, p_0 + Jz/N) = 0$ and $\tilde{u}_J^s = \tilde{u}_{J-1}^s$. The case for J even is similar. This proves (68) and (69) for J .

Before we proceed, we give some comments on how to handle the case when the first best is reached at some point of the game. Once we reach y^* , that is, once $\tilde{u}_j^b + \tilde{u}_j^s = u(y^*) - v(y^*)$, the sequence $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$ is constant afterwards and in equilibrium there is no trade in rounds after j . Note that this is consistent with our definition of simple SPE. Thus, we may only consider the case where

$$\tilde{u}_{J-1}^b + \tilde{u}_{J-1}^s < S^*. \quad (70)$$

Equilibrium Intermediate Payoffs

We now construct the sequence of intermediate payoffs (and the corresponding allocations and offers) that will lead to final payoffs. We emphasize that the sequence $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$ used to construct the final payoffs

is distinct from the sequence of intermediate payoffs, as we will illustrate shortly. To do so, we expand the notation slightly to explicate the recursive nature of the sequence $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$. As mentioned, at each step according to (68)-(69), the next payoff is computed by either a rightward or upward shift to the next Pareto frontier. Formally, we define two operators, $F_r(u^b, u^s)$ and $F_u(u^b, u^s)$ given by

$$F_r(u^b, u^s) = (u^{b'}, u^{s'}) \text{ such that } u^{s'} = u^s \text{ and } H(u^{b'}, u^s, p + z/N) = 0, \quad (71)$$

$$F_u(u^b, u^s) = (u^{b'}, u^{s'}) \text{ such that } u^{b'} = u^b \text{ and } H(u^b, u^{s'}, p + z/N) = 0, \quad (72)$$

where $p = u[S^{-1}(u^b + u^s)] - u^b$. The operator $F_r(u^b, u^s)$ moves from (u^b, u^s) to the next Pareto frontier by a rightward shift, and $F_u(u^b, u^s)$ moves upward. It then follows directly from (68) and (69) that, for all j even,

$$(\tilde{u}_{j+1}^b, \tilde{u}_{j+1}^s) = F_r(\tilde{u}_j^b, \tilde{u}_j^s), \quad (73)$$

$$(\tilde{u}_{j+2}^b, \tilde{u}_{j+2}^s) = F_u(\tilde{u}_{j+1}^b, \tilde{u}_{j+1}^s) = (F_u \circ F_r)(\tilde{u}_j^b, \tilde{u}_j^s). \quad (74)$$

Our construction of equilibrium intermediate payoffs follows backward induction from the final payoffs constructed in Lemma 6. Consider a game $[(u_0^b, u_0^s), J]$ with J even. Lemma 6 shows that the final payoffs to the agents are given by $(\tilde{u}_J^b, \tilde{u}_J^s)$. Let $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ denote the equilibrium intermediate payoff for the agents at the end of round- $(J-1)$ bargaining. Applying Lemma 6 to the game $[(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s), 1]$, the equilibrium payoff to that game is given by $F_r(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$. Thus, subgame perfection requires

$$F_r(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s) = (\tilde{u}_J^b, \tilde{u}_J^s). \quad (75)$$

The solution to (75) is to move from $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ leftward to the next Pareto frontier: formally, it is given by

$$H[\hat{u}_{J-1}^b, \hat{u}_{J-1}^s, p_0 + (J-1)z/N] = 0, \quad \hat{u}_{J-1}^s = \tilde{u}_J^s. \quad (76)$$

In general, the same argument shows that the equilibrium intermediate payoff at the end of round- $(J-j)$ bargaining, denoted by $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)$, must satisfy

$$\begin{aligned} (F_u \circ F_r)^{j/2}(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s) &= (\tilde{u}_J^b, \tilde{u}_J^s) \text{ for } j \text{ even,} \\ F_r[(F_u \circ F_r)^{(j-1)/2}(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)] &= (\tilde{u}_J^b, \tilde{u}_J^s) \text{ for } j \text{ odd.} \end{aligned} \quad (77)$$

According to (77), for j even, if we start with $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)$, it should reach the final payoffs, $(\tilde{u}_J^b, \tilde{u}_J^s)$, by $j/2$ rightward and upward shifts to next Pareto frontiers. Now, by repeated use of (74), we have that $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s) = (\tilde{u}_{J-j}^b, \tilde{u}_{J-j}^s)$ for all j even. For j odd, if we start with $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)$, it should reach the final payoffs, $(\tilde{u}_J^b, \tilde{u}_J^s)$, by $(j-1)/2$ rightward and upward shifts to next Pareto frontiers, plus one more rightward

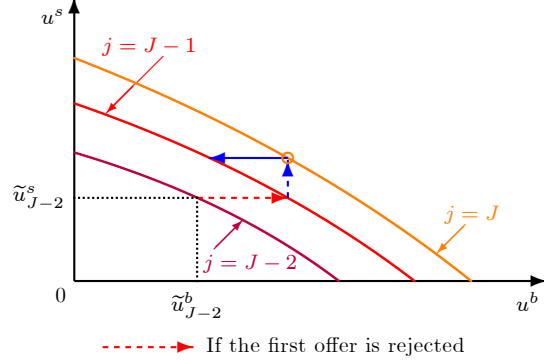


Figure 15: Backward induction

shift. Hence, $(\tilde{u}_{j-2}^b, \tilde{u}_{j-2}^s)$ can be obtained from $(\tilde{u}_j^b, \tilde{u}_j^s)$ by first a leftward shift to the previous Pareto frontier, followed by $(j-1)/2$ downward and leftward shifts to previous frontiers. Figure 15 illustrates this process for $j=2$. We have the following lemma.

Lemma 7 Consider a game $[(u_0^b, u_0^s), J]$ be given with J even that satisfies (70). There is a unique sequence, $\{(\hat{u}_{j-1}^b, \hat{u}_{j-1}^s)\}_{j=0}^{J-1}$, with corresponding sequence of allocation denoted by $\{\hat{y}_{j-1}\}_{j=0}^{J-1}$, possibly except for $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$, that satisfies (77), which also enjoys the following properties:

$$\hat{u}_{j-1}^b > \hat{u}_{j-2}^b \text{ for all } j = 0, \dots, J-2; \hat{u}_1^b > u_0^b; \quad (78)$$

$$\hat{u}_{j-1}^s > \hat{u}_{j-2}^s \text{ for all } j = 1, \dots, J-2; \hat{u}_1^s > u_0^s; \quad (79)$$

$$\hat{y}_j > \hat{y}_{j-1} \text{ for all } j = 2, \dots, J; \hat{y}_1 > y_0. \quad (80)$$

The proof of Lemma 7 is based on induction on j and uses the fact that $u(y) - v(y)$ is strictly concave. The proof is rather straightforward but tedious and the detailed proof is available upon request. Moreover, since $(\hat{u}_{j-1}^b, \hat{u}_{j-1}^s) = (\tilde{u}_{j-1}^b, \tilde{u}_{j-1}^s)$ for all j even, (78) and (79) imply that the two sequences, $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=1}^{J-1}$ and $\{(\hat{u}_{j-1}^b, \hat{u}_{j-1}^s)\}_{j=1}^{J-1}$ in fact nests one another, and hence, if one sequence converges to some limit, the other also converges to the same limit. We also remark that while we have assumed J to be even, an analogous lemma for J odd holds as well. In that case, $(\hat{u}_{j-1}^b, \hat{u}_{j-1}^s) = (\tilde{u}_{j-1}^b, \tilde{u}_{j-1}^s)$ for all j odd, but we need to compute $(\hat{u}_{j-1}^b, \hat{u}_{j-1}^s)$ for j even with an alternative sequence analogous to the one we constructed for the case with J even and j odd.

Uniqueness of SPE

Here we prove our uniqueness claim. First we show that, for any subgame, $[(u_0^b, u_0^s), J]$, the equilibrium final payoffs in any SPE is given by (68)-(69), denoted by $(\tilde{u}_j^b, \tilde{u}_j^s)$. For $J=1$ this is the standard ultimatum game

and the uniqueness is standard. Suppose that we have uniqueness for $J - 1$, $J \geq 2$. Then, fix a SPE and consider the game at first bargaining round, and, without loss of generality, assume that producer is making an offer and J is even. We show that the consumer can guarantee a final payoff of \tilde{u}_J^b and the producer can guarantee \tilde{u}_J^s at the first round. First, by rejecting the producer offer, by the induction hypothesis, the unique equilibrium payoff to the consumer is $\tilde{u}_{J-1}^b = \tilde{u}_J^b$, and hence any offer that leads to a final payoff lower than \tilde{u}_J^b will be rejected. For the consumer, Lemma 7 shows that there exists a unique intermediate payoff, (u_1^b, u_1^s) , such that $F_r \circ (F_u \circ F_r)^{(J-2)/2}(u_1^b, u_1^s) = (\tilde{u}_J^b, \tilde{u}_J^s)$, and such intermediate payoff is achievable with some offer (y_1, d_1) . By offering $(y_1 + \varepsilon, d_1)$ for ε small the producer can guarantee consumer acceptance and hence, taking ε to zero, the producer can guarantee a final payoff of \tilde{u}_J^s . Since the payoffs, $(\tilde{u}_J^b, \tilde{u}_J^s)$, lie on the Pareto frontier achievable by the two agents with total assets the consumer has, and each can guarantee the payment, this final payoff is unique.

Now we show that the intermediate payoffs we constructed are unique in a simple SPE. Note first that in a simple SPE, the game effectively ends when active rounds end. Let J be the number of active rounds and the final payoffs are given by $(\tilde{u}_J^b, \tilde{u}_J^s)$, and, by backward induction, (77) must hold. Lemma 7 implies that there is a unique solution to that except for $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$. However, that payoff can be pinned down by the fact that buyer has to spend z/N in a simple SPE in round $J - 1$. Finally, when the output corresponding to $(\tilde{u}_J^b, \tilde{u}_J^s)$ is less than y^* , then $J = N$, and the solution to (77) is unique for all j . Since y^* is not achievable in any subgame, it follows that the SPE is unique.

Convergence to Gradual Nash Solution

We consider convergence of games with N even. The limit will be the same for N odd and hence we have convergence. Here we show that the limit intermediate payoffs converge as N approaches infinity in simple SPE in the following sense. Now, for each N and each $n \in \{1, 2, \dots, N\}$, define

$$[u_N^b(\tau), u_N^s(\tau)] = (u_n^b, u_n^s) \text{ if } \tau \in [(n-1)z/N, nz/N],$$

where (u_n^b, u_n^s) is an equilibrium intermediate payoff in the game with N rounds. We then show that $[u_N^b(\tau), u_N^s(\tau)]$ converges (pointwise) to $[u^b(\tau), u^s(\tau)]$, the solution to (6) and (7).

As we have seen, the sequence of intermediate equilibrium payoffs, $\{(\hat{u}_n^b, \hat{u}_n^s)\}_{n=1}^N$, satisfies $(\hat{u}_n^b, \hat{u}_n^s) = (\tilde{u}_n^b, \tilde{u}_n^s)$ for n even. Consider two bargaining rounds, $n - 1$ and $n + 1$, where n is an odd number. So, $(\hat{u}_{n-1}^b, \hat{u}_{n-1}^s)$ and $(\hat{u}_{n+1}^b, \hat{u}_{n+1}^s)$ are corresponding equilibrium intermediate payoffs.

Fix some τ and let $nz/N \rightarrow \tau$ as N goes to infinity. Let $\Delta u^b = \hat{u}_{n+1}^b - \hat{u}_{n-1}^b$ (note that $\hat{u}_{n+1}^b = \tilde{u}_n^b$) denote the buyer's incremental payoffs (on the equilibrium path) in rounds $n - 1$ and $n + 1$, and $\Delta u^s = \hat{u}_{n+1}^s - \hat{u}_{n-1}^s$

(note that $\tilde{u}_n^s = \tilde{u}_{n-1}^s$) denote the producer's incremental payoff (on the equilibrium path) in rounds $n - 1$ and $n + 1$. Similarly, let $\Delta z = 2z/N$. Then we have

$$H(\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s; \frac{n-1}{N}z) = 0 \quad (81)$$

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s; \frac{n-1}{N}z + \frac{\Delta z}{2}) = 0 \quad (82)$$

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s + \Delta u^s; \frac{n-1}{N}z + \Delta z) = 0. \quad (83)$$

According to (81) and (82), the producer's intermediate payoff is unchanged at \tilde{u}_{n-1}^s while the consumer's intermediate payoff increases by Δu^b . The amount of assets up for negotiation on the n^{th} frontier are nz/N . According to (83), at the end of round $n + 1$ the intermediate payoffs are obtained by moving vertically from the n^{th} frontier to the $(n + 1)^{\text{th}}$ frontier (since $n + 1$ is even).

A first-order Taylor series expansion of (82) in the neighborhood of $(u^b, u^s, \tau) = (\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s, \frac{n-1}{N}z)$ yields:

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s; \frac{n}{N}z) = H_1 \Delta u^b + H_3 \frac{\Delta z}{2} + o(\Delta u^b) + o(\frac{1}{N}),$$

where $\lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N \rightarrow \infty} N o(\frac{1}{N}) = 0$, we used that $H(\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s; \frac{n-1}{N}z) = 0$ from (81), and the partial derivatives H_1 , H_2 , and H_3 are evaluated at $(\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s, \frac{n-1}{N}z)$. Similarly, a first-order Taylor series expansion of (83) yields

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s + \Delta u^s; \frac{n+1}{N}z) = H_1 \Delta u^b + H_2 \Delta u^s + H_3 \Delta z + o(\Delta u^b) + o(\Delta u^s) + o(\frac{1}{N}),$$

where $\lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^s)}{\Delta u^s} = \lim_{N \rightarrow \infty} N o(\frac{1}{N}) = 0$. Using that $H = 0$ for payoffs on the Pareto frontiers, we obtain that

$$\begin{aligned} H_1 \Delta u^b + o(\Delta u^b) &= -H_3 \frac{\Delta z}{2} + o(\frac{1}{N}), \\ H_1 \Delta u^b + o(\Delta u^b) + H_2 \Delta u^s + o(\Delta u^s) &= -H_3 \Delta z + o(\frac{1}{N}), \\ o(\Delta u^b) + H_2 \Delta u^s + o(\Delta u^s) &= -H_3 \frac{\Delta z}{2} + o(\frac{1}{N}). \end{aligned}$$

From the first one and rearranging terms, we obtain

$$\frac{\Delta u^b}{\Delta z} = -\frac{H_3}{2H_1} + \frac{o(\Delta u^b)}{H_1 \Delta z} + \frac{o(\frac{1}{N})}{H_1 \Delta z}.$$

Note that

$$\lim_{N \rightarrow \infty} \frac{o(\frac{1}{N})}{H_1 \Delta z} = \frac{o(\frac{1}{N})N}{H_1 z} = 0 \text{ and } \lim_{N \rightarrow \infty} \frac{o(\Delta u^b)}{H_1 \Delta z} = \lim_{N \rightarrow \infty} \frac{o(\Delta u^b)}{H_1 z \Delta u^b} (\Delta u^b N) = 0,$$

where

$$\Delta u^b N = (\tilde{u}_{n+1}^b - \tilde{u}_{n-1}^b)N \in [[1 - v'(\tilde{y}_{n+1})/u'(\tilde{y}_{n+1})]z, [1 - v'(\tilde{y}_{n-1})/u'(\tilde{y}_{n-1})]z]$$

and hence its limit exists and is bounded away from zero by the concavity of the function $S(\bullet)$. Thus,

$$\frac{\partial u^b}{\partial \tau} = \lim_{N \rightarrow \infty} \frac{\Delta u^b}{\Delta z} = -\frac{1}{2} \frac{H_3}{H_1} = -\frac{1}{2} \frac{\partial H / \partial \tau}{\partial H / \partial u^b}.$$

Similarly, combining these two equations and rearranging, we obtain

$$\frac{\Delta u^s}{\Delta z} = -\frac{H_3}{2H_2} + \frac{o(\Delta u^b)}{H_2 \Delta z} + \frac{o(\Delta u^s)}{H_2 \Delta z} + \frac{o(\frac{1}{N})}{H_2 \Delta z}.$$

By the same arguments, we have

$$\frac{\partial u^s}{\partial \tau} = \lim_{N \rightarrow \infty} \frac{\Delta u^s}{\Delta z} = -\frac{1}{2} \frac{H_3}{H_2} = -\frac{1}{2} \frac{\partial H / \partial \tau}{\partial H / \partial u^s}.$$

These correspond to (6) and (7).

Extensions

Here we introduce a time frame within which the negotiation can occur. Suppose that the two players are given a specific amount of time, $\bar{\tau}$, to negotiate their trades. Each unit of the asset takes $1/\delta$ units of time and hence the maximum amount of assets that can be traded is $\bar{\tau}\delta$. Our target is the continuous time model but here we provide a discrete time foundation. So suppose that there are M rounds of bargaining, and hence each round of bargaining takes $\eta = \bar{\tau}/M$ units of time and in each round at most $\delta\eta$ units of assets can be put up for negotiation.

Let z be the consumer's asset holding. If $z \geq \bar{\tau}\delta$, then the game is exactly the same as in the last section with asset holding $z' = \bar{\tau}\delta$. So suppose that $z < \bar{\tau}\delta$. For simplicity we assume that there exists N such that $N = \frac{z}{\delta\eta} < M$, and hence it takes exactly N rounds to negotiate the whole asset holdings, and at each round up to z/N units of assets can be negotiated. As before, we use (y_n, p_n) to denote cumulative offers accepted up to round n .

Given these background assumptions, we now analyze the game. As before, we consider the case where the consumer makes the very last offer, at round M . We denote such a game by $(z, M, \bar{\tau}, \delta)$. The following is our proposition.

Proposition 13 *The game $(z, M, \bar{\tau}, \delta)$ has a unique SPE final payoffs that coincide with the final payoff of the game $[(u_0^b, u_0^s), N]$ with $u_0^b = 0 = u_0^s$ as constructed in Lemma 6, where $N = \frac{z}{\delta\eta}$.*

We prove this by induction. Indeed, if $N = M$, then the result follows directly from Lemma 6. Thus we shall prove this by induction on $M - N$. For this exercise we shall fix z and N , and increase M . Notice that for any M , once we reach round- $(M - N + 1)$, we are in the game $[(u_{M-N}^b, u_{M-N}^s), N]$, where

(u_{M-N}^b, u_{M-N}^s) is the intermediate payoffs reached at the end of round- $(M - N)$. As mentioned, when $M - N = 0$, $(u_{M-N}^b, u_{M-N}^s) = (0, 0)$ and hence the equilibrium final payoffs are given by $(\tilde{u}_N^b, \tilde{u}_N^s)$ computed by (68)-(69).

Suppose that N is even. Now consider $M = N + 1$ and hence the consumer is the first to make the offer. Then, at the first stage, the producer can secure a final payoff of \tilde{u}_N^s by rejecting any offer from the consumer. Moreover, we also know that in this game, any final payoff (u^b, u^s) must satisfy

$$H(u^b, u^s, z) \geq 0. \quad (84)$$

Since there is no other pair of final payoff (u^b, u^s) that satisfies both (84) and that $u^b > \tilde{u}_N^b$ and $u^s \geq \tilde{u}_N^s$, it follows that it is optimal for the consumer to offer $(0, 0)$ at the first stage, and hence $(\tilde{u}_N^b, \tilde{u}_N^s)$ is achievable; moreover, it is the unique equilibrium final payoff, as in any equilibrium we would have $u^s \geq \tilde{u}_N^s$ and $u^b \geq \tilde{u}_N^b$.

Suppose, by induction, that the result holds for some $M - 1$, $M > 0$. Then consider the game with M stages and suppose that M is even and hence the producer is the first to make the offer. By induction, we know that the consumer can secure a final payoff of \tilde{u}_N^b by rejecting any offer from the producer. As before, we also know that in this game, any final payoff (u^b, u^s) must satisfy (84). The rest of the argument then follows.

Note that there are other SPEs sharing the same SPE payoffs. For example, it is also an SPE that they finish bargaining in the initial N active rounds and then there is no trade in the remaining $M - N$ rounds.

Asymmetric bargaining powers

Here we revise our game to support gradual Nash solution with asymmetric bargaining power, denoted by θ . The parameter θ affects the game as follows. We assume that the number of rounds is an even number N , and the producer is the one making the first offer and the consumer is making the last offer.

1. In each round $n \in \{1, 3, \dots, N - 1\}$, it is the producer's turn to make an offer, with asset transfer within the range $[0, 2(1 - \theta)z/N]$; the consumer then decides to accept or reject the offer.
2. In each round $n \in \{2, 4, \dots, N\}$, it is the consumer's turn to make an offer, with asset transfer within the range $[0, 2\theta z/N]$; the producer then decides to accept or reject the offer.

Note that at the end of odd round n , the maximum cumulative asset transfer is $[2(n - 1) + 2(1 - \theta)]z/N$, and at the end of even round n , the maximum cumulative asset transfer is nz/N , for all $n = 2, 4, \dots, N$.

As before, to solve the game, we need to solve all possible subgames. Also, such subgame can still be characterized by $[(u_0^b, u_0^s), J]$, where (u_0^b, u_0^s) is the intermediate payoff at the beginning of the subgame and J is the number of remaining bargaining rounds.

Proposition 14 Fix some $\theta \in [0, 1]$. There exists a SPE in each alternating-ultimatum offer game, and all SPE share the same final payoffs. When the output level corresponding to the final payoffs is less than y^* , the SPE is unique and is simple; otherwise, there is a unique simple SPE. Moreover, in any simple SPE, the intermediate payoffs, $\{(u_n^b, u_n^s)\}_{n=1,2,\dots,N}$, converge to the solution $[u^b(\tau), u^s(\tau)]$ to the differential equations (13) and (14) as N approaches ∞ and $[(n-1) + 2(1-\theta)]z/N$ or nz/N approaches τ .

Note that Proposition 1 is a special case of Proposition 14 with $\theta = 1/2$. However, the proof of Proposition 14 follows exactly the same outline as that of Proposition 1. In particular, we will use the same technique to compute the final payoffs for any subgame, but with necessary modification to accommodate the fact that the consumer has control over θ fraction of assets to be negotiated every two rounds. As before, we can denote an arbitrary subgame by $[(u_0^b, u_0^s), J]$ with $0 \leq u_0^b + u_0^s < u(y^*) - v(y^*)$.

The final payoff is computed as follows. Define $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$ as $(\tilde{u}_0^b, \tilde{u}_0^s) = (u_0^b, u_0^s)$, and

$$H(\tilde{u}_j^b, \tilde{u}_{j-1}^s, p_0 + 2\theta z/N + (j-1)z/N) = 0, \text{ and } \tilde{u}_j^s = \tilde{u}_{j-1}^s, \text{ for } j \geq 1 \text{ odd}, \quad (85)$$

$$H(\tilde{u}_{j-1}^b, \tilde{u}_j^s, p_0 + jz/N) = 0, \text{ and } \tilde{u}_j^b = \tilde{u}_{j-1}^b, \text{ for } j \geq 2 \text{ even}, \quad (86)$$

where $p_0 = u[S^{-1}(u_0^b + u_0^s)] - u_0^b$. Below we show that the final equilibrium payoffs for the agents are given by $(\tilde{u}_J^b, \tilde{u}_J^s)$.

The logic behind this construction is exactly the same as the symmetric case, except for the fact that the consumer and the producer controls different shares of assets up for negotiation. In particular, when $J = 1$, the game $[(u_0^b, u_0^s), 1]$ is a standard take-it-or-leave-it offer game (with the consumer making the offer). Since the consumer can offer up to additional $2\theta z/N$ units of assets, the final payoff is computed by a rightward shift to next Pareto frontier with intermediate payments $p_0 + 2\theta z/N$, as in (85) with $j = 0$. When $J = 2$, the producer makes the first offer and take the final payoff for consumer in case he rejects the offer as given. Note that with $J = 2$ the final Pareto frontier has intermediate payment of $p_0 + 2z/N$, as in (86) with $j = 0$.

To compute the intermediate payoffs, we first define the functions F_r and F_u analogous to (71) and (72):

$$F_r(u^b, u^s) = (u^{b'}, u^{s'}) \text{ such that } u^{s'} = u^s \text{ and } H(u^{b'}, u^s, p + 2\theta z/N), \quad (87)$$

$$F_u(u^b, u^s) = (u^{b'}, u^{s'}) \text{ such that } u^{b'} = u^b \text{ and } H(u^b, u^{s'}, p + 2(1-\theta)z/N), \quad (88)$$

where $p = u[S^{-1}(u^b + u^s)] - u^b$. Now we are ready to explain how to compute intermediate payoffs. Consider a game $[(u_0^b, u_0^s), J]$ with J even. Using the same backward induction argument as in the symmetric case, if $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ is the equilibrium intermediate payoff for the agents at the end of round- $(J-1)$ bargaining, then

$$F_r(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s) = (\hat{u}_J^b, \hat{u}_J^s). \quad (89)$$

As before, the solution would be obtained by a leftward shift, but, under θ , to the lower Pareto frontier with intermediate payment lowered by $2\theta z/N$; that is,

$$H(\hat{u}_{J-1}^b, \hat{u}_J^s, p_0 + Jz/N - 2\theta z/N) = 0, \quad \hat{u}_{J-1}^s = \hat{u}_J^s. \quad (90)$$

Note that in this case, $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ and $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ do not lie on the same Pareto frontier unless $\theta = 1/2$.

In general, we can still use (77) to compute the equilibrium intermediate payoff at the end of round- $(J-j)$ bargaining, denoted by $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)$, with F_r and F_u defined by (87)-(88), and we have an analogous result to that of Lemma 7 for the existence and uniqueness of such a sequence. For j even the terms are obtained as before. For j odd, we need a second sequence, $\{(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)\}_{j=0}^{J-1}$ as follows: $(\bar{u}_J^b, \bar{u}_J^s) = (\hat{u}_J^b, \hat{u}_J^s)$, and

$$H(\bar{u}_{J-j}^b, \bar{u}_{J-j+1}^s, p_0 + (J-j-1)z/N + 2(1-\theta)z/N) = 0, \quad \text{and } \bar{u}_{J-j}^s = \bar{u}_{J-j+1}^s \text{ for } j \geq 1 \text{ odd}, \quad (91)$$

$$H(\bar{u}_{J-j+1}^b, \bar{u}_{J-j}^s, p_0 + (J-j)z/N) = 0, \quad \text{and } \bar{u}_{J-j}^b = \bar{u}_{J-j+1}^b \text{ for } j \geq 1 \text{ even}. \quad (92)$$

Graphically, for j odd, $(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)$ is obtained from $(\bar{u}_{J-j+1}^b, \bar{u}_{J-j+1}^s)$ by moving toward left to the next lower Pareto frontier, with a decrease of incremental transfer of $2\theta z/N$; for j even, $(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)$ is obtained from $(\bar{u}_{J-j+1}^b, \bar{u}_{J-j+1}^s)$ by moving downward to the next lower Pareto frontier, with a decrease of incremental transfer of $2(1-\theta)z/N$. Note that $(\bar{u}_{J-1}^b, \bar{u}_{J-1}^s) = (\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ given by (90). Note also that, in contrast to the symmetric case, $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)$ is situated in the same Pareto frontier as $(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)$ if and only if j is even; for j odd, $(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)$ lies on a different frontier.

Now we show that the intermediate payoffs converge to the same limit. As in the symmetric case, consider convergence of games with N even. The limit will be the same for N odd and hence we have convergence. By the above arguments we have that the sequence of intermediate equilibrium payoffs at the end of each round is given by $\{(\hat{u}_n^b, \hat{u}_n^s)\}_{n=1}^N$ with $(u_0^b, u_0^s) = (0, 0)$, and that $(\hat{u}_n^b, \hat{u}_n^s) = (\bar{u}_n^b, \bar{u}_n^s)$ for n even. Consider two bargaining rounds, n and $n+2$ with n even. So, $(\hat{u}_n^b, \hat{u}_n^s)$ and $(\bar{u}_{n+2}^b, \bar{u}_{n+2}^s)$ are corresponding equilibrium intermediate payoffs. Let $\Delta u^b = \bar{u}_{n+2}^b - \hat{u}_n^b$ denote the buyer's incremental payoffs (on the equilibrium path) in rounds n and $n+2$, and $\Delta u^s = \bar{u}_{n+2}^s - \hat{u}_n^s$ denote the producer's incremental payoff (on the equilibrium path) in rounds n and $n+2$. Let $\Delta z = 2z/N$ be the corresponding change in assets. Then we have

$$H(\hat{u}_n^b, \hat{u}_n^s; nz/N) = 0 \quad (93)$$

$$H(\hat{u}_n^b + \Delta u^b, \hat{u}_n^s; \theta\Delta z + \frac{n}{N}z) = 0 \quad (94)$$

$$H(\hat{u}_n^b + \Delta u^b, \hat{u}_n^s + \Delta u^s; nz/N + \Delta z) = 0. \quad (95)$$

A first-order Taylor series expansion of (94) in the neighborhood of $(u^b, u^s, \tau) = (\hat{u}_n^b, \hat{u}_n^s, \frac{n}{N}z)$ yields:

$$H(\hat{u}_n^b + \Delta u^b, \hat{u}_n^s; \theta\Delta z + \frac{n}{N}z) = H_1\Delta u^b + H_3\theta\Delta z + o(\Delta u^b) + o(\frac{1}{N}),$$

where $\lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N \rightarrow \infty} No(\frac{1}{N}) = 0$, we used that $H(\tilde{u}_n^b, \tilde{u}_n^s; \frac{n}{N}z) = 0$ from (93), and the partial derivatives H_1 , H_2 , and H_3 are evaluated at $(\tilde{u}_n^b, \tilde{u}_n^s, \frac{n}{N}z)$. Similarly, a first-order Taylor series expansion of (95) yields

$$H(\tilde{u}_n^b + \Delta u^b, \tilde{u}_n^s + \Delta u^s; \frac{n+2}{N}z) = H_1 \Delta u^b + H_2 \Delta u^s + H_3 \Delta z + o(\Delta u^b) + o(\Delta u^s) + o(\frac{1}{N}),$$

where $\lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^s)}{\Delta u^s} = \lim_{N \rightarrow \infty} No(\frac{1}{N}) = 0$. Using that $H = 0$ for payoffs on the Pareto frontiers, we obtain that

$$\begin{aligned} H_1 \Delta u^b + o(\Delta u^b) &= -H_3 \theta \Delta z + o(\frac{1}{N}), \\ H_1 \Delta u^b + o(\Delta u^b) + H_2 \Delta u^s + o(\Delta u^s) &= -H_3 \Delta z + o(\frac{1}{N}), \\ H_2 \Delta u^s + o(\Delta u^s) + o(\Delta u^b) &= -(1-\theta)H_3 \Delta z + o(\frac{1}{N}) \end{aligned}$$

From the first equation with rearranging, we obtain

$$\frac{\Delta u^b}{\Delta z} = -\theta \frac{H_3}{H_1} + \frac{o(\Delta u^b)}{H_1 \Delta z} + \frac{o(\frac{1}{N})}{H_1 \Delta z}.$$

Similarly, from the third equation with rearranging, we obtain

$$\frac{\Delta u^s}{\Delta z} = -(1-\theta) \frac{H_3}{H_2} + \frac{o(\Delta u^b)}{H_2 \Delta z} + \frac{o(\Delta u^s)}{H_2 \Delta z} + \frac{o(\frac{1}{N})}{H_2 \Delta z}.$$

Thus, we have

$$\begin{aligned} \frac{\partial u^b}{\partial \tau} &= \lim_{N \rightarrow \infty, 2n/N \rightarrow \tau} \frac{\Delta u^b}{\Delta z} = -\theta \frac{H_3}{H_1} = -\theta \frac{\partial H / \partial \tau}{\partial H / \partial u^b}, \\ \frac{\partial u^s}{\partial \tau} &= \lim_{N \rightarrow \infty, 2n/N \rightarrow \tau} \frac{\Delta u^s}{\Delta z} = -(1-\theta) \frac{H_3}{H_2} = -(1-\theta) \frac{\partial H / \partial \tau}{\partial H / \partial u^s}. \end{aligned}$$

Proof of Proposition 5

The proof follows the same logic as that of Proposition 1, and we only highlight key differences. Let z be consumer's asset holding. As before, we assume that consumer is the last to make an offer. The final payoff can be characterized in the same way as Lemma 6, but with some modifications. First, the function that characterizes the Pareto frontier is now $H(u^b, u^s, \bar{y})$ given by (19). The final payoffs are computed as follows. Consider a game $[(u_0^b, u_0^s), J]$ with $0 \leq u_0^b + u_0^s < u(y^*) - v(y^*)$. Define $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$ as $(\tilde{u}_0^b, \tilde{u}_0^s) = (u_0^b, u_0^s)$, and

$$H(\tilde{u}_j^b, \tilde{u}_{j-1}^s, y_0 + jy^*/N) = 0, \text{ and } \tilde{u}_j^s = \tilde{u}_{j-1}^s, \text{ for } j \geq 1 \text{ odd}, \quad (96)$$

$$H(\tilde{u}_{j-1}^b, \tilde{u}_j^s, y_0 + jy^*/N) = 0, \text{ and } \tilde{u}_j^b = \tilde{u}_{j-1}^b, \text{ for } j \geq 2 \text{ even}, \quad (97)$$

where $y_0 = S^{-1}(u_0^b + u_0^s)$. The key difference from Lemma 6 is that, since the payment is constrained by the total asset holding, z , the sequence may become constant before J . We use \tilde{J} to denote the number of rounds needed to reach the final payoff. It only requires minimal modification when $\tilde{J} < J$ and that can be handled in the same way as in Proposition 13, and construct a SPE in which no trade occurs until round $J - \tilde{J} + 1$. We also need to compute the intermediate payoffs. Since we focus on equilibria where no trade occurs until round $J - \tilde{J} + 1$, we can assume that $J = \tilde{J}$, and suppose that J is even. For each $j < J$ odd, define $\hat{u}_{J-j}^s = \tilde{u}_{J-j+1}^s$, and define \hat{u}_{J-j}^b be such that

$$\hat{u}_{J-j}^b + \hat{u}_{J-j}^s = u(\tilde{y}_{J-j}) - v(\tilde{y}_{J-j}).$$

For each $j \leq J - 2$ even, define

$$(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s) = (\tilde{u}_{J-j}^b, \tilde{u}_{J-j}^s).$$

Since the difference between the Pareto frontiers is decreasing, the intermediate payoff always lies to the northwest of the starting point.

Back to the original game with N rounds, we use \tilde{N} to denote the number of rounds needed to finish bargaining, which is an endogenous object. Convergence of intermediate payoffs follows exactly the same arguments as before, but we also need to consider convergence of \tilde{N}/N . The equations corresponding to (81)-(83) now should be modified to

$$H(\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s; \frac{n-1}{N}y^*) = 0 \quad (98)$$

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s; \frac{n-1}{N}y^* + \frac{\Delta y}{2}) = 0 \quad (99)$$

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s + \Delta u^s; \frac{n-1}{N}y^* + \Delta y) = 0, \quad (100)$$

and we require $n \leq \tilde{N}$, the number of rounds of negotiation needed before the assets are depleted. The rest of the argument is similar.

We also need to give the convergence result of \tilde{N} . Given z , (96)-(97) imply that \tilde{N} is determined by

$$\begin{aligned} & u\left(\frac{\tilde{N}}{N}y^*\right) - \sum_{j \leq \tilde{N} \text{ odd}} \left[S\left(\frac{j}{N}y^*\right) - S\left(\frac{j-1}{N}y^*\right) \right] \leq z \\ & < u\left(\frac{\tilde{N}+1}{N}y^*\right) - \sum_{j \leq \tilde{N}+1 \text{ odd}} \left[S\left(\frac{j}{N}y^*\right) - S\left(\frac{j-1}{N}y^*\right) \right]. \end{aligned} \quad (101)$$

Now, let $\xi_N = \tilde{N}/N$, and, by concavity of S ,

$$u(\xi_N y^*) - \frac{1}{2}S(\xi_N y^*) = z + O(1/N),$$

where $\lim_{N \rightarrow \infty} O(1/N) = 0$. Hence, $\lim_{N \rightarrow \infty} \xi_N = \xi$ with $\xi = y/y^*$ and

$$u(y) - \frac{1}{2}S(y) = z.$$

Finally, we show that the solution coincides with the axiomatic solution determined by

$$u^{b'}(\bar{y}) = -\frac{1}{2} \frac{\partial H(u^b, u^s, \bar{y}) / \partial \bar{y}}{\partial H(u^b, u^s, \bar{y}) / \partial u^b} \quad (102)$$

$$u^{s'}(\bar{y}) = -\frac{1}{2} \frac{\partial H(u^b, u^s, \bar{y}) / \partial \bar{y}}{\partial H(u^b, u^s, \bar{y}) / \partial u^s}. \quad (103)$$

Note that by (19), as long as the payment constraint does not bind, $p < z$, then $u^s < z - v(\bar{y})$ and the equation of the Pareto frontier is linear, at least locally. In that case the gradual solution gives:

$$u^{b'}(\bar{y}) = u^{s'}(\bar{y}) = \frac{1}{2} [u'(\bar{y}) - v'(\bar{y})]. \quad (104)$$

It then follows that the change in the payment over the gradual bargaining path is given by:

$$u^{b'}(\bar{y}) = u'(\bar{y}) - \frac{\partial p}{\partial \bar{y}} = \frac{1}{2} [u'(\bar{y}) - v'(\bar{y})].$$

Hence,

$$\frac{\partial p}{\partial \bar{y}} = \frac{1}{2} [u'(\bar{y}) + v'(\bar{y})].$$

Integrating from $\bar{y} = 0$ to $\bar{y} = y$, we obtain $p(y) = \frac{1}{2} [u(y) + v(y)]$, the payment function in the proposition.

We verify that the payment constraint does not bind up to z . Let $\tilde{y} = \min\{y^*, p^{-1}(z)\}$ with $p(y) = \frac{1}{2} [u(y) + v(y)]$. If $\tilde{y} = y^*$, then it is easy to see that the constraint never binds. Otherwise, for all $\bar{y} < \tilde{y}$, the constraint $p(\bar{y}) \leq z$ is not binding and hence the differential equations (102)-(103) apply.

Appendix C: Proof of Proposition 3

We use backward induction to prove Proposition 3.

Round N

Consider the alternating-offer game in the last round, N . The cumulated offer up to round N is (y_{N-1}, d_{N-1}) with associated payoff (u_{N-1}^b, u_{N-1}^s) . So, if no agreement is reached in round N , the terminal payoffs are (u_{N-1}^b, u_{N-1}^s) . The maximum wealth that can be negotiated at the end of round N is $z_N = d_{N-1} + z/N$. We will show that at the limit, when ξ_N goes to 1 (the risk of breakdown vanishes), the unique SPE payoffs of the subgame starting at the beginning of round N are determined according to the symmetric Nash solution:

$$\max_{u_N^b, u_N^s} (u_N^b - u_{N-1}^b) (u_N^s - u_{N-1}^s) \quad \text{s.t.} \quad H(u_N^b, u_N^s, z_N) = 0. \quad (105)$$

The terminal payoffs maximize the Nash product subject to the constraint that they belong to the Pareto frontier associated with z_N units of wealth. The ratio of the first-order conditions give

$$\frac{u_N^b - u_{N-1}^b}{u_N^s - u_{N-1}^s} = \frac{H_2(u_N^b, u_N^s, z_N)}{H_1(u_N^b, u_N^s, z_N)}. \quad (106)$$

At the optimum the slope of the Nash product is equal to the slope of the Pareto frontier.

We focus on the existence of the SPE and its construction. For simplicity we assume that y^* is never achieved. For the proof of uniqueness, see Rubinstein (1982). When it is his turn to make an offer the consumer proposes the (cumulative) offer (y^b, d^b) and the producer proposes (y^s, d^s) . The consumer and the producer have a reservation surplus to accept offers, u^b and u^s , respectively. The consumer's offer solves:

$$u_N^b = \max_{y^b, d^b} \{u(y^b) - d^b\} \quad \text{s.t.} \quad -v(y^b) + d^b \geq u^s \quad \text{and} \quad d^b \leq z_N. \quad (107)$$

The consumer maximizes his surplus subject to the constraint that his offer must generate a surplus for the producer that is at least equal to u^s and the offer must be feasible, $d^b \leq z_N$. Hence, u_N^b satisfies $H(u_N^b, u^s, z_N) = 0$. A solution to (107) exists provided that $u(y) - v(y) \geq u^s$ where $y = \min\{u^{-1}(z_N), y^*\}$. The reservation surplus of the producer solves

$$u^s = (1 - \xi_N)u_{N-1}^s + \xi_N [-v(y^s) + d^s]. \quad (108)$$

If the producer rejects the offer, his expected utility is equal to the weighted average of u_{N-1}^s , if the negotiation ends, and $-v(y^s) + d^s$ if the producer has the opportunity to make a counter-offer. Similarly, the producer's offer solves:

$$u_N^s = \max_{y^s, d^s} \{-v(y^s) - d^s\} \quad \text{s.t.} \quad u(y^s) - d^s = u^b \quad \text{and} \quad d^s \leq z_N, \quad (109)$$

where the reservation surplus of the consumer solves:

$$u^b = (1 - \xi_N)u_{N-1}^b + \xi_N [u(y^b) - d^b]. \quad (110)$$

Hence, u_N^s satisfies $H(u^b, u_N^s, z_N) = 0$. A solution to (109) exists provided that $u(y) - v(y) \geq u^b$ where $y = \min\{v^{-1}(z_N), y^*\}$. Substituting u^b and u^s by their expressions given by (108) and (110), the equilibrium payoffs, (u_N^b, u_N^s) , solve the following system of equations:

$$H[u_N^b, (1 - \xi_N)u_{N-1}^s + \xi_N u_N^s, z_N] = 0, \quad (111)$$

$$H[(1 - \xi_N)u_{N-1}^b + \xi_N u_N^b, u_N^s, z_N] = 0. \quad (112)$$

It is standard to check that for all $\xi_N < 1$ this system admits a unique solution. See Figure 16. By virtue of the one-stage-deviation principle, the proposed strategies form a SPE.

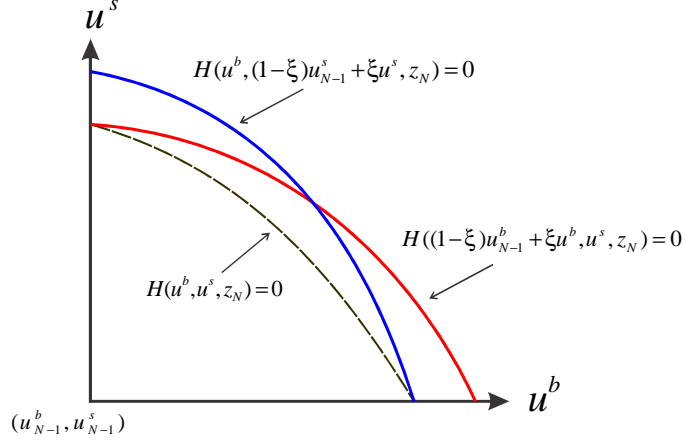


Figure 16: Determination of equilibrium payoffs

Let us consider the limit as ξ_N approaches to 1. Using a first-order Taylor series expansion we can rewrite (111)-(112) as:

$$\begin{aligned} H(u_N^b, u_N^s, z_N) - H_2(u_N^b, u_N^s, z_N)(1 - \xi_N)(u_N^s - u_{N-1}^s) &= o[(1 - \xi_N)], \\ H(u_N^b, u_N^s, z_N) - H_1(u_N^b, u_N^s, z_N)(1 - \xi_N)(u_N^b - u_{N-1}^b) &= o[(1 - \xi_N)], \end{aligned}$$

where H_j is the partial derivative with respect to the j^{th} argument, and $o[(1 - \xi_N)]/(1 - \xi_N)$ converges to 0 as ξ_N converges to 1. Rearranging the terms and take limits, we obtain:

$$\lim_{\xi_N \rightarrow 1} H_2(u_N^b, u_N^s, z_N)(u_N^s - u_{N-1}^s) - H_1(u_N^b, u_N^s, z_N)(u_N^b - u_{N-1}^b) = \lim_{\xi_N \rightarrow 1} o[(1 - \xi_N)]/(1 - \xi_N) = 0. \quad (113)$$

This equation coincides with the FOC for (105). Hence, the solution to the alternating-offer round game corresponds to the Nash solution with disagreement points (u_{N-1}^b, u_{N-1}^s) .

Terminal payoffs

We now make the following proposition for the determination of the terminal payoffs starting from any arbitrary round, and let N be the total number of rounds. Let the probability that the negotiation continues at the end of round n be denoted by ξ_n , $n = 1, \dots, N$. When solving the game with N rounds, we take the limit on the probability of negotiation breakdown. We solve the game by taking ξ_N to one first and obtain the solution to the subgame beginning from round N . Then we solve round $N - 1$, taking the limit of ξ_N at 1 as given. Then we take ξ_{N-1} to one, and so on.

We also need to expand the notation slightly. Let (u_{n-1}^b, u_{n-1}^s) be a given intermediate payoff at the beginning of round n , and let d_{n-1} be the corresponding cumulative transfer of assets; i.e., $H(u_{n-1}^b, u_{n-1}^s, d_{n-1}) =$

0. Define $F(u_{n-1}^b, u_{n-1}^s) = (u_n^b, u_n^s)$ to be the solution of

$$\max_{u_n^b, u_n^s} (u_n^b - u_{n-1}^b) (u_n^s - u_{n-1}^s) \quad \text{s.t.} \quad H(u_n^b, u_n^s, d_{n-1} + z/N) = 0. \quad (114)$$

Proposition 15 *Consider the subgame starting from the beginning of round $n \in \{1, \dots, N\}$ with intermediate payoffs, (u_{n-1}^b, u_{n-1}^s) , where $H(u_{n-1}^b, u_{n-1}^s, d_{n-1}) = 0$. Take limits in the following order: $\xi_N \rightarrow 1$, $\xi_{N-1} \rightarrow 1, \dots$, $\xi_n \rightarrow 1$. The terminal payoffs, (u_N^b, u_N^s) , are obtained recursively from (u_{n-1}^b, u_{n-1}^s) according to:*

$$\max_{u_{n+j}^b, u_{n+j}^s} (u_{n+j}^b - u_{n+j-1}^b) (u_{n+j}^s - u_{n+j-1}^s) \quad \text{s.t.} \quad H(u_{n+j}^b, u_{n+j}^s, z_{n+j}) = 0, \quad j = 0 \dots N - n, \quad (115)$$

where $z_{n+j} = d_{n-1} + (1+j)z/N$.

The recursion (115) generates a sequence of payoffs, $\{(u_{n+j}^b, u_{n+j}^s)\}_{j=0}^{N-n}$, where each element, (u_{n+j}^b, u_{n+j}^s) , corresponds to the Nash solution of a bargaining problem with endogenous disagreement points, $(u_{n+j-1}^b, u_{n+j-1}^s)$, and Pareto frontier corresponding to the wealth z_{n+j} . We illustrate this construction in Figure 5 for the subgame starting in $N - 2$.

We prove the proposition by induction. We have shown that the proposition holds for round N . We now show that if the proposition holds for some arbitrary round n , then it holds for round $n - 1$. Consider the beginning of round $n - 1$ with intermediate payoffs, (u_{n-2}^b, u_{n-2}^s) , where $H(u_{n-2}^b, u_{n-2}^s, d_{n-2}) = 0$. We also assume that at round $n - 1$, it is the consumer to make the first offer.

In order to characterize the outcome of the alternating offer bargaining game in round $n - 1$ we need to compute the payoffs in case the negotiation ends without an agreement. In the event of a breakdown in round $n - 1$, then the players move to round n but keep the same intermediate payoffs, (u_{n-2}^b, u_{n-2}^s) . By inductive assumption, since the proposition holds for round n , the terminal payoffs in that subgame, denoted (u_{N-1}^b, u_{N-1}^s) , are given by $(u_{N-1}^b, u_{N-1}^s) = F^{N-n}(u_{n-1}^b, u_{n-1}^s)$, if we take the limits $\xi_N \rightarrow 1$, $\xi_{N-1} \rightarrow 1, \dots$, $\xi_n \rightarrow 1$, in that order.

Since our induction hypothesis allows us to compute the terminal payoffs from any intermediate payoffs in the beginning of round n , for any outcome from round $n - 1$, we can compute the continuation value. First let

$$\mathcal{H}_N = \{(u_N^b, u_N^s) \geq 0 : H(u_N^b, u_N^s, z_N) \geq 0\}$$

be the set of all possible individually rational final payoffs given the initial disagreement point. We use $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$ to denote the set of all terminal payoffs, (u_N^b, u_N^s) , attainable from (u_{n-2}^b, u_{n-2}^s) , for which the corresponding allocation is given by (y_{n-2}, d_{n-2}) , according to the induction hypothesis, if an offer at

round $n - 1$ is accepted:

$$\begin{aligned} \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) &= \{F^{N-n+1}(u_{n-1}^b, u_{n-1}^s) : \exists(y_{n-1}, d_{n-1}) \geq (y_{n-2}, d_{n-2}), d_{n-1} - d_{n-2} \leq z/N \\ &\quad \text{such that } u_{n-1}^b = u(y_{n-1}) - d_{n-1}, u_{n-1}^s = -v(y_{n-1}) + d_{n-1}\}. \end{aligned}$$

Note that $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) \subset \mathcal{H}_N$ is nonempty, as $(u_{N-1}^b, u_{N-1}^s) \equiv F^{N-n+1}(u_{n-2}^b, u_{n-2}^s) \in \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$, which is attained if no trade is offered. Moreover, $(\hat{u}_N^b, \hat{u}_N^s) = F^{N-n+2}(u_{n-2}^b, u_{n-2}^s) \in \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$ as well, which is attained if the offer corresponding to $(\hat{u}_{n-1}^b, \hat{u}_{n-1}^s) = F(u_{n-2}^b, u_{n-2}^s)$, denoted by $(\hat{y}_{n-1}, \hat{d}_{n-1})$, is offered and accepted. Moreover, since the cumulative offer, $(\hat{y}_{n-1}, \hat{d}_{n-1})$, is interior, i.e., $(\hat{y}_{n-1}, \hat{d}_{n-1}) > (y_{n-2}, d_{n-2})$, by continuity, there exists a neighborhood \mathcal{O} around $(\hat{u}_N^b, \hat{u}_N^s)$ such that

$$\mathcal{O} \cap \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) \tag{116}$$

is open relative to \mathcal{H}_N .

Thus, using these terminal payoffs, the game in round $n - 1$ can be reduced to the following game: the two players take turns to make an offer $(u_N^b, u_N^s) \in \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$. If accepted, the game ends with the terminal payoff (u_N^b, u_N^s) . Otherwise, with probability ξ_{n-1} the other player makes an offer; with probability $1 - \xi_{n-1}$ the game ends with payoff (u_{N-1}^b, u_{N-1}^s) . Note that only payoffs $(u_N^b, u_N^s) \geq (u_{N-1}^b, u_{N-1}^s)$ are relevant, for offers that lead to other payoffs are dominated by them. We claim that for ξ_{n-1} sufficiently large, the equilibrium payoffs, (u_N^b, u_N^s) , solve the following system of equations:

$$H[u_N^b, (1 - \xi_{n-1})u_{N-1}^s + \xi_{n-1}u_N^s, z_N] = 0, \tag{117}$$

$$H[(1 - \xi_{n-1})u_{N-1}^b + \xi_{n-1}u_N^b, u_N^s, z_N] = 0. \tag{118}$$

First we note that if $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) = \mathcal{H}_N^{IR} \equiv \{(u_N^b, u_N^s) \in \mathcal{H}_N : (u_N^b, u_N^s) \geq (u_{N-1}^b, u_{N-1}^s)\}$, then this follows from the same argument as that for round N . The set \mathcal{H}_N^{IR} consists of all individually rational final payoffs relative to the disagreement point (u_{N-1}^b, u_{N-1}^s) . Now, since $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) \subset \mathcal{H}_N$ and anything that is not individually rational is dominated by (u_{N-1}^b, u_{N-1}^s) , the proof is still valid as long as the final payoffs correspond to the solutions, $[u_N^b, (1 - \xi_{n-1})u_{N-1}^s + \xi_{n-1}u_N^s]$ and $[(1 - \xi_{n-1})u_{N-1}^b + \xi_{n-1}u_N^b, u_N^s]$, belong to $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$. By earlier argument we know that those solutions converge to $(\hat{u}_N^b, \hat{u}_N^s)$. Thus, for ξ_{n-1} sufficiently large, such solutions also belong to \mathcal{O} given by (116). Finally, the fact that the solution converges to the Nash solution as ξ_n approaches 1 follows exactly the same argument as round N . This proves that the proposition holds at $n - 1$. Given that it holds at N , by induction it holds for all $n \geq 0$.

Intermediate payoffs

We determine the equilibrium terminal payoffs at the start of the whole game by using the initial condition $(u_0^b, u_0^s) = (0, 0)$ and (115), i.e.,

$$\max_{u_n^b, u_n^s} (u_n^b - u_{n-1}^b) (u_n^s - u_{n-1}^s) \quad \text{s.t.} \quad H\left(u_n^b, u_n^s, \frac{n}{N}z\right) = 0.$$

We obtain a sequence $\{(u_n^b, u_n^s)\}_{n=0}^N$ where the last term corresponds to the terminal payoffs. Let's now denote $\{(\tilde{u}_n^b, \tilde{u}_n^s)\}_{n=0}^N$ the sequence of intermediate payoffs along the SPE. We determine this sequence by backward induction starting from $(\tilde{u}_N^b, \tilde{u}_N^s) = (u_N^b, u_N^s)$. Consider the alternating offer game in round N . Its solution is given by

$$(u_N^b, u_N^s) = \arg \max_{u_N^b, u_N^s} (u_N^b - \tilde{u}_{N-1}^b) (u_N^s - \tilde{u}_{N-1}^s) \quad \text{s.t.} \quad H(u_N^b, u_N^s, z) = 0.$$

By the definition of $\{(u_n^b, u_n^s)\}_{n=0}^N$ it follows that $(\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s) = (u_{N-1}^b, u_{N-1}^s)$.

Let's now move to round $N-1$. The disagreement point is $(\hat{u}_{N-1}^b, \hat{u}_{N-1}^s)$ solution to

$$(\hat{u}_{N-1}^b, \hat{u}_{N-1}^s) = \arg \max_{u_{N-1}^b, u_{N-1}^s} (u_{N-1}^b - \tilde{u}_{N-2}^b) (u_{N-1}^s - \tilde{u}_{N-2}^s) \quad \text{s.t.} \quad H\left(u_{N-1}^b, u_{N-1}^s, \frac{N-1}{N}z\right) = 0.$$

Given this disagreement point the terminal payoffs solve:

$$\max_{u_N^b, u_N^s} (u_N^b - \hat{u}_{N-1}^b) (u_N^s - \hat{u}_{N-1}^s) \quad \text{s.t.} \quad H(u_N^b, u_N^s, z) = 0.$$

It follows that $(\hat{u}_{N-1}^b, \hat{u}_{N-1}^s) = (u_{N-1}^b, u_{N-1}^s)$ and hence $(\tilde{u}_{N-2}^b, \tilde{u}_{N-2}^s) = (u_{N-2}^b, u_{N-2}^s)$. We can iterate this procedure and obtain that $(\tilde{u}_n^b, \tilde{u}_n^s) = (u_n^b, u_n^s)$ for all n . This then proves (17).

Gradual bargaining: limit as $N \rightarrow \infty$

The FOCs of the Nash problems above give

$$\frac{u_n^s - u_{n-1}^s}{u_n^b - u_{n-1}^b} = \frac{H_1\left(u_n^b, u_n^s, \frac{n}{N}z\right)}{H_2\left(u_n^b, u_n^s, \frac{n}{N}z\right)}.$$

Denote $\tau = nz/\delta N$. Divide both the numerator and the denominator of the left side by $z/\delta N$ and take the limit as N tends to infinity to obtain $u^{s'}(\tau)/u^{b'}(\tau)$. This gives:

$$\frac{u^{s'}(\tau)}{u^{b'}(\tau)} = \frac{H_1(u_\tau^b, u_\tau^s, \delta\tau)}{H_2(u_\tau^b, u_\tau^s, \delta\tau)}.$$

This differential equation coincides with (8).

Appendix D. Gradual bargaining and prices in OTC markets

In order to illustrate the versatility of our approach we now reinterpret our model as one where agents, called investors, have idiosyncratic valuations for an illiquid asset that can only be traded through pairwise meetings, similar to Duffie et al. (2005, 2007). At the end of each period, each agent receives an equal endowment of Lucas trees, Ω , that pay off at the end of the following period. The payoff from holding ω units of trees is $\varepsilon f(\omega)$ where $\varepsilon \in \{\varepsilon_h, \varepsilon_\ell\}$ is an idiosyncratic valuation with $\varepsilon_h > \varepsilon_\ell > 0$. Upon entering the DM half of the agents draw ε_h while the other half draw ε_ℓ . These Lucas trees can only be traded in an OTC market, through pairwise meetings, in the DM. The efficient trade size between an h -investor and an ℓ -investor is such that $\varepsilon_h f'(\Omega + y^*) = \varepsilon_\ell f'(\Omega - y^*)$.

In accordance with the literature on OTC markets, investors can either meet directly or they can trade through dealers. Dealers are risk-neutral agents with linear preferences for the numeraire who have access to a competitive interdealer market in the DM. Upon contact with a dealer, investors can buy and sell assets at the competitive interdealer price in terms of the numeraire, q , in exchange for the payment of an intermediation fee, φ , also expressed in the numeraire.

Investors, who cannot commit, must accumulate liquid assets to pay for illiquid Lucas trees. The liquid asset takes the form of fiat money with $d = 0$. The supply of money grows at rate $\pi \in (\beta - 1, \infty)$, $A_{t+1} = (1 + \pi)A_t$, where new money distributed in a lump-sum fashion. We denote $i \equiv (1 + \rho)(1 + \pi) - 1$ as the cost of holding money. We assume that dealers can commit to deliver the assets they purchase on behalf of investors in the interdealer market.

The matching technology in the OTC market is described as follows. We denote α^u the product of the probability of drawing a high (low) valuation times the probability of being matched with a low (high) valuation investor. We denote α^d the probability of drawing a high (low) valuation times the probability of meeting a dealer.

We need to make assumptions on how agents bargain in these different meetings. For simplicity we assume that $\lambda = 0$, which corresponds to the case where the time constraint never binds. In matches between investors, we follow our approach in Section 5 and assume that agents bargain gradually over the liquid asset, here fiat money. We later compare the equilibrium outcome to the one where agents bargain gradually over the illiquid asset. In matches between a dealer and an investor, we assume that the investor sells gradually the asset he is offering, i.e., money in matches with h -investors and the illiquid asset in matches with ℓ -investors. As shown in Proposition 6, this choice corresponds to each investor's preferred agenda.

Consider a match between an h -investor and an ℓ -investor. The solutions from the previous sections apply, where we define $u(y) \equiv \varepsilon_h [f(\Omega + y) - f(\Omega)]$ and $v(y) \equiv \varepsilon_\ell [f(\Omega) - f(\Omega - y)]$. It follows that the payment function for the illiquid asset is

$$p^u(y) = \int_0^y \frac{2\varepsilon_\ell f'(\Omega - x)\varepsilon_h f'(\Omega + x)}{\varepsilon_h f'(\Omega + x) + \varepsilon_\ell f'(\Omega - x)} dx. \quad (119)$$

Hence, at the margin, the price of an illiquid asset is

$$p^{u'}(y) = 2 \left(\frac{1}{\varepsilon_\ell f'(\Omega - y)} + \frac{1}{\varepsilon_h f'(\Omega + y)} \right)^{-1}.$$

The price is the harmonic mean of the marginal productivities of the buyer and the producer.

We now turn to a match between an h -investor holding z real balances and a dealer. An allocation, (y, φ^a) , specifies a quantity of assets purchased by the dealer on behalf of the investor and a payment (in real balances) equal to $qy + \varphi^a$, where $q + \varphi^a/y$ is interpreted as an average ask price, and φ^a is the intermediation fee to the dealer associated with this ask price. The allocation is subject to the feasibility constraint, $qy + \varphi^a \leq z$. (This feasibility constraint differs from the one in Lagos and Zhang (2018) where it is assumed that $qy \leq z$ and φ^a is financed with credit repaid in the CM. This formulation makes their model with linear f and Nash bargaining more tractable.) The surplus of the investor is $u^b = \varepsilon_h f(\Omega + y) - qy - \varphi^a - \varepsilon_h f(\Omega)$ while the dealer's profits are $u^d = \varphi^a$. Applying the gradual bargaining solution where the agenda specifies that the h -investor sells his real balances gradually over time, the marginal surplus of the buyer is

$$u^{b'}(z) = \frac{\varepsilon_h f'(\Omega + y)/q - 1}{2}, \quad (120)$$

if $y \leq \tilde{y}_q^h$ where $\varepsilon_h f'(\Omega + \tilde{y}_q^h) = q$ and $u^{b'}(z) = 0$ otherwise. According to (120) the increase in the buyer's surplus from an additional unit of real balances is half of the gains that the buyer would enjoy by purchasing assets in the interdealer market directly. By the definition of the buyer's payoff, $u^{b'}(z) = \varepsilon_h f'(\Omega + y) \partial y / \partial z - 1$. Substituting this expression into (120) and integrating, the total payment for y units of assets is

$$p^a(y) = \varphi^a(y) + qy = q \int_0^y \frac{2\varepsilon_h f'(\Omega + x)}{\varepsilon_h f'(\Omega + x) + q} dx,$$

for all $y \leq \tilde{y}_q^h$. This payment function is increasing and concave in y . Hence, the average ask price decreases with trade size and increases with the investor's valuation, ε_h .

In a match between an ℓ -investor and a dealer, an allocation, (y, φ^b) , specifies the quantity y of assets purchased by the dealer in exchange for a payment $qy - \varphi^b$, where $q - \varphi^b/y$ is the average bid price and φ^b is the intermediation fee to the dealer associated with this bid price. The investor's surplus is $u^s =$

$\varepsilon_\ell f(\Omega - y) + qy - \varphi^b - \varepsilon_\ell f(\Omega)$ and the dealer's profits are φ^b . If the ℓ -investor sells his assets gradually over time, then the total payment function is given by the egalitarian solution:

$$p^b(y) = qy - \varphi^b(y) = \frac{qy + \varepsilon_\ell f(\Omega) - \varepsilon_\ell f(\Omega - y)}{2},$$

for all $y \leq \tilde{y}_q^\ell$ where $\varepsilon_\ell f'(\Omega - \tilde{y}_q^\ell) = q$. This function is increasing and convex in y . Hence, the average bid price is increasing in y . The optimal y maximizes $\varepsilon_\ell f(\Omega - y) + p^b(y)$, i.e., assuming an interior solution,

$$q = \varepsilon_\ell f'(\Omega - y^d), \quad (121)$$

where we use y^d to denote the amount of assets traded between an ℓ -investor and a dealer. In equilibrium, this will also be the amount traded between an h -investor and a dealer.

The investor's optimal choice of real balances, assuming an interior solution, satisfies a generalized version of (26), i.e.,

$$i = \frac{\alpha^u}{2} \left[\frac{\varepsilon_h f'(\Omega + y^u)}{\varepsilon_\ell f'(\Omega - y^u)} - 1 \right] + \frac{\alpha^d}{2} \left[\frac{\varepsilon_h f'(\Omega + y^d)}{\varepsilon_\ell f'(\Omega - y^d)} - 1 \right], \quad (122)$$

where $y^u = \min\{y^*, (p^u)^{-1}(z)\}$ is the amount of asset traded in direct trades, $y^d = \min\{\tilde{y}_q^h, (p^a)^{-1}(z; q)\}$ is the amount of asset traded in intermediated trades, and we have replaced q by its expression above. The first term on the right side of (122) is the marginal benefit of real balances to the investor in direct trades. The second term is the marginal benefit in intermediated trades. An equilibrium is a list (z, y^u, y^d, q) solution to (121), (122), and the bargaining outcomes.

Consider first an OTC market without dealers, $\alpha^d = 0$. The trade size is uniquely determined by (122) and it is such that $\partial y^u / \partial i < 0$. Moreover, as i approaches 0, y^u approaches y^* . The same results hold if agents bargain gradually over the illiquid asset since in that case the bargaining solution coincides with the proportional solution. However, the trade size is larger if agents bargain gradually over the liquid asset instead of the illiquid one. This is another illustration of how the agenda of the negotiation matters for allocations and welfare. If agents bargain according to Nash, then $y^u < y^*$ even when i is driven to 0. So trade volume is inefficiently low. Gradual bargaining leads to larger trade sizes and larger trade volume by allowing agents to capture some of the gains from trade that each unit of real balances generates. We summarize these results in the following proposition.

Proposition 16 (Gradual bargaining in OTC markets) *Suppose $\alpha^d = 0$.*

1. **(Gradual bargaining over real balances)** *If $(\varepsilon_h - \varepsilon_\ell)/(2\varepsilon_\ell) > i/\alpha$, then there exists a unique steady-state monetary equilibrium. It is such that y approaches y^* as i approaches 0.*

2. (**Gradual bargaining over illiquid assets**) If $(\varepsilon_h - \varepsilon_\ell)/(\varepsilon_h + \varepsilon_\ell) > i/\alpha$, then there exists a unique steady-state monetary equilibrium. It is such that y approaches y^* as i approaches 0. The trade size, y , is lower if agents bargain gradually over the illiquid asset instead of bargaining gradually over real balances.

3. (**Nash bargaining**) In any steady-state monetary equilibrium, $y < y^*$.

Proof. (1) The equilibrium condition is given by (122) with $\alpha^d = 0$, which can be rewritten as

$$\frac{f'(\Omega + y)}{f'(\Omega - y)} = \frac{\varepsilon_\ell}{\varepsilon_h} \left(1 + \frac{2i}{\alpha^u} \right), \quad (123)$$

To have a solution with $y > 0$, it is necessary and sufficient that $\frac{\varepsilon_\ell}{\varepsilon_h} \left(1 + \frac{2i}{\alpha^u} \right) < 1$, that is, $(\varepsilon_h - \varepsilon_\ell)/2\varepsilon_\ell > i/\alpha^u$.

(2) First we derive the equilibrium condition as in (122). When the agents bargain over DM asset, the payment is determined by Egalitarian solution and hence

$$p^{DM}(y) = \frac{\varepsilon_h [f(\Omega + y) - f(\Omega)] + \varepsilon_\ell [f(\Omega) - f(\Omega - y)]}{2}.$$

Thus, the FOC for the consumer is given by

$$-ip^{DM'}(y) + \alpha^u [u'(y) - p^{DM'}(y)] = 0,$$

which can be rewritten as

$$\frac{f'(\Omega + y)}{f'(\Omega - y)} = \frac{\varepsilon_\ell}{\varepsilon_h} \left(\frac{i + \alpha^u}{\alpha^u - i} \right), \quad (124)$$

To have a solution with $y > 0$, it is necessary and sufficient that $\frac{\varepsilon_\ell}{\varepsilon_h} \left(\frac{i + \alpha^u}{\alpha^u - i} \right) < 1$, that is, $(\varepsilon_h - \varepsilon_\ell)/(\varepsilon_h + \varepsilon_\ell) > i/\alpha^u$. Moreover, since

$$\left(\frac{i + \alpha^u}{\alpha^u - i} \right) > \left(1 + \frac{2i}{\alpha^u} \right),$$

the y that solves (123) is larger than that that solves (124).

(3) Following Proposition 4, the payment determined by Nash solution is given by

$$p^N(y) = \frac{\varepsilon_\ell f'(\Omega - y)}{\varepsilon_h f'(\Omega + y) + \varepsilon_\ell f'(\Omega - y)} \varepsilon_h [f(\Omega + y) - f(\Omega)] + \frac{\varepsilon_h f'(\Omega - y)}{\varepsilon_h f'(\Omega + y) + \varepsilon_\ell f'(\Omega - y)} \varepsilon_\ell [f(\Omega) - f(\Omega - y)],$$

Hence, the payoff of the h -investor is

$$\frac{\varepsilon_h f'(\Omega + y) \{ \varepsilon_h [f(\Omega + y) - f(\Omega)] - \varepsilon_\ell [f(\Omega) - f(\Omega - y)] \}}{\varepsilon_h f'(\Omega + y) + \varepsilon_\ell f'(\Omega - y)}.$$

It is easy to check that close to y^* this surplus is decreasing. Hence, under Nash bargaining the trade size is inefficiently low for all $i \geq 0$. ■

Consider the other polar case of a pure dealer market where all trades are intermediated, $\alpha^u = 0$ (This corresponds to the version of the model by Lagos and Rocheteau (2007, 2009), and Lagos and Zhang (2018)). From (122), the equilibrium trade size is the solution to:

$$\frac{\varepsilon_h f'(\Omega + y^d)}{\varepsilon_\ell f'(\Omega - y^d)} \leq 1 + \frac{2i}{\alpha^d}, \quad " = " \text{ if } y^d > 0. \quad (125)$$

The trade size decreases with i and increases with α^d . As i goes to 0 then y^d tends to y^* . In accordance with Proposition 16, the Friedman rule implements the first best trade size under gradual bargaining while it fails to do so under Nash bargaining. From (121) the interdealer price decreases with i because as i goes up, investors reduce their real balances, which reduces the demand for illiquid assets.

Finally, consider an economy with both $\alpha^u > 0$ and $\alpha^d > 0$. First, replacing q by its expression given by (121) into $p^a(y)$, we obtain $p^u(y) < p^a(y)$ for all $y \leq y^d$. For the same trade size, buyers pay less in direct trades than in intermediated trades. It follows that for i close to 0, investors trade the first best in direct trades, $y^u = y^*$, while they are liquidity constrained in trades with dealers, i.e., y^d solves (125). So for low interest rates, an increase in i does not affect prices and trade sizes in direct trades but it reduces trade sizes in intermediated trades.